Modeling and Analysis of DYNAMIC SYSTEMS

Second Edition

Ramin S. Esfandiari
Bei Lu

CRC Press
Taylor & Francis Group

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20140220

International Standard Book Number-13: 978-1-4665-7495-3 (eBook - PDF)

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http://www.taylorandfrancis.com

and the CRC Press Web site at
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To my wife Haleh,
my sisters Mandana and Roxana,
and my parents to whom I owe it all.

R. E.

To my husband Qifu,
and my parents.

B. L.

Contents

Preface

The principal goal of this edition essentially is the same as that of the first edition, that is, to provide the reader with a thorough knowledge of mathematical modeling and analysis of dynamic systems. MATLAB®, Simulink®, and Simscape™ are introduced at the outset and are utilized throughout the book to perform symbolic, graphical, numerical, and simulation tasks. The textbook, written at the junior level, meticulously covers techniques for modeling dynamic systems, methods of response analysis, and an introduction to vibration and control systems.

The book comprises 10 chapters and two appendices. Chapter 1 methodically introduces MATLAB, Simulink, and Simscape to the reader. The essential mathematical background is covered in Chapter 2 (complex analysis, differential equations, and Laplace transformation) and Chapter 3 (matrix analysis). Different forms of system model representation (state-space form, transfer function, input–output equation, block diagram, etc.), as well as linearization, are discussed in Chapter 4. Each topic is also handled using MATLAB. Block diagrams are constructed and analyzed by using Simulink.

Chapter 5 treats translational, rotational, and mixed mechanical systems. The free-body diagram approach is greatly emphasized in the derivation of the systems' equations of motion. Electrical and electromechanical systems are covered in Chapter 6. Also included are operational amplifiers and impedance methods. Chapter 7 discusses pneumatic, liquid-level, and thermal systems. Modeling and analysis of dynamic systems ranging from mechanical to thermal using Simulink and Simscape are fully integrated in Chapters 5 through 7.

Chapter 8 deals with time-domain and frequency-domain analysis of dynamic systems. Time-domain analysis entails transient response of first-, second-, and higher-order systems. The sinusoidal transfer function (frequency response function) is introduced and utilized in obtaining the system's frequency response, as well as the Bode diagram. Analytical solution of the state equation is also included in this chapter. MATLAB and Simulink play significant roles in determining and simulating system response, and are used throughout the chapter.

Chapter 9 presents an introduction to vibrations and includes free and forced vibrations of single and multiple-degrees-of-freedom systems, vibration suppression including vibration isolators and absorbers, modal analysis, and vibration testing. Also included are some applications of vibrations: logarithmic decrement for experimental determination of the damping ratio, rotating unbalance, and harmonic base excitation.

Chapter 10 gives an introduction to control systems analysis and design in the time and frequency domains. Basic concepts and terminology are presented first, followed by stability analysis, types of control, root-locus analysis, Bode plot, and full-state feedback. All these analytical techniques are implemented using MATLAB, Simulink, and Simscape.

APPENDICES

Appendix A gives a summary of systems of units and conversion tables. Appendix B contains useful formulas such as trigonometric identities and integrals.

EXAMPLES AND EXERCISES

Each covered topic is followed by at least one example for a better comprehension of the subject matter at hand. More complex topics will be accompanied by multiple, painstakingly worked-out examples. Each section of each chapter is followed by several exercises so that the reader can

immediately apply the ideas just learned. End-of-chapter review exercises help in learning how a combination of different ideas can be used to analyze a problem.

CHAPTER SUMMARIES

Chapter summaries provide succinct reviews of the key aspects of each chapter.

INSTRUCTOR'S SOLUTIONS MANUAL

A solutions manual, featuring complete solution details of all exercises, is prepared by the authors and will be available to instructors adopting the book.

NEW TO THIS EDITION

The main new features of this edition revolve around its full integration of the MATLAB Simscape Toolbox, as well as the usage of Simulink for new purposes. In particular,

1. Modeling and analysis of dynamic systems ranging from mechanical to thermal using Simscape is fully covered in Chapters 5 through 7.
2. Simulink is utilized for linearization, as well as simulation of nonlinear dynamic systems.
3. Integration of Simscape into Simulink for control system analysis and design gives the reader better insight into the involvement of actual physical components rather than their mathematical representations.

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Bei Lu, PhD

Acknowledgment

The authors express their deep gratitude to Jonathan Plant (Senior Editor, Mechanical, Aerospace, Nuclear and Energy Engineering) at Taylor & Francis/CRC Press for his assistance in various stages of the development of this book.

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1 Introduction to MATLAB®, Simulink®, and Simscape®

This chapter introduces the fundamental features of MATLAB, Simulink, and Simscape that are important in modeling and analysis of dynamic systems. These include the descriptions and applications of several commonly used built-in functions (commands) in MATLAB and the basics of building block diagrams for the purpose of simulation of dynamic systems using Simulink and Simscape. MATLAB, Simulink, and Simscape are fully integrated throughout the book, and the fundamental features and capabilities introduced in this chapter will play a crucial role in better understanding the more advanced applications in the subsequent chapters.

1.1 MATLAB COMMAND WINDOW AND COMMAND PROMPT

Once a MATLAB session is opened, commands can be entered at the MATLAB command prompt `">>"` (Figure 1.1). For example, $\sqrt{\sin\left(\frac{3\pi}{4}\right)}$ can be calculated as

```
>> sqrt(sin(3*pi/4))
ans =
    0.8409
```

The outcome of a calculation can be stored under a variable name, and suppressed by using a semicolon at the end of the statement:

```
>> s = sqrt(sin(3*pi/4));
```

Commands such as `sqrt` (square root) and `sin` (sine of an angle in radians) are built-in functions in MATLAB. Each of these functions is accompanied by a brief but sufficient description through the `help` command.

```
>> help sqrt
 SQRT   Square root.
    SQRT(X) is the square root of the elements of X. Complex
    results are produced if X is not positive.
    See also sqrtm, realsqrt, hypot.
    Overloaded methods:
       sym/sqrt
    Reference page in Help browser
       doc sqrt
```

Other elementary functions—assuming the variable name is "a"—include `abs(a)` for $|a|$, `log(a)` for $\ln a$, `log10(a)` for $\log_{10}(a)$, `exp(a)` for e^a, and many more. Descriptions of these functions are available through the help command.

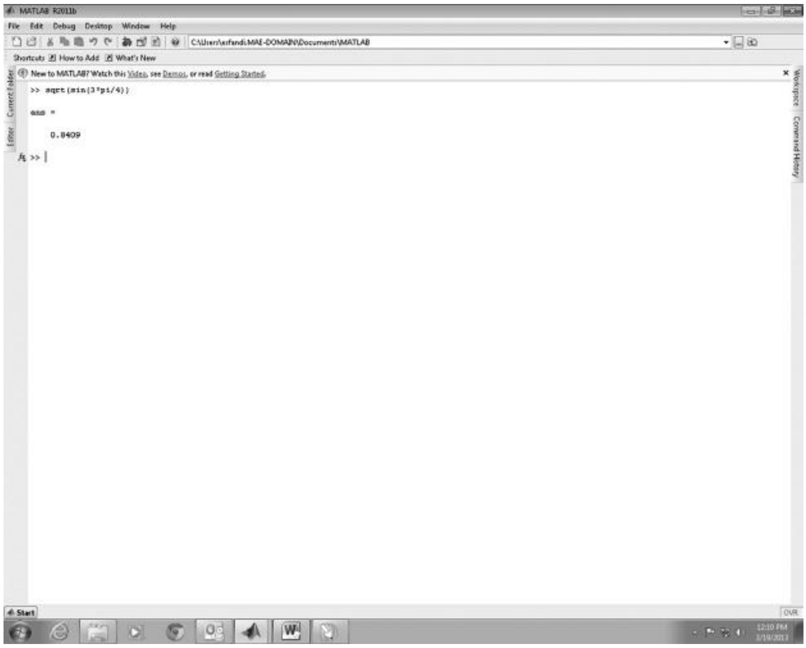


FIGURE 1.1 Screen capture of a MATLAB session.

1.2 USER-DEFINED FUNCTIONS AND SCRIPT FILES

User-defined M file functions and scripts may be created, saved, and edited in MATLAB using the `edit` command. For example, suppose we want to create a function (say, `Cone`) that returns the surface area and volume of a cone with specified base radius and height. The function can be saved in a folder on the MATLAB path or in the current directory. The current directory can be viewed or changed using the drop-down menu at the top of the MATLAB Command Window. Once the current directory has been properly selected, type:

```
>> edit Cone
```

This will open a new window in which the function is constructed by typing the following code:

```
function [S V] = Cone(r,h)
%
% Cone calculates the surface area and volume of a cone with base
% radius r and height h.
S = pi*r*sqrt(h^2+r^2) + pi*r^2;
V = (1/3)*pi*r^2*h;
```

To execute this successfully, the current directory must be where the function was saved. Also, the two input arguments (`r,h`) must be supplied. For example, for a cone with a base radius of 0.25 m and a height of 0.5 m, we find:

```
>> [S V] = Cone(0.25,0.5)
S =
    0.6354    % square meters
V =
    0.0327    % cubic meters
```

1.2.1 Creating a Script File

A script file is composed of a list of commands as if they were typed at the command line. Script files can be created in the MATLAB Editor, and saved as an M file. For example, typing

```
>> edit My_script_file
```

opens the Editor Window, where the script can be created and saved under the name `My_script_file`. It is recommended that a script start with the functions `clear` and `clc`. The first one clears all the previously generated variables, and the second one clears the Command Window. Suppose we type the following lines in the Editor Window:

```
clear
clc
x = 1; N = 10;
Q = sin(x)^2*N;
```

While in the Editor Window, select "Run My_script_file.m" under the Debug pull-down menu. This will execute the lines in the script file and return the Command Prompt. Simply type `Q` at the prompt to see the result.

```
>> My_script_file
>> Q
Q =
    7.0807
```

This can also be done by highlighting the contents and selecting "Evaluate Selection." An obvious advantage of creating a script file is that it allows us to simply make changes to the contents without having to retype all the commands.

1.3 DEFINING AND EVALUATING FUNCTIONS

The built-in function `inline` is ideal for defining and evaluating functions of one or more variables. For instance, consider $h(t) = t^2$, which is a function of a single variable t and hence must be defined as such:

```
>> h = inline('t^2','t')
h =
     Inline function:
     h(t) = t^2
```

For the case of one independent variable, as above, dependence on t may simply be omitted:

```
>> h = inline('t^2');
```

For functions of two or more independent variables, for example, $g = y\sin x$, the desired order of the variables must be specified. Otherwise, MATLAB will list them in alphabetical order.

```
>> g = inline('y*sin(x)')
g =
     Inline function:
     g(x,y) = y*sin(x)
```

If $g(y,x)$ is desired, then

```
>> g = inline('y*sin(x)','y','x')
g =
     Inline function:
     g(y,x) = y*sin(x)
```

To evaluate $g(y,x)$ when $x = \pi/3$ and $y = 1.1$,

```
>> g(1.1,pi/3)
ans =
    0.9526
```

Another way to define and evaluate a function is as follows. Let $g = y\sin x$ as before. Then, function g is defined symbolically in one of two ways:

```
>> g = 'y*sin(x)'
g =
y*sin(x)
```

or

```
>> g = sym('y*sin(x)')
g =
y*sin(x)
```

The evaluation of a function can be achieved by using the `subs` command:

```
>> help subs
   Utilities for obsolete MUTOOLS commands.
subs is both a directory and a function.
SUBS   Symbolic substitution.
   SUBS(S) replaces all the variables in the symbolic expression S with
   values obtained from the calling function, or the MATLAB workspace.
   SUBS(S,NEW) replaces the free symbolic variable in S with NEW.
   SUBS(S,OLD,NEW) replaces OLD with NEW in the symbolic expression S.
```

For instance, to evaluate g when $x = \pi/3$ and $y = 1.1$

```
>> x = pi/3; y = 1.1;
>> subs(g)
ans =
    0.9526   % Agrees with the earlier result
```

1.4 ITERATIVE CALCULATIONS

Iterations involve statements that are repeated a specific or indefinite number of times. For a specific number of repeated statements, the command `for` is used. As an example, to generate the sequence $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}\right\}$, we proceed as follows:

```
>> for n = 1:5,
x(n) = 1/(2^n);
end
```

```
>> x
x =
    0.5000   0.2500   0.1250   0.0625   0.0313
```

The situations entailing an indefinite number of repeated statements can be handled in several ways. One way is to use the command `for` in conjunction with the command `if`, as follows. The code below will generate the elements of the sequence $x_n = 1/n$ ($n = 1, 2, 3, \ldots$), terminating the process as soon as $\left|x_n - x_{n-1}\right| < 0.1$ is met.

```
>> x(1) = 1;   % Define the first element
>> for n = 2:20,   % Maximum number of iterations is 20
x(n) = 1/n;  % Generate subsequent elements
if abs(x(n) - x(n-1)) < 0.1, break; end   % Terminating condition
end
>> x
x =
    1.0000   0.5000   0.3333   0.2500
```

Note that not all 20 values of the index n were used here because the terminating condition was satisfied when $n = 4$.

1.5 MATRICES AND VECTORS

A matrix can be created by using brackets enclosing all elements of the matrix, rows separated by a semicolon.

```
>> A = [-1 1 2;2 0 -1;0 1 5]      % 3-by-3 matrix A
A =
    -1    1    2
     2    0   -1
     0    1    5
```

An entry can be accessed by using the row and column number of the location of that entry.

```
>> A(3,2)   % Entry at the intersection of the 3rd row and 2nd column
ans =
     1
```

An entire row or an entire column of a matrix is accessed via a colon operator.

```
>> Row2 = A(2,:)    % Second row of A and any column
Row2 =
     2    0   -1

>> Col3 = A(:,3)    % Third column of A and any row
Col3 =
     2
    -1
     5
```

To replace an entire column of matrix `A` by a given vector `v`, we proceed as follows.

```
>> v = [-4;1;2]    % Define the given vector v
v =
    -4
     1
     2
```

```
>> Anew = A;    % Preallocate matrix Anew

>> Anew(:,2) = v    % Assign vector v to the second column of Anew

Anew =
    -1   -4    2
     2    1   -1
     0    2    5   % Same as A except for the 2nd column, which is v
```

The $m \times n$ zero matrix is created by using `zeros(m,n)`; for instance, the 2×3 zero matrix:

```
>> A = zeros(2,3)
A =
     0    0    0
     0    0    0
```

This is a common way of preallocating memory for a matrix. Now, any entry can be altered whereas others remain unchanged.

```
>> A(2,2) = -1; A(1,3) = 2;
>> A

A =
     0    0    2
     0   -1    0
```

Suppose we wish to construct a 4×4 matrix whose diagonal elements are all –1's, entries directly above the diagonal are 2's, entries directly below are –2's, and all other entries are zeros. This can be achieved by using flow control commands `if`, `elseif`, and `for`.

```
>> for m = 1:4,
for n = 1:4,
if m == n,
A(m,n) = -1;
elseif m-n == 1
A(m,n) = -2;
elseif n-m == 1
A(m,n) = 2;
end
end
end
>> A
A =
    -1    2    0    0
    -2   -1    2    0
     0   -2   -1    2
     0    0   -2   -1
```

1.6 DIFFERENTIATION AND INTEGRATION

Differentiation of a function with respect to an independent variable is done through the `"diff"` command. Suppose $x = t^2 - \sin t$ and we are interested in dx/dt. We first need to define the function symbolically. As mentioned in Section 1.3, this can be done in one of two ways:

```
>> x = sym('t^2-sin(t)');
```

or

```
>> x = 't^2-sin(t)';
```

Then, dx/dt is calculated as

```
>> dxdt = diff(x)
dxdt =
2*t-cos(t)
```

To evaluate `dxdt` when `t = 1`, we use the `subs` command as before.

```
>> eval(subs(dxdt,'1'))
ans =
    1.4597
```

The second derivative d^2x/dt^2 can be obtained as

```
>> diff(x,2)
ans =
2+sin(t)
```

Now consider a function of two variables, say, $g = t^2 + a\sin s^2$, where a = const. The partial derivatives of g with respect to its independent variables t and s are found as follows:

```
>> syms a     % Define variable a symbolically
>> g = 't^2+a*sin(s^2)';     % Define function g
>> diff(g,'t')    % Differentiate with respect to t
ans =
2*t
>> diff(g,'s')    % Differentiate with respect to s
ans =
2*a*cos(s^2)*s
```

Definite (and indefinite) integrals are calculated in MATLAB via the "`int`" command.

```
INT   Integrate.
  INT(S) is the indefinite integral of S with respect to its symbolic
    variable as defined by FINDSYM. S is a SYM (matrix or scalar).
    If S is a constant, the integral is with respect to 'x'.
  INT(S,v) is the indefinite integral of S with respect to v. v is a
    scalar SYM.
  INT(S,a,b) is the definite integral of S with respect to its
    symbolic variable from a to b. a and b are each double or
    symbolic scalars.
  INT(S,v,a,b) is the definite integral of S with respect to v
    from a to b.
```

The definite integral $\int_0^1 (a + 2t)\,dt$, where a = const, is evaluated as

```
>> syms a t
>> int(a+2*t,t,0,1)
ans =
a+1
```

1.7 PLOTTING IN MATLAB

1.7.1 Plotting Data Points

Plotting a vector of values versus another vector of values is done by using the `plot` command.

```
>> help plot
PLOT   Linear plot.
   PLOT(X,Y) plots vector Y versus vector X. If X or Y is a matrix,
   then the vector is plotted versus the rows or columns of the matrix,
   whichever line up. If X is a scalar and Y is a vector, disconnected
   line objects are created and plotted as discrete points vertically at X.
```

For instance, to plot $y = e^{-t/2}\sin t$ versus $t \in \left[0,\ \frac{7}{2}\pi\right]$ using 100 points, we proceed as follows:

```
>> t = linspace(0,7*pi/2);    % Generate 100 (default) equally spaced points
>> y = exp(-t/2).*sin(t);     % Evaluate the function at these points
>> plot(t,y)    % Generate Figure 1.2
>> grid on    % Add grid
```

Because `t` is an array, we have used `".*"` to allow for element-by-element multiplication. The `plot` command can also be used to generate multiple plots. For example, to plot $x_1 = e^{-t/3}\sin 2t$ and $x_2 = e^{-t/2}\sin t$ versus $0 \le t \le 10$, the code below may be executed.

```
>> t = linspace(0,10);
>> x1 = exp(-t/3).*sin(2*t); x2 = exp(-t/2).*sin(t);
>> plot(t,x1,t,x2)    % Figure 1.3
>> grid on
```

The built-in command `subplot` is designed to create multiple figures in tiled positions.

```
>> help subplot
SUBPLOT Create axes in tiled positions.
   H = SUBPLOT(m,n,p), or SUBPLOT(mnp), breaks the Figure window into an
   m-by-n matrix of small axes, selects the p-th axes for the current
```

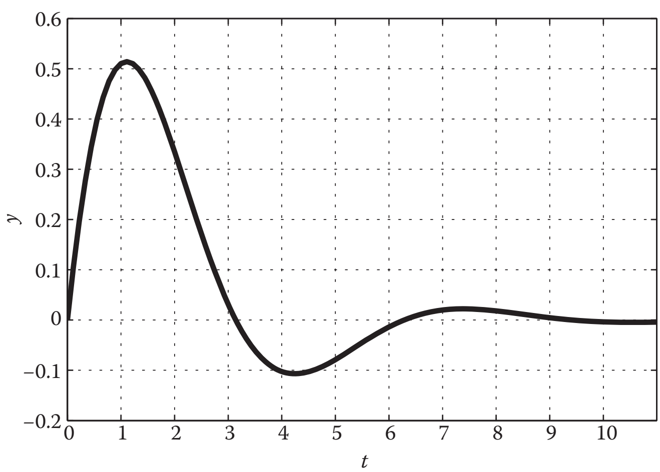


FIGURE 1.2 Single plot.

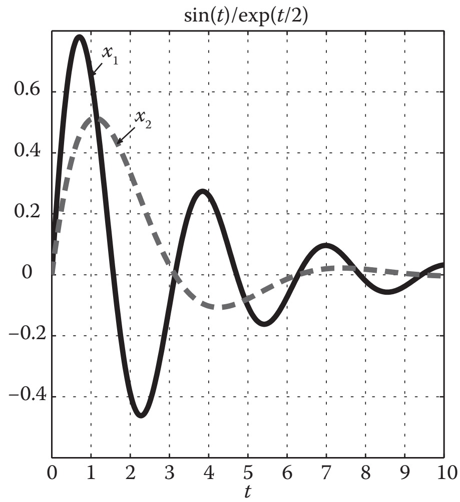


FIGURE 1.3 Multiple plots.

```
plot, and returns the axis handle. The axes are counted along the top
row of the Figure window, then the second row, etc.
```

Let $u(x, t) = e^{-x}\sin(t + 3x)$ where $0 \le x \le 5$. To plot $u(x,t)$ versus x for $t = 0, 1, 2, 3$, we execute the following code:

```
>> x = linspace(0,5);
>> t = 0:1:3;
>> for i = 1:4,
for j = 1:100,
u(j,i) = exp(-x(j))*sin(t(i)+3*x(j));
end
end
>> subplot(2,2,1), plot(x,u(:,1)), grid on    % Initiate Figure 1.4
>> title('t = 0')
>> subplot(2,2,2), plot(x,u(:,2)), grid on
>> title('t = 1')
>> subplot(2,2,3), plot(x,u(:,3)), grid on
>> title('t = 2')
>> subplot(2,2,4), plot(x,u(:,4)), grid on
>> title('t = 3')    % Complete Figure 1.4
```

1.7.2 Plotting Analytical Expressions

An alternative method is to use the `ezplot` command, which plots the function without requiring data generation.

```
>> syms t
>> y = exp(-t/2)*sin(t);
>> ezplot(y,[0,7*pi/2])    % Also generates Figure 1.2
```

Multiple plots can also be generated by using `ezplot`. Reconsidering an earlier example (which led to Figure 1.3), we plot the two functions $x_1 = e^{-t/3}\sin 2t$ and $x_2 = e^{-t/2}\sin t$ versus $0 \le t \le 10$ as follows:

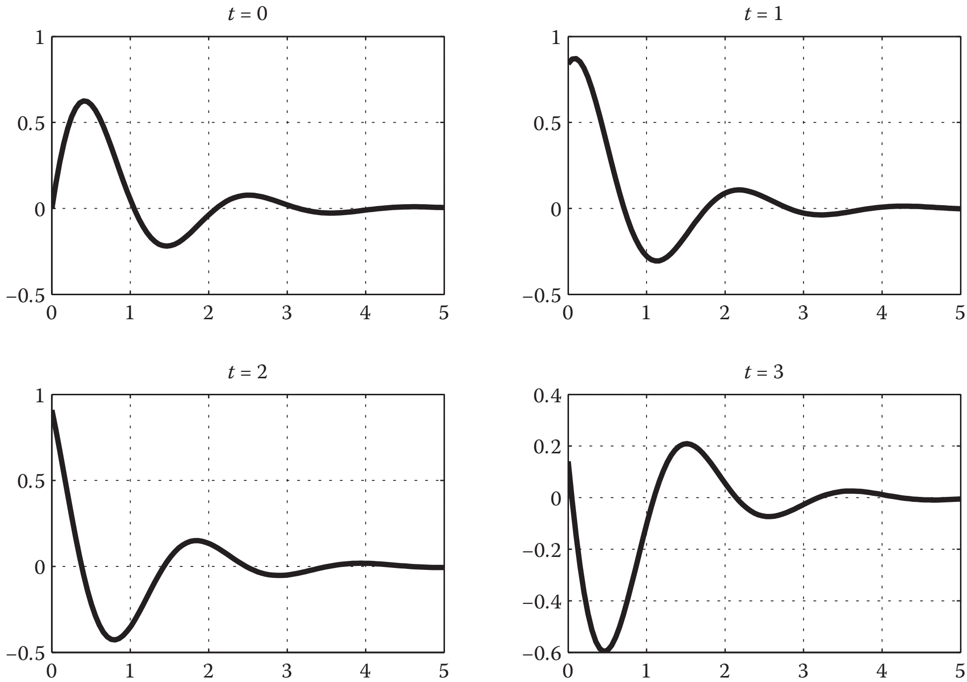


FIGURE 1.4 Subplot.

```
>> syms t
>> x1=exp(-t/3)*sin(2*t);
>> x2=exp(-t/2)*sin(t);
>> ezplot(x1,[0,10])     % Initiate plot
>> hold on
>> ezplot(x2,[0,10])     % Complete plot
>> axis([0 10 -0.6 0.8])     % Set the limits to resemble Figure 1.3
>> grid on
```

Note that the limits for the horizontal and vertical axes have been reset so that the resulting graph matches that in Figure 1.3.

1.8 SIMULINK

Simulink is a powerful software package widely used in academia and industry for modeling, analysis, and simulation of dynamic systems. In the modeling phase, models are built as block diagrams (Section 4.5) using a graphical user interface (GUI). Once a model is built, it can be simulated with the aid of Simulink menus or by entering commands in MATLAB's Command Window. One of the greatest advantages of Simulink is the fact that it allows for the analysis and simulation of the more realistic nonlinear models rather than the idealized linear ones.

1.8.1 Block Library

Typing `Simulink` at the MATLAB command prompt opens a new window labeled Simulink Library Browser (Figure 1.5), which includes a complete block library of sinks, sources, components, and connectors. Clicking on any of the categories reveals the list of blocks it contains. For instance, clicking on Commonly Used Blocks results in the window shown in Figure 1.6.

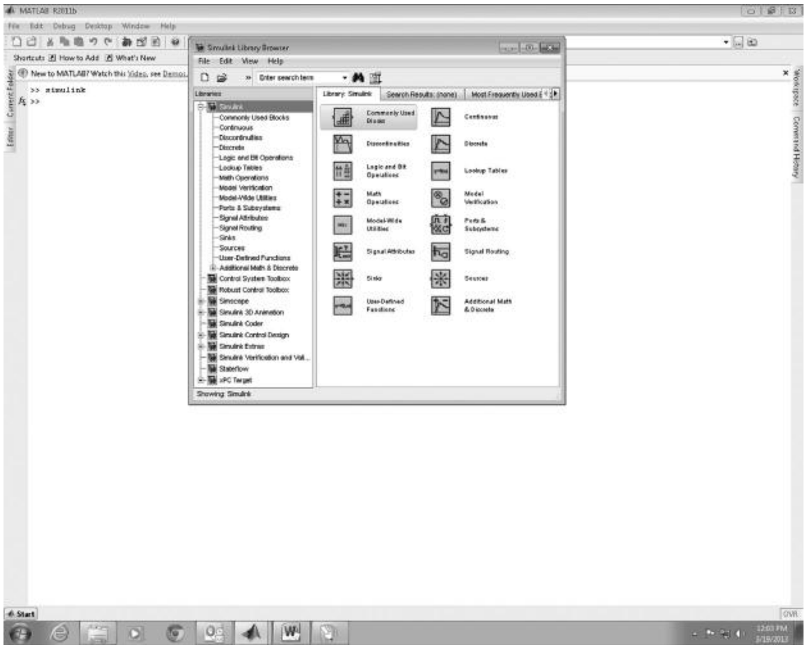


FIGURE 1.5 Simulink library.

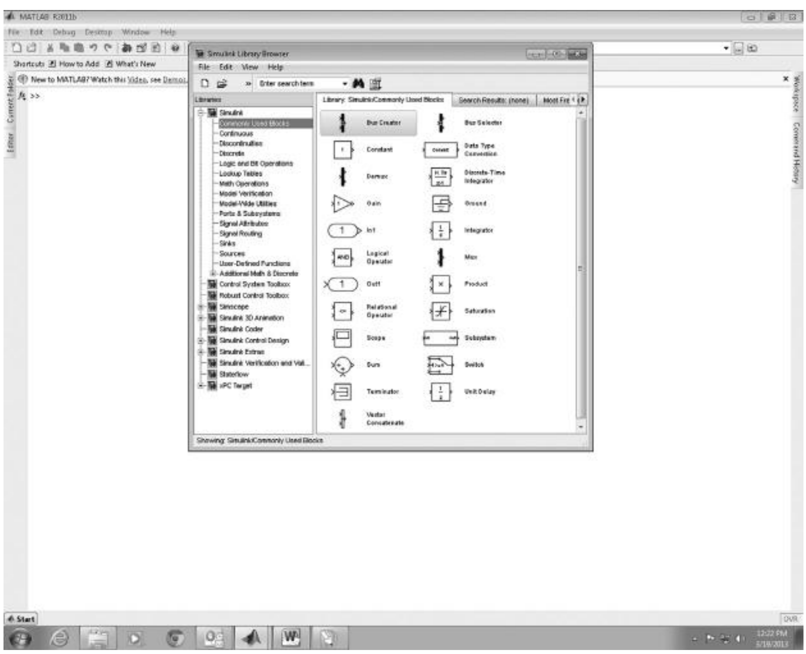


FIGURE 1.6 Commonly used blocks.

1.8.2 Building a New Model

To create a new model, select **Model** from the **New** submenu of the Simulink library window's **File** menu or simply press the **New Model** button on the Library Browser's toolbar. This will open a new model window (Figure 1.7). Suppose we want to build a model that integrates a step signal and displays the result, along with the step signal itself. The block diagram of this model will resemble Figure 1.8. To create this model, we will need to copy (drag and drop) the following blocks into our new model:

- The step signal (from Sources library)
- The Integrator block (from Continuous library)
- The Mux block (from Signal Routing library)
- The Scope block (from Sinks library)

This results in what is shown in Figure 1.9. Note that double-clicking on each block reveals more detailed properties for that block. The ">" symbol pointing out of a block is an output port. The symbol ">" pointing to a block is an input port. A signal travels through a connecting line out of an output port of a block and into an input port of another block. The port symbols disappear as soon as the blocks are connected. Let us now connect the blocks. First, connect the Step block to the top

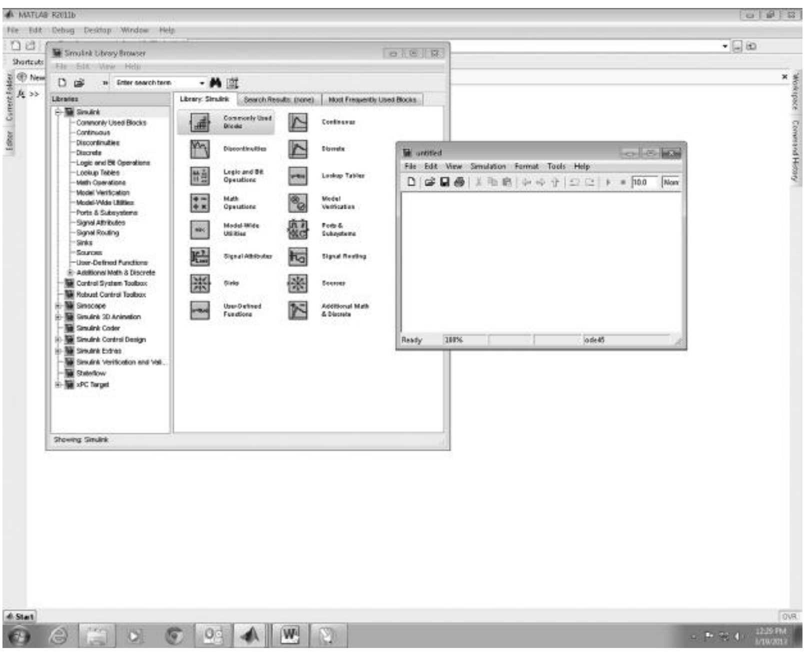


FIGURE 1.7 A new model window.

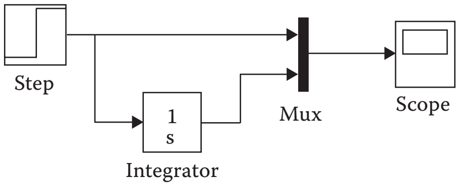


FIGURE 1.8 New model.

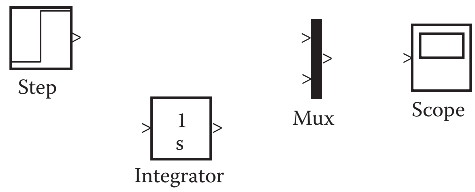


FIGURE 1.9 Basic blocks involved in a new model.

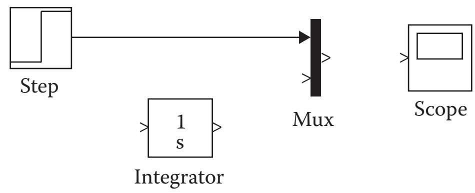


FIGURE 1.10 Connecting line.

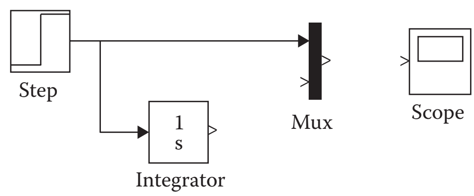


FIGURE 1.11 Branch line.

input port of the Mux block: place the pointer over the output port of the Step block and note that the cursor shape changes to crosshairs. Hold down the mouse button and move the cursor to the top input port of the Mux block, then release the mouse button. If the connecting line is not straight, simply drag the Step block up or down until the ports are lined up (Figure 1.10). The signal going from the Step block to the Mux block must also go through the integrator. This may be done using a branch line. Place the pointer on the connecting line between the Step and Mux blocks. Press and hold down the **Ctrl** key. Press the mouse button and drag the pointer to the Integrator block's input port, then release the mouse button (this results in Figure 1.11). Finally, connect from the output port of the Integrator to the bottom input port of the Mux block, and draw a connecting line from the output port of the Mux to the input port of the Scope. The completed block diagram will look like the original in Figure 1.8.

1.8.3 Simulation

The simulation parameters can be set by choosing **Configuration Parameters** from the **Simulation** menu. Note that the default stop time is 10.0. Next, choose **Start** from the **Simulation** menu. Double-click on the Scope block and then choose the binocular option (auto scale) to see the step signal as well as its integral (Figure 1.12). Obviously, the simulation stops when the stop time of 10.0 is reached. The simulation thus obtained cannot be copied and pasted into a document and is only for observation while in a MATLAB session. To gain access to the actual output data, the following needs be done. Select the Out1 block from the Commonly Used Blocks library and copy it onto the existing model. Then, draw the appropriate branch line to obtain the completed diagram shown in

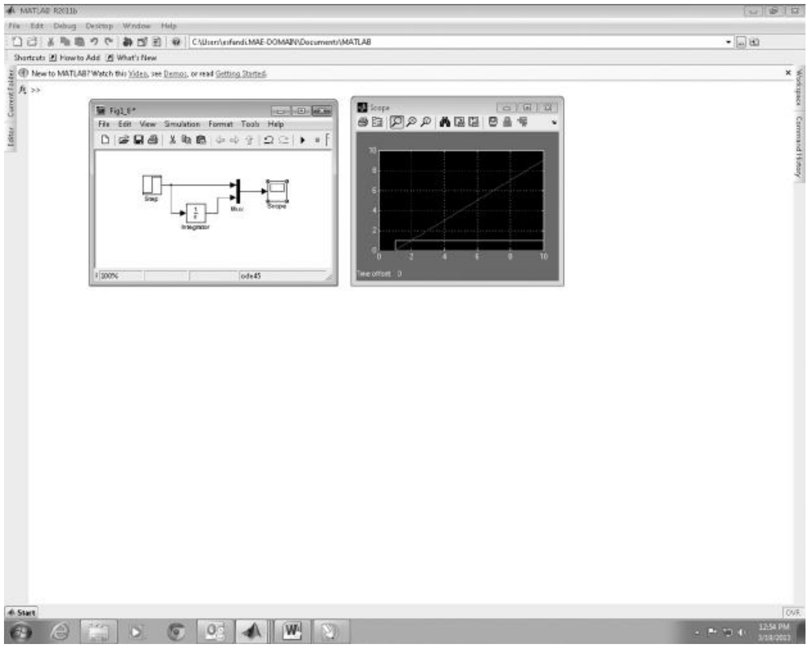


FIGURE 1.12 Simulation.

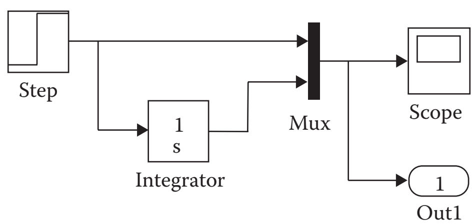


FIGURE 1.13 Storing output data.

Figure 1.13. Once again, run the simulation. As a result, the time vector is automatically saved in `tout`, whereas the output is saved in `yout`. Next, at the MATLAB command prompt, type

```
>> plot(tout,yout)
```

This yields Figure 1.14. Note that this is the same result as observed in the simulation.

1.9 SIMSCAPE

Simscape is a powerful software package that extends the Simulink product line with tools for modeling and simulation of physical systems, such as those with mechanical, electrical, hydraulic, thermal, and pneumatic components. Unlike other Simulink blocks, which represent mathematical operations or operate on signals, Simscape blocks directly represent physical components or relationships. With Simscape blocks, a system model is built the same way a physical system is assembled.

Simscape models use a Physical Network approach to model building: components (blocks) corresponding to physical elements such as pumps, motors, and op-amps, are joined by lines corresponding to the physical connections that transmit power. This approach allows for the description

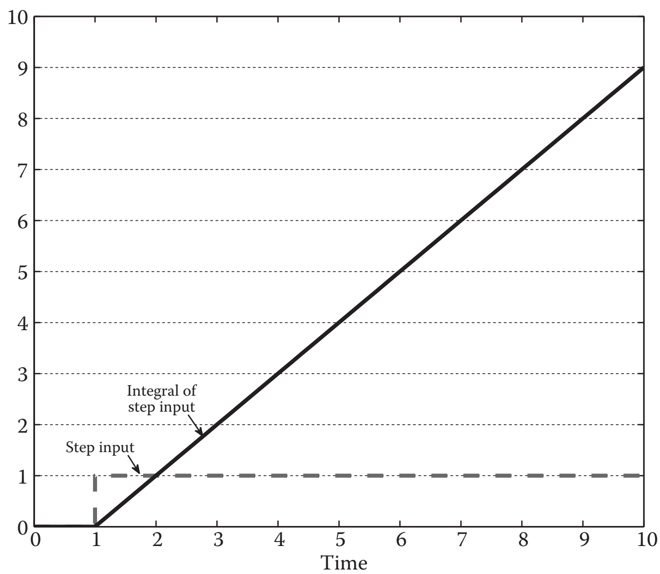


FIGURE 1.14 Output data.

of the physical structure of a system rather than the causal mathematics. Simscape automatically constructs, from the model, equations that characterize the system behavior, which are in turn integrated with the rest of the Simulink model. Simscape functions and utilities support functionality common to other Simulink products that use physical connections between their blocks. Simscape serves as the platform product for these add-on products of the Physical Modeling family:

- SimHydraulics®, for modeling and simulating hydraulic systems
- SimDriveline™, for modeling and simulating power train systems
- SimMechanics™, for modeling and simulating general mechanical systems
- SimElectronics®, for modeling and simulating electromechanical and electronic systems
- SimPowerSystems™, for modeling and simulating electrical power systems

These products can be used together to model physical systems in the Simulink environment.

1.9.1 Block Library

The Simscape block library contains two top-level libraries: Foundation Library and Utilities. If any add-on Physical Modeling products have been installed, they will appear under the Simscape library. Type `Simulink` at the MATLAB command prompt and expand the Simscape entry in the contents tree (Figure 1.15). Each library can be expanded by double-clicking on its icon. Double-clicking on the Foundation Library icon and Utilities icon results in the menus displayed in Figures 1.16 and 1.17, respectively.

1.9.2 Building a New Model and Simulation

A physical model can be built by using a combination of blocks from the Simscape Foundation and Utilities libraries. Each Simscape diagram must contain a Solver Configuration block from the Simscape Utilities library (Figure 1.17). Regular Simulink blocks (such as sources or scopes) can be connected to the physical network diagram using connector blocks; the Simulink-PS Converter

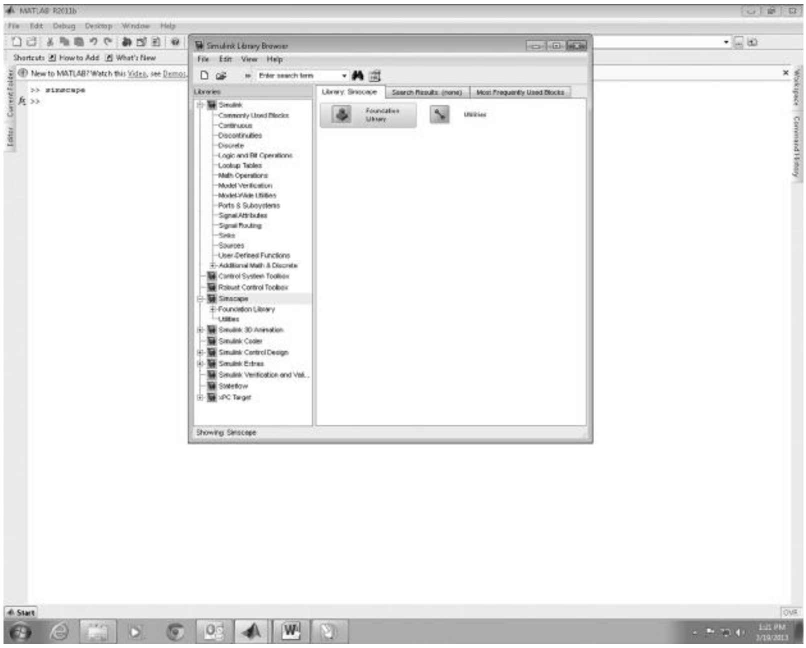


FIGURE 1.15 Simscape libraries.

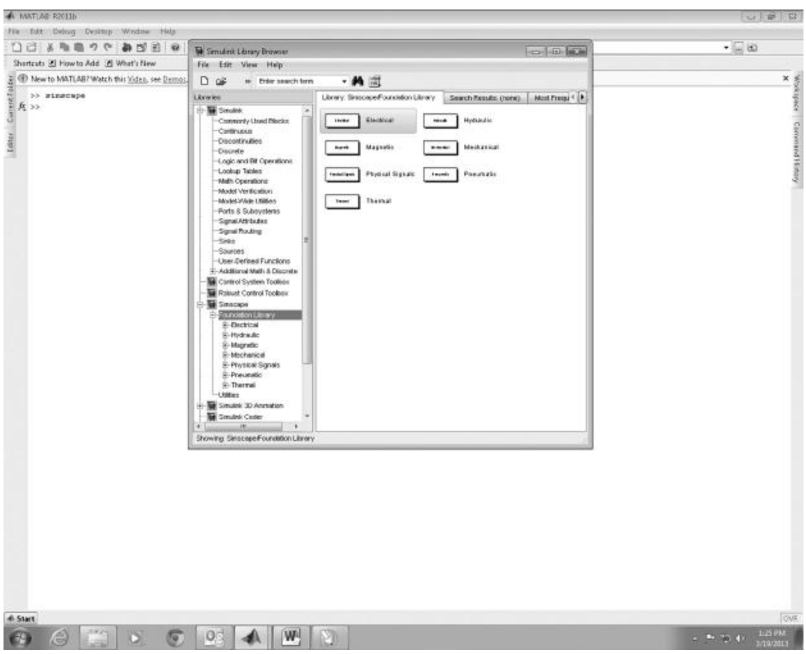


FIGURE 1.16 Simscape Foundation library contents.

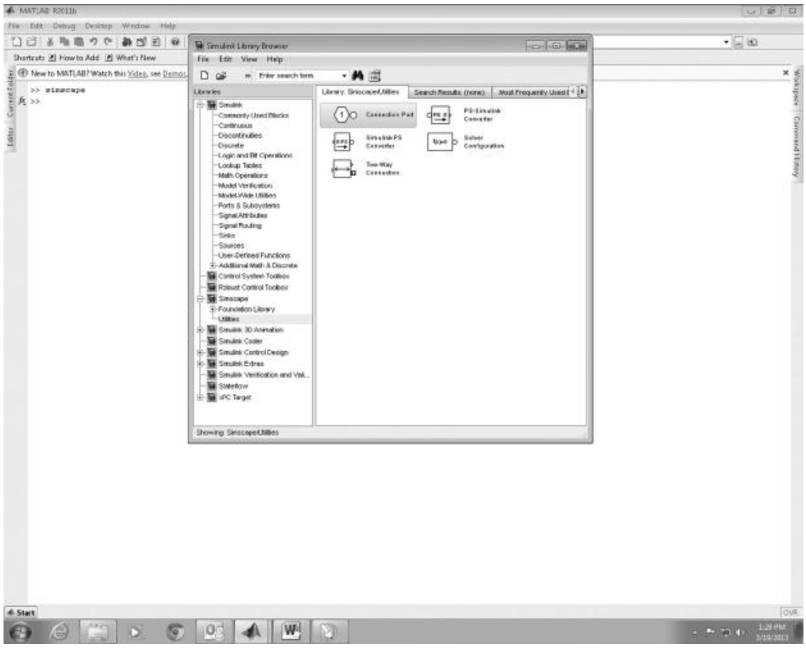


FIGURE 1.17 Simscape Utilities library contents.

block is used to connect Simulink outports to Physical Signal inports; the PS-Simulink Converter block is used to connect Physical Signal outports to Simulink inports. Physical Signals, unlike Simulink signals, have units, which may be specified via the Simscape block dialogs. Input and output signal units can be specified through the converter blocks.

Suppose we want to create a Simscape diagram equivalent to the simple RLC electrical circuit in Figure 1.18, which consists of a resistor, an inductor, and a capacitor, and is driven by an applied voltage. We can vary the model parameters, such as the resistance or the applied voltage, and view the subsequent changes to the electric current.

To create a new model, select **Model** from the **New** submenu of the Simulink library window's **File** menu. This will open a new Model Editor Window. Open the Simscape > Foundation Library > Electrical > Electrical Elements library. Drag the Capacitor, Resistor, Inductor, and (one) Electrical Reference blocks into the model window. The representation of the applied voltage can be added by opening the Simscape > Foundation Library > Electrical > Electrical Sources library and adding the Controlled Voltage Source to the diagram. The current sensor can be added by opening the Simscape > Foundation Library > Electrical > Electrical Sensors library. Note that the *current sensor must be connected with the electrical elements in series.* The Solver Configuration block, and

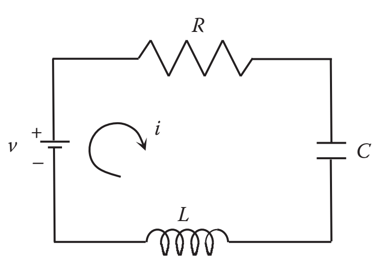


FIGURE 1.18 An RLC circuit.

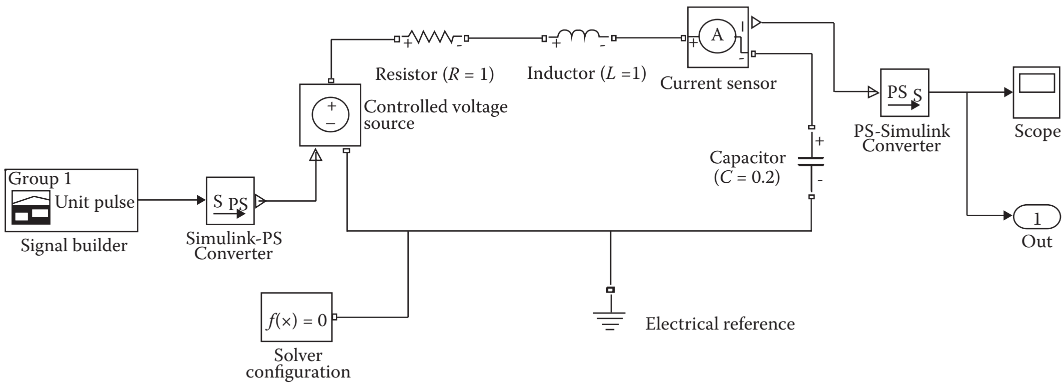


FIGURE 1.19 Simscape model for an RLC circuit.

the PS-Simulink and Simulink-PS converters are found in the Simscape Utilities library. For simulation purposes, a Signal Builder block from the Sources menu of Simulink is added to the model. This block will be used to define the input signal. The corresponding output—the outcome of simulation—will be stored in the Scope block. However, as explained in Section 1.8, the simulation thus obtained cannot be copied and pasted into a document and is only for observation while in a MATLAB session. To have access to the actual output data, the Out1 block from the Commonly Used Blocks library must be added to the existing model. The completed new model is shown in Figure 1.19.

1.9.3 Simulation

Suppose the input is a unit pulse between t = 1 and t = 2. The parameter values are assumed to be R = 1 Ω, L = 1 H, and C = 0.2°F. Select a Simulink solver. On the top menu bar of the model

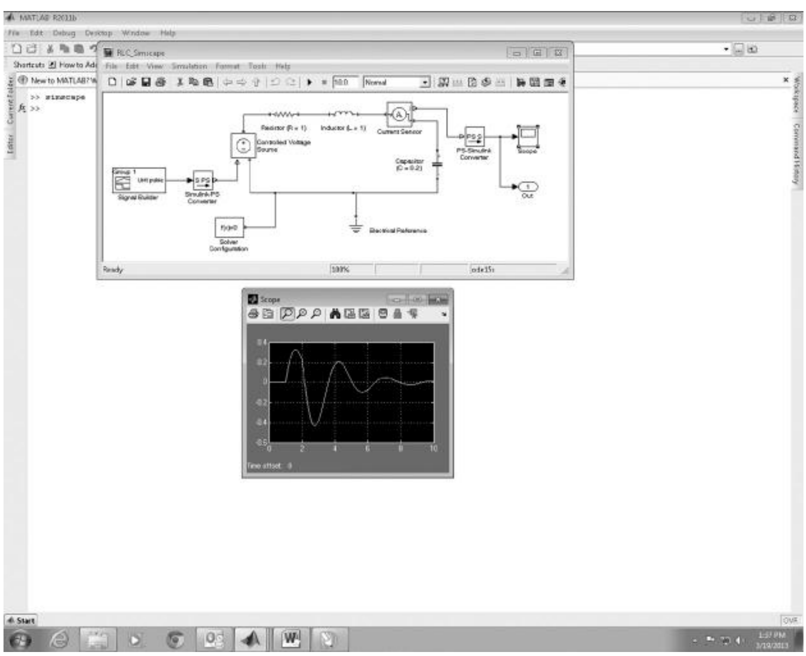


FIGURE 1.20 Simulation of the RLC circuit.

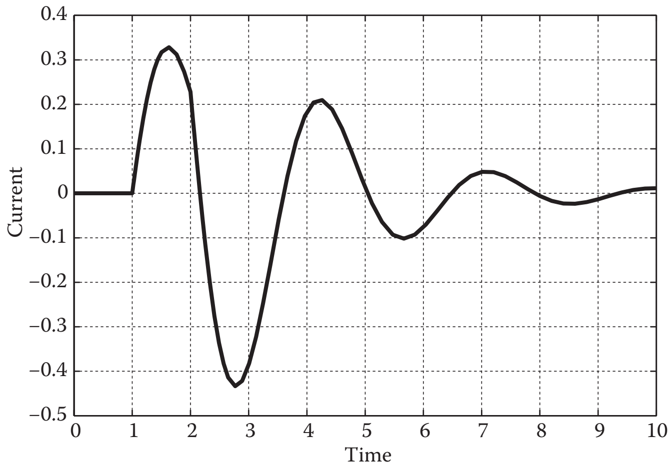


FIGURE 1.21 Output of the simulation of the RLC circuit.

window, select Simulation > Configuration Parameters. The Configuration Parameters dialog box opens. Under Solver options, set Solver to ode15s (Stiff/NDF), which is recommended for solving stiff differential equations, and set max step size to 0.2. Choose **Start** from the **Simulation** menu. Double-click on the Scope block and then choose the binocular option (auto scale) to see the output (Figure 1.20). Typing `plot(tout,yout)` at the MATLAB command prompt yields Figure 1.21.

REVIEW PROBLEMS

1. Write a user-defined function with function call `C = temp_conv(F)` that converts the temperature from Fahrenheit (`F`) to Celsius (`C`). Execute the function for the case of `F = 86`.
2. Write a user-defined function with function call `[P A] = circ(r)` that computes the perimeter (`P`) and area (`A`) of a circle of radius (`r`). Execute the function to calculate the perimeter and area of a circle with radius `r = 1.70`.
3. Write a user-defined function with function call `val = evalf(f,a,b)` where `f` is an inline function, and `a` and `b` are constants such that `a < b`. The function calculates the midpoint m of the interval $[a, b]$ and returns the value of $\frac{1}{2}f(a)+\frac{1}{3}f(m)+\frac{1}{4}f(b)$. Execute the function for $f(x) = e^{-x}\cos 2x$, $a = -1$, $b = 3$.
4. Write a user-defined function with function call `Q = laplace_eval(f,a,b)` where `f` is a function defined symbolically, and `a` and `b` are constants. The function calculates $f_{xx} + f_{yy}$, and evaluates the result at $x = a$, $y = b$. Execute the function for $f = x^2\cos y - 1/y$, $a = 0$, $b = 1$.
5. Write a user-defined function with function call `P = partial_eval(f,g,a)` where `f` and `g` are functions defined symbolically, and `a` is a constant. The function returns the value of $f' + g'$ at $x = a$. Execute the function for $f = x^2 + e^{-x/3}$, $g = \cos x$, and $a = 0.65$.
6. Write a user-defined function with function call `m = mid_point(a,b,e)` where `a` and `b` are constants, and `e` is a tolerance. The function calculates the midpoint of $[a, b]$, called m_1, then the midpoint of $[a, m_1]$, called m_2, then the midpoint of $[a, m_2]$, called m_3, and so on. The process terminates when $|m_k - m_{k-1}| < e$ is met. Allow a maximum of 20 iterations. The function output will be the sequence of generated midpoints $m_1, m_2, \ldots$. Execute the function for the case of $a = 1$, $b = 8$, and $e = 10^{-2}$ (in MATLAB, written as `1e-2`).

7. Plot $\int_0^t e^{x-t}\sin\left(\frac{1}{3}x\right)dx$ versus $0.1 \le t \le 7$.
8. Plot $\int_0^t (x+t)^2 e^{-(2t-x)}\,dx$ versus $-0.5 \le t \le 1$.
9. Evaluate $\int_0^\infty \frac{\sin x}{x}dx$.
10. Differentiate $h(x) = 3^{x-2}\sin x - e^{3-2x}$ with respect to x, and evaluate at $x = 0.75$.
11. Write a script file that uses any combination of the flow control commands to generate

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 & 0 \\ -1 & 0 & 3 & 0 & 3 & 0 \\ 0 & -1 & 0 & 4 & 0 & 3 \\ 0 & 0 & -1 & 0 & 5 & 0 \\ 0 & 0 & 0 & -1 & 0 & 6 \end{bmatrix}$$

12. Write a script file that uses any combination of the flow control commands to generate

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -3 & 2 & 0 & 0 \\ 0 & 4 & -1 & 3 & 2 & 0 \\ 0 & 0 & 4 & 1 & -3 & 2 \\ 0 & 0 & 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

13. Plot the two functions $x_1(t) = \frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$ and $x_2(t) = te^{-t}$ versus $0 \le t \le 10$ in the same graph. Adjust the limits of the vertical axis to -0.1 and 0.4. Add grid and label.
14. Plot the three functions $y_{1,2,3}(t) = e^{-\alpha/2}\cos\left(\frac{1}{2}t\right)$, corresponding to $\alpha = 1, 1.5, 2$, versus $0 \le t \le 10$ in the same graph. Adjust the limits of the vertical axis to -0.8 and 0.8. Add grid and label.
15. Plot $w(x,t) = \sin\left(\frac{1}{2}\pi x\right)\cos\left(\frac{\sqrt{3}}{2}\pi t\right)$ versus $0 \le x \le 4$ for $t = 0.1, 0.5, 1, 1.5$ in a 2×2 tile and add title.
16. Plot $u(x, t) = (1 - \sin x)e^{-(t+1)}$ versus $0 \le x \le 5$ for $t = 1,2$ in a 1×2 tile. Add grid and title.
17. Create the Simulink model shown in Figure 1.22. Double-clicking on each block allows you to explore its properties. Choose the signal generator as a square wave with amplitude of $\frac{1}{2}$ and frequency of 1 rad/s. To flip the gain block (Commonly Used Blocks), right click on it, then go to Format and choose the "Flip Block" option. Perform a simulation and generate a figure that can be imported into a document.
18. Repeat Problem 17 for the model shown in Figure 1.23, where the input signal is a sine wave. Note that double-clicking on the sum (Commonly Used Blocks) allows control over the list of desired signs.
19. Build the model shown in Figure 1.24, perform a simulation and generate a figure that can be imported into a document.

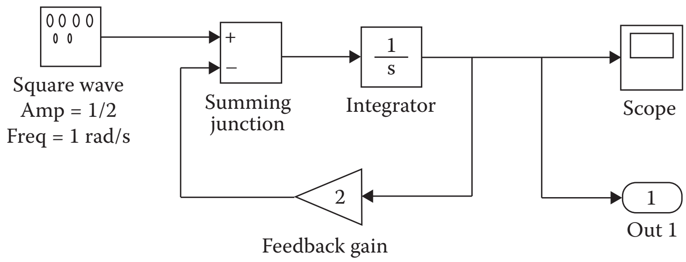


FIGURE 1.22 Problem 17.

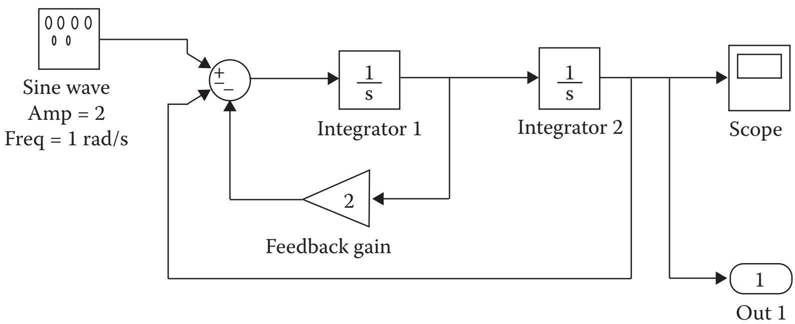


FIGURE 1.23 Problem 18.

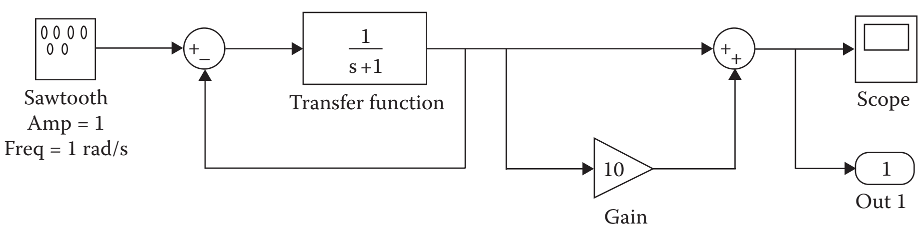


FIGURE 1.24 Problem 19.

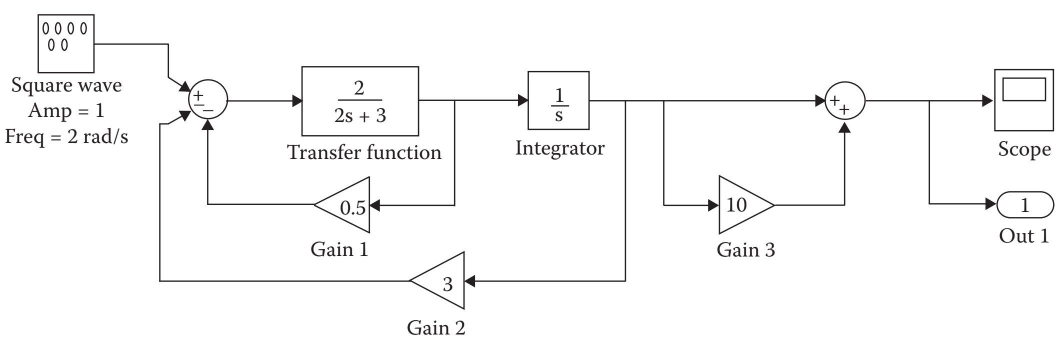


FIGURE 1.25 Problem 20.

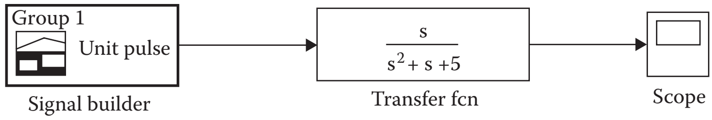


FIGURE 1.26 Problem 21.

20. Build the model shown in Figure 1.25, perform a simulation, and generate a figure that can be imported into a document.
21. Figure 1.26 shows the Simulink model of the RLC circuit considered in this chapter and is equivalent to the Simscape model presented in Figure 1.19. Perform the simulation to confirm that both models yield the same response.
22. Consider an RL circuit with parameters and input signal identical to the RLC circuit considered in this chapter, but with the capacitor removed. Build the Simscape model, run the simulation, and generate the response plot.

2 Complex Analysis, Differential Equations, and Laplace Transformation

This chapter comprises complex analysis, differential equations, and Laplace transformation, essential tools that facilitate the comprehension of various ideas and the implementation of many techniques involved in the analysis of dynamic systems. Complex analysis refers to the study of complex numbers, variables, and functions. Ordinary differential equations (ODEs) arise in situations involving the rate of change of a function with respect to its sole independent variable. Differential equations of various orders—with constant coefficients—are generally difficult to solve. To that end, Laplace transformation is employed to transform the data from time domain to the *s* domain, in which equations are algebraic and hence easier to work with. Back transformation of the information from the *s* domain to the time domain ultimately describes the solution of the differential equation.

2.1 COMPLEX ANALYSIS

Complex analysis consists of complex numbers, variables, and functions, which occur in a wide range of areas in dynamic systems analysis from the calculation of a system's natural frequencies to the analysis of a system's frequency response.

2.1.1 Complex Numbers in Rectangular Form

A complex number z in rectangular form is expressed as

$$z = x + jy, \quad j = \sqrt{-1} = \text{imaginary number} \tag{2.1}$$

where x and y are real numbers, known as the real and imaginary parts of z, respectively, and are denoted by $x = \text{Re}\{z\}$, $y = \text{Im}\{z\}$. For example, if $z = -1 + 2j$, then $x = \text{Re}\{z\} = -1$ and $y = \text{Im}\{z\} = 2$. A complex number with zero real part is called pure imaginary, for example, $z = 2j$. Two complex numbers are equal if and only if their respective real and imaginary parts are equal. The addition of complex numbers is performed component-wise, that is, if $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then

$$\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + j(y_1 + y_2) \end{aligned}$$

Multiplication of two complex numbers is performed in the same way as two binomials with the provision that $j^2 = -1$, that is,

$$\begin{aligned} z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2) &= x_1 x_2 + jy_1 x_2 + jx_1 y_2 + j^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) \end{aligned}$$

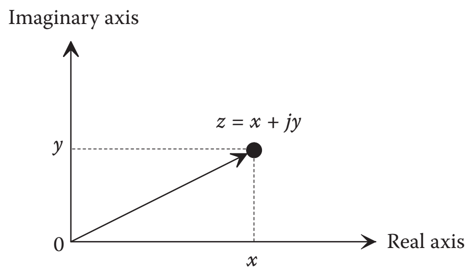


FIGURE 2.1 Geometry of a complex number.

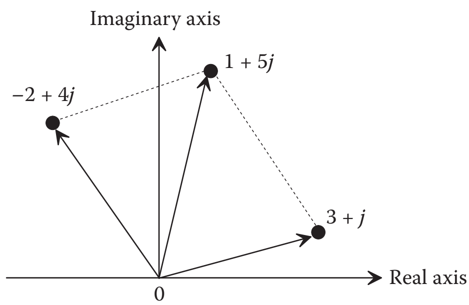


FIGURE 2.2 Addition of complex numbers.

Because they consist of a real part and an imaginary part, complex numbers have a two-dimensional character. Therefore, they may be represented geometrically as points in a Cartesian coordinate system, known as the complex plane. The x axis of the complex plane is the real axis, and the y axis is the imaginary axis (Figure 2.1). Because $z = x + jy$ is uniquely identified by an ordered pair (x,y) of real numbers, it can be represented as a position vector in the complex plane, with initial point 0 and terminal point $z = x + jy$. The concept of vector addition therefore applies to the addition of complex numbers. For example, consider $z_1 = 3 + j$ and $z_2 = -2 + 4j$ as in Figure 2.2. It is then readily seen that the sum $z_1 + z_2 = 1 + 5j$ agrees precisely with the resultant of the position vectors of z_1 and z_2.

2.1.1.1 Magnitude

The magnitude (or modulus) of a complex number $z = x + jy$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{x^2 + y^2} \tag{2.2}$$

Referring to Figure 2.1, the magnitude of z is simply the distance from the origin to z. If z is a real number ($z = x$), it is on a real axis, and its magnitude is equal to its absolute value. If z is pure imaginary ($z = jy$), it is on the imaginary axis and $|z| = |y|$. The distance between two complex numbers z_1 and z_2 is given by $|z_1 - z_2|$. The addition of complex numbers follows the triangle inequality rule,

$$|z_1 + z_2| \le |z_1| + |z_2| \tag{2.3}$$

Example 2.1: Magnitude

Let $z_1 = -1 - 2j$, $z_2 = 3 - 4j$.

a. Calculate the distance between z_1 and z_2, and verify the triangle inequality.
b. Perform Part (a) in MATLAB®.

Solution

a. $z_1 - z_2 = -4 + 2j$ and $|z_1 - z_2| = \sqrt{(-4)^2 + 2^2} = \sqrt{20}$. The triangle inequality (Equation 2.3) can also be verified as follows. First,

$$|z_1| = |-1-2j| = \sqrt{5}, \quad |z_2| = |3-4j| = 5, \quad |z_1 + z_2| = |2-6j| = \sqrt{40}$$

Subsequently, it is observed that $\sqrt{40} \le \sqrt{5} + 5$.

b.

```
>> z1 = -1-2*j; z2 = 3-4*j;
% MATLAB recognizes both i and j as the imaginary number
>> abs(z1 - z2)
ans =
   4.4721    % sqrt(20)
>> abs(z1 + z2)
ans =
   6.3246    % sqrt(40)
>> sqrt(5) + 5
ans =
   7.2361    % Triangle inequality is verified
```

2.1.1.2 Complex Conjugate

The complex conjugate of $z = x + jy$ is denoted by $\bar{z}$ and is defined as $\bar{z} = x - jy$. The product of a complex number ($z \neq 0$) and its conjugate is a positive, real number, equal to the square of the magnitude of z,

$$z\bar{z} = (x + jy)(x - jy) = x^2 + y^2 = |z|^2 \tag{2.4}$$

Geometrically, $\bar{z}$ is the reflection of z through the real axis. Conjugates play an important role in the division of complex numbers. Let us consider z_1/z_2, where $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ ($z_2 \neq 0$). Multiplication of the numerator and the denominator by the conjugate of the denominator ($\bar{z}_2 = x_2 - jy_2$) results in

$$\frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} \underset{\text{in the denominator}}{\overset{\text{Using Equation 2.4}}{=}} \frac{(x_1x_2 + y_1y_2) + j(y_1x_2 - y_2x_1)}{x_2^2 + y_2^2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j\,\frac{y_1x_2 - y_2x_1}{x_2^2 + y_2^2}$$

Note that the resulting complex number has been expressed in standard rectangular form.

Example 2.2: Conjugation

a. Perform $(3 + 2j)/(1 - 3j)$ and express the result in rectangular form.
b. Repeat (a) in MATLAB.

Solution

a. $\dfrac{3+2j}{1-3j} = \dfrac{(3+2j)\overline{(1-3j)}}{(1-3j)\overline{(1-3j)}} = \dfrac{(3+2j)(1+3j)}{(1-3j)(1+3j)} = \dfrac{-3+11j}{10} = \dfrac{-3}{10} + \dfrac{11}{10}j.$

b.

Following the exact steps as in the solution above, and using the `conj` command, we find

```
>> (3 + 2*j)*conj(1 - 3*j)/((1 - 3*j)*conj(1 - 3*j))

ans =

   -0.3000 + 1.1000i    % Note that MATLAB returns i instead of j
```

Alternatively, we can let MATLAB perform the division directly.

```
>> (3 + 2*j)/(1 - 3*j)

ans =

   -0.3000 + 1.1000i    % Both results agree with (a)
```

2.1.2 Complex Numbers in Polar Form

The standard rectangular form $z = x + jy$ turns out to be very inefficient in many situations; for instance, a simple calculation such as $(3 - 2j)^5$. To remedy this, the polar form of a complex number is utilized. As its name suggests, the polar form uses polar coordinates to represent a complex number in the complex plane. The location of any point $z = x + jy$ in the complex plane can be determined by a radial coordinate r and an angular coordinate θ. The relationships between the rectangular and polar coordinates are given by (see Figure 2.3)

$$x = r\cos\theta, \quad y = r\sin\theta$$

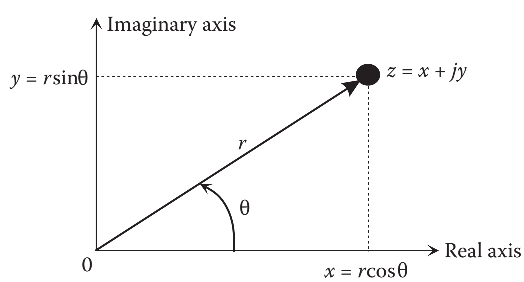


FIGURE 2.3 Connection between polar and rectangular coordinates.

We first introduce Euler's formula,

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{2.5}$$

Consequently,

$$z = x + jy = \underbrace{r\cos\theta}_{x} + j\underbrace{r\sin\theta}_{y} = r\underbrace{(\cos\theta + j\sin\theta)}_{e^{j\theta}} = re^{j\theta} \tag{2.6}$$

The result, $z = re^{j\theta}$, is known as the polar form of z. Here, r is the magnitude of z given by

$$r = |z| = \sqrt{(\mathrm{Re}\{z\})^2 + (\mathrm{Im}\{z\})^2} = \sqrt{x^2 + y^2}$$

and the phase (argument) of z is determined by

$$\theta = \arg z = \tan^{-1}\left(\frac{\mathrm{Im}\{z\}}{\mathrm{Re}\{z\}}\right) = \tan^{-1}\left(\frac{y}{x}\right) \tag{2.7}$$

The angle θ is measured from the positive real axis and, by convention, is regarded as positive in the counterclockwise direction. It is measured in radians (rad) and is determined in terms of integer multiples of 2π. The specific value of θ that lies in the interval $(-\pi,\pi]$ is called the principal value of arg z and is denoted by Arg z. In engineering applications, it is also common to express the polar form as

$$z = r \angle \theta$$

where $\angle$ denotes the angle.

Example 2.3: Polar Form

a. Express $z = -2 - j$ in polar form.
b. Repeat in MATLAB.

Solution

a. Location of z (Figure 2.4), will facilitate the phase calculation. By Equation 2.7,

$$\theta = \tan^{-1}\left(\frac{-1}{-2}\right) \cong 0.4636 \text{ rad} \cong 26.5651°$$

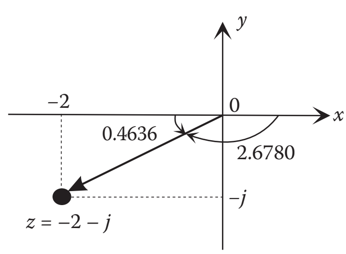


FIGURE 2.4 Phase calculation in Example 2.3.

This, however, is not the phase. Because z is in the third quadrant, the phase in the positive sense (counterclockwise) is calculated as $0.4636 + \pi \cong 3.6052$ rad, whereas in the negative sense (clockwise) it is $\pi - 0.4636 \cong 2.6780$ rad, which has the smaller measure between the two. Noting that $r = \sqrt{5}$, and using the phase in the negative direction, we find

$$z = -2 - j = \sqrt{5}\, e^{-2.6780j}$$

b.

```
>> z = -2-j;
>> mag = abs(z);
>> theta = pi - atan(imag(z)/real(z));
>> mag*exp(-j*theta)

ans =

   -2.0000 - 1.0000i    % Agrees with the original z
```

2.1.2.1 Complex Algebra Using the Polar Form

Working with the polar form considerably simplifies complex algebra. Consider two complex numbers, in their respective polar forms, $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$z_1 z_2 = r_1 r_2 e^{j(\theta_1+\theta_2)} \qquad \text{or alternatively,} \qquad r_1 r_2 \angle(\theta_1 + \theta_2)$$

This means that the magnitude and phase of the product $z_1 z_2$ are

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2| \qquad \text{and} \qquad \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

Similarly, in the case of division,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)} \quad \text{or} \quad \frac{r_1}{r_2} \angle(\theta_1 - \theta_2)$$

Therefore, the magnitude and phase of z_1/z_2 are

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

Example 2.4: Division Using Polar Form

Express the result in polar form:

$$\frac{-2+j}{1-j}$$

Solution

The numerator and the denominator are in the second and fourth quadrants, respectively. Hence, their polar forms are

$$-2+j=\sqrt{5}\,e^{2.6780j}, \quad 1-j=\sqrt{2}\,e^{-(\pi/4)j}$$

As a result,

$$\frac{-2+j}{1-j}=\frac{\sqrt{5}\,e^{2.6780j}}{\sqrt{2}\,e^{-(\pi/4)j}}=\sqrt{\frac{5}{2}}\,e^{3.4634j}$$

This may be verified as follows:

$$\frac{-2+j}{1-j}\cdot\frac{1+j}{1+j}=\frac{-3-j}{2}=\frac{-3}{2}-\frac{1}{2}j \overset{\text{Polar form}}{\underset{\text{third quadrant}}{=}} \sqrt{\frac{5}{2}}\,e^{3.4634j} \overset{\text{Equivalently}}{=} \sqrt{\frac{5}{2}}\,e^{-2.8198j}$$

Given $z = re^{j\theta}$, its conjugate is derived as

$$\bar{z}=x-jy=r\cos\theta-j(r\sin\theta)=r(\cos\theta-j\sin\theta)=r[\cos(-\theta)+j\sin(-\theta)] \overset{\text{Euler's formula}}{=} re^{-j\theta}$$

This makes sense geometrically because a complex number and its conjugate are reflections of one another through the real axis. Hence, they are equidistant from the origin, that is, $|z|=|\bar{z}|=r$, and their phases are equal but opposite in sign, that is, $\arg(z)=-\arg(\bar{z})$ (Figure 2.5). The magnitude property of complex conjugation (Equation 2.4) can now be confirmed as

$$z\bar{z}=[re^{j\theta}][re^{-j\theta}]=r^2=|z|^2$$

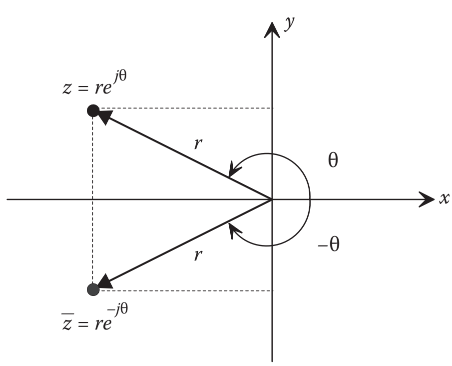


FIGURE 2.5 A complex number and its conjugate.

2.1.2.2 Integer Powers of Complex Numbers

As mentioned at the outset of this section, one area that demonstrates the efficacy of the polar form is in dealing with expressions in the form z^n for integer n. The idea is simple and shown below:

$$\begin{aligned} z^n &\overset{\text{Polar form}}{=} [re^{j\theta}]^n = r^n e^{jn\theta} \overset{\text{Euler's formula}}{=} r^n(\cos n\theta + j\sin n\theta) \\ &\overset{\text{Rectangular form}}{=} r^n \cos n\theta + jr^n \sin n\theta \end{aligned} \tag{2.8}$$

Example 2.5: Integer Power

Simplify $(-2 - j)^4$.

Solution

Using the result of Example 2.3, and following Equation 2.8, we have

$$(-2-j)^4 = (\sqrt{5}\ e^{-2.6780j})^4 = 25e^{-10.7120j} = -7+24j$$

2.1.2.3 Roots of Complex Numbers

In real calculus, if a is a real number, then $\sqrt[n]{a}$ has a single value. On the contrary, given a complex number $z \neq 0$, and a positive integer n, the nth root of z, written $\sqrt[n]{z}$, is *multivalued*. In fact, there are n different values of $\sqrt[n]{z}$ corresponding to each value of $z \neq 0$. If $z = re^{j\theta}$, it can then readily be shown that

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + j\sin\frac{\theta + 2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1 \tag{2.9}$$

Geometrically, these n values lie on a circle centered at the origin with a radius of $\sqrt[n]{r}$ and are the n vertices of an n-sided regular polygon.

Example 2.6: Fourth Roots of 1

Calculate all four values of $\sqrt[4]{1}$.

Solution

The objective is to find $w = \sqrt[4]{z}$ where $z = 1$. Noting that $z = 1$ is located on the positive real axis, one unit from the origin, we have $r = 1$ and $\theta = 0$, hence $z = 1 = e^{j(0)}$. Using Equation 2.9 with $n = 4$, $r = 1$, and $\theta = 0$, we find

$$\sqrt[4]{1} = \sqrt[4]{1}\left(\cos\frac{0+2k\pi}{4} + j\sin\frac{0+2k\pi}{4}\right), \quad k = 0,1,2,3$$

This yields the four values of ±1, $\pm j$. Note that all four roots lie on a circle of radius $\sqrt[4]{1}=1$ centered at the origin, and are the vertices of a regular four-sided polygon, as asserted.

2.1.3 Complex Variables and Functions

If x or y or both vary, then $z = x + jy$ is called a complex variable. The Laplace variable (Section 2.3) is a recognized example of a complex variable. A complex function F defined on a set S is a rule which assigns a complex number w to each $z \in S$. The notation is $w = F(z)$ and set S is the domain of definition of F. For instance, the domain of the function $F(z) = z/(z-1)$ is any region that excludes the point $z = 1$. Because z assumes different values from set S, it is clearly a complex variable. Transfer functions (Section 4.4) and frequency response functions (Section 8.3) are examples of complex functions that frequently arise in the analysis of dynamic systems.

PROBLEM SET 2.1

In Problems 1 through 4,

a. Perform z_1/z_2 and express the result in rectangular form.
b. Verify that $|z_1/z_2| = |z_1|/|z_2|$.
c. Repeat Part (a) in MATLAB.

1. $\dfrac{-3-j}{2j}$
2. $\dfrac{2+j}{1-2j}$
3. $\dfrac{-3j}{2+3j}$
4. $\dfrac{4}{-4+3j}$

In Problems 5 through 8, express each complex number in its polar form.

5. $-\sqrt{3}-3j$
6. $1-\frac{3}{2}j$
7. $3+j\sqrt{3}$
8. $-1+\frac{1}{2}j$

In Problems 9 through 16, perform the operations using polar form and express the result in rectangular form.

9. $\dfrac{3+2j}{-1+3j}$
10. $\dfrac{\sqrt{3}+3j}{3-j\sqrt{3}}$
11. $\dfrac{3-5j}{2j}$

12. $\dfrac{3j}{1-j}$
13. $(4 + 3j)^3$
14. $(0.9511 + 0.3090j)^{10}$
15. $\dfrac{(1+3j)^3}{(-1+2j)^2}$
16. $\dfrac{5j}{(1+4j)^3}$

In Problems 17 through 20, find all possible values for each expression.

17. $(-1)^{1/6}$
18. $(-1 + j)^{1/3}$
19. $(\sqrt{3} - 3j)^{1/2}$
20. $\sqrt{1 + j\sqrt{3}}$

2.2 DIFFERENTIAL EQUATIONS

Differential equations are divided into two general categories: ordinary differential equations (ODEs) and partial differential equations (PDEs). An equation involving an unknown function and one or more of its derivatives is called a differential equation. When there is only one independent variable, the equation is called an ODE. For example, $\dot{x} + 2x = e^{-t}$ is an ODE involving the unknown function $x(t)$, its first derivative $\dot{x} = dx/dt$, as well as a given function e^{-t}. Similarly, $t\ddot{x} + x\dot{x} = \cos t$ is an ODE relating $x(t)$ and its first and second derivatives with respect to t, as well as the function $\cos t$. In dynamic system models, the independent variable is generally time t. The derivative of the highest order of the unknown function $x(t)$ with respect to t is the order of the ODE. For instance, $\dot{x} + 2x = e^{-t}$ is of order 1 and $t\ddot{x} + x\dot{x} = \cos t$ is of order 2. Consider an nth-order ODE in the form

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_1 \dot{x} + a_0 x = F(t) \tag{2.10}$$

where $x = x(t)$ and $x^{(n)} = d^n x/dt^n$. If all coefficients $a_0, a_1, \ldots, a_n$ are either constants or functions of the independent variable t, then the ODE is linear. Otherwise, the ODE is nonlinear. If $F(t) \equiv 0$, the ODE is homogeneous. Otherwise, it is nonhomogeneous. Therefore, $\dot{x} + 2x = e^{-t}$ is linear, whereas $t\ddot{x} + x\dot{x} = \cos t$ is nonlinear, and both are nonhomogeneous. If the unknown function is a function of more than one independent variable, the equation is called a partial differential equation.

2.2.1 Linear, First-Order Differential Equations

In accordance with Equation 2.10, linear, first-order ODEs are expressed as

$$a_1 \dot{x} + a_0 x = F(t) \xrightarrow[\text{and rewrite as}]{\text{Divide by } a_1} \dot{x} + g(t)x = f(t) \tag{2.11}$$

with a general solution in the form

$$x(t) = e^{-h}\left[\int e^h f(t)\,dt + c\right], \quad h = \int g(t)\,dt \tag{2.12}$$

where c is a constant. To derive a particular solution, an initial condition must be specified. Assuming that the initial time is $t = t_0$, the initial condition for x refers to the value of x immediately *prior to* the initial time and is denoted by $x(t_0^-)$. On the other hand, the initial value of x is its value immediately after the initial time and is expressed as $x(t_0^+)$. Although initial condition and initial value of a quantity are almost always the same, there are rare instances in dynamic systems analysis in which they are different (see Section 2.3). In the meantime, we simply assume that the two are the same and are denoted by $x(t_0)$. That said, a first-order initial-value problem (IVP) is described by

$$\dot{x} + g(t)x = f(t), \qquad x(t_0) = x_0$$

Example 2.7: Initial-Value Problem

Solve $t\dot{x} + x = e^t$, $x(1) = -1$.

Solution

Rewrite the ODE as $\dot{x} + \frac{1}{t}x = \frac{e^t}{t}$ so that $g(t) = \frac{1}{t}$ and $f(t) = \frac{e^t}{t}$ in Equation 2.11. Then, by Equation 2.12, we have $h = \int \frac{dt}{t} = \ln t$, and a general solution is

$$x(t) = e^{-\ln t}\left[\int e^{\ln t}\frac{e^t}{t}dt + c\right] = \frac{1}{t}\left[\int e^t dt + c\right] = \frac{1}{t}(e^t + c)$$

Using the initial condition, we find $c = -1 - e$, and consequently $x(t) = \frac{1}{t}(e^t - 1 - e)$.

2.2.2 Second-Order Differential Equations with Constant Coefficients

A special class of linear, second-order ODEs that arises in dynamic system models is in the form

$$\underbrace{a_2\ddot{x} + a_1\dot{x} + a_0x}_{\text{Constant coefficients}} = F(t) \xrightarrow{\text{Rewrite}} \ddot{x} + b\dot{x} + cx = f(t) \qquad (2.13)$$

This equation is normally accompanied by a set of two initial conditions, $x(t_0)$ and $\dot{x}(t_0)$, where t_0 denotes the initial time and is usually zero. A general solution $x(t)$ of Equation 2.13 is a superposition of the homogeneous solution $x_h(t)$, sometimes called the complementary solution $x_c(t)$, and the particular solution $x_p(t)$, that is, $x(t) = x_h(t) + x_p(t)$.

2.2.2.1 Homogeneous Solution

To obtain $x_h(t)$, we solve the homogeneous equation

$$\ddot{x} + b\dot{x} + cx = 0$$

as follows: Assume $x = e^{\lambda t}$, with λ to be determined, and substitute in the previous equation to find

$$(\lambda^2 + b\lambda + c)e^{\lambda t} = 0 \xrightarrow{e^{\lambda t} \neq 0} \underbrace{\lambda^2 + b\lambda + c = 0}_{\text{Characteristic equation}} \xrightarrow{\text{Solve}} \begin{matrix}\lambda_1 \\ \lambda_2\end{matrix}$$

The characteristic values λ_1 and λ_2 establish the two linearly independent solutions that form $x_h(t)$.

Case (1) $\lambda_1 \neq \lambda_2$ real
The two independent solutions are $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$, and a linear combination of the two yields

$$x_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad c_1, c_2 = \text{const}$$

Case (2) $\lambda_1 = \lambda_2 = \lambda$
The independent solutions are $e^{\lambda t}$ and $te^{\lambda t}$ so that

$$x_h(t) = (c_1 + c_2 t)e^{\lambda t}, \quad c_1, c_2 = \text{const}$$

Case (3) $\lambda_2 = \bar{\lambda}_1$ complex conjugates
If $\lambda_1 = \alpha + j\beta$, then the two independent solutions are $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$, leading to

$$x_h(t) = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t), \quad c_1, c_2 = \text{const}$$

2.2.2.2 Particular Solution

The particular solution $x_p(t)$ of Equation 2.13 is obtained by using the method of undetermined coefficients. The method is limited in its applications and only handles cases in which $f(t)$ is a polynomial, exponential, sinusoidal, or some combination of these. Table 2.1 contains the recommended $x_p(t)$ for different scenarios of $f(t)$. These recommended expressions are subject to modification in some special cases as follows. If x_p contains a term that coincides with a solution of the homogeneous equation, and the solution corresponds to a nonrepeated characteristic value, then the recommended x_p must be multiplied by t. If the characteristic value is repeated—as in Case (2)—then x_p must be multiplied by t^2.

Example 2.8: Second-Order ODE

Solve $\ddot{x} + 3\dot{x} + 2x = e^{-t}$, $x(0) = 1$, $\dot{x}(0) = 0$.

TABLE 2.1
Selection of Particular Solution (Method of Undetermined Coefficients)

Term in $f(t)$	Recommended $x_p(t)$
$A_n t^n + \ldots + A_1 t + A_0$	$K_n t^n + \ldots + K_1 t + K_0$
Ae^{at}	Ke^{at}
$A\sin\omega t$ or $A\cos\omega t$	$K_1\cos\omega t + K_2\sin\omega t$
$Ae^{\sigma t}\sin\omega t$ or $Ae^{\sigma t}\cos\omega t$	$e^{\sigma t}(K_1\cos\omega t + K_2\sin\omega t)$

Solution

The characteristic equation, $\lambda^2 + 3\lambda + 2 = 0$, yields $\lambda = -1,-2$ so that $x_h = c_1e^{-t} + c_2e^{-2t}$. Based on Table 2.1, we choose $x_p = Ke^{-t}$. However, e^{-t} coincides with the independent solution in x_h associated with $\lambda = -1$. Therefore, x_p is modified to $x_p = Kte^{-t}$. Inserting into the original ODE and equating the two sides, we find $K = 1$ so that $x_p = te^{-t}$. This means a general solution is $x = c_1e^{-t} + c_2e^{-2t} + te^{-t}$. Finally, application of the initial conditions results in $c_1 = 1$, $c_2 = 0$, hence $x(t) = e^{-t}(1 + t)$.

Example 2.9: Second-Order ODE

Solve $\ddot{x} + \omega^2 x = 0$ ($\omega = \text{const}$), $x(0) = 0$, $\dot{x}(0) = 1$.

Solution

The characteristic equation $\lambda^2 + \omega^2 = 0$ has roots $\lambda = \pm j\omega$ so that $x(t) = c_1\cos\omega t + c_2\sin\omega t$. Applying the initial conditions leads to $c_1 = 0$, $c_2 = 1/\omega$, hence $x(t) = \sin\omega t/\omega$.

2.2.2.2.1 Expressing $A\cos\omega t + B\sin\omega t$ as $D\sin(\omega t + \phi)$

In systems analysis, we often encounter expressions in the form $A\cos\omega t + B\sin\omega t$, where the sine and cosine waves have the same frequency ω. In these situations, it is preferred to replace this expression with a single trigonometric term such as $D\sin(\omega t + \phi)$, where D is the amplitude and ϕ is the phase shift. Using a trigonometric expansion,

$$D\sin(\omega t + \phi) = D\sin\omega t\cos\phi + D\cos\omega t\sin\phi$$

Comparing with $A\cos\omega t + B\sin\omega t$, we find

$$\begin{array}{l} D\sin\phi = A \\ D\cos\phi = B \end{array} \quad \text{Divide the two equations} \quad \tan\phi = \frac{A}{B}$$

Next, we construct a right triangle in which the angle ϕ satisfies $\tan\phi = A/B$ (Figure 2.6). Then,

$$\sin\phi = \frac{A}{\sqrt{A^2 + B^2}}, \quad \cos\phi = \frac{B}{\sqrt{A^2 + B^2}}$$

Noting that $\sin\phi = A/D$ and $\cos\phi = B/D$, we find $D = \sqrt{A^2 + B^2}$. In summary,

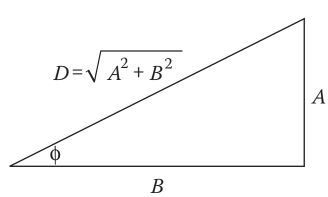


FIGURE 2.6 Phase shift.

$$A\cos\omega t + B\sin\omega t = D\sin(\omega t + \phi), \quad D = \sqrt{A^2 + B^2}, \quad \phi = \tan^{-1}\frac{A}{B} \tag{2.14}$$

Example 2.10: Amplitude, Phase Shift

Express $2\sin t - \cos t$ as $D\sin(t + \phi)$.

Solution

Comparing with the general form, we have $A = -1$ and $B = 2$ so that $D\sin\phi = A < 0$ and $D\cos\phi = B > 0$. But because $D = \sqrt{5} > 0$, it is clear that $\sin\phi < 0$ and $\cos\phi > 0$, hence ϕ is in the fourth quadrant and $\phi = \tan^{-1}(-1/2) = -26.5651° = -0.4636$ rad agrees with the location of ϕ. In conclusion,

$$2\sin t - \cos t = \sqrt{5}\sin(t - 0.4636)$$

PROBLEM SET 2.2

In Problems 1 through 10, solve the IVP.

1. $\dot{x} + x = \sin t,\ x(0) = -1$
2. $\frac{1}{3}\dot{x} + x = 0,\ x(0) = \frac{1}{3}$
3. $2\dot{y} + ty = t,\ y(0) = 2$
4. $\dot{u} = (1 - u)\sin t,\ u(\pi/2) = 2$
5. $(t - 1)\dot{y} + ty = 2t,\ y(0) = 1$
6. $\ddot{x} + 2\dot{x} + x = e^{-2t},\ x(0) = 1,\ \dot{x}(0) = 1$
7. $\ddot{x} + 4\dot{x} = 17\cos t,\ x(0) = -1,\ \dot{x}(0) = 0$
8. $\ddot{u} + u = \sin 2t,\ u(0) = 1,\ \dot{u}(0) = 0$
9. $\ddot{u} + 4\dot{u} + 3u = 4e^{-t},\ u(0) = 0,\ \dot{u}(0) = -1$
10. $2\ddot{y} + 3\dot{y} + y = 0,\ y(0) = 0,\ \dot{y}(0) = \frac{1}{2}$

In Problems 11 through 14, write the expression in the form $D\sin(\omega t + \phi)$.

11. $\cos t + 3\sin t$
12. $\cos 2t - \sin 2t$
13. $-\sin 2t - \frac{1}{2}\cos 2t$
14. $3\sin\omega t - \cos\omega t$

In Problems 15 and 16, write the expression in the form $D\cos(\omega t + \phi)$.

15. $\frac{2}{3}\cos t - \sin t$
16. $4\cos t + 3\sin t$

2.3 LAPLACE TRANSFORMATION

In Section 2.2, we solved ODEs with constant coefficients entirely in the time domain using the method of undetermined coefficients, which has limitations in its applications. In this section, we introduce an alternative and systematic approach to solve these equations with the advantage that the arbitrary constants in a general solution do not need to be obtained separately. The idea is to transform an IVP to the s domain, in which the transformed problem is an algebraic one, and is thus much easier to handle. This algebraic problem is then properly solved, and the data is ultimately transformed back to the time domain to find the solution of the original problem. If a function $x(t)$ is defined for all $t \geq 0$, its Laplace transform is defined by

$$X(s) \overset{\text{Notation}}{=} \mathcal{L}\{x(t)\} \overset{\text{Definition}}{=} \int_0^\infty e^{-st} x(t)\, dt \tag{2.15}$$

provided that the integral exists. The complex variable s is the Laplace variable, and $\mathcal{L}$ is the Laplace transform operator. It is common practice to denote a time-dependent function by a lowercase letter such as $x(t)$ and its Laplace transform by the same letter in capital, $X(s)$. Transforming the data back to the time domain is done through inverse Laplace transformation $\mathcal{L}^{-1}$ as (see Figure 2.7)

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

Laplace transforms of several functions are listed in Table 2.2 and will be referred to frequently.

Example 2.11: Laplace Transform

a. Find $\mathcal{L}\{e^{at}\}$, a = const.
b. Repeat in MATLAB.

Solution

a. $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \dfrac{1}{s-a}$.

b.

```
>> syms a t
>> F = laplace(exp(a*t))
F =
-1/(a - s)
```

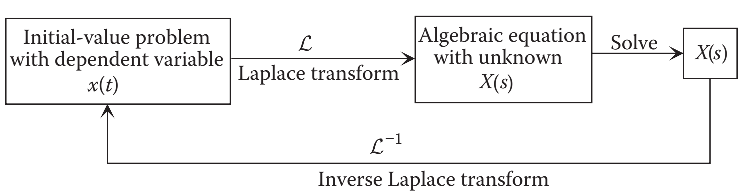


FIGURE 2.7 Solution process for IVPs using Laplace transformation.

TABLE 2.2
Laplace Transform Pairs

No.	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	1, Unit step $u_s(t)$	$\frac{1}{s}$
3	t, Unit ramp $u_r(t)$	$\frac{1}{s^2}$
4	$\delta(t-a)$	e^{-as}
5	$u(t-a)$	e^{-as}/s
6	t^{n-1}, $n = 1,2,3, \ldots$	$(n-1)!/s^n$
7	t^{a-1}, $a > 0$	$\Gamma(a)/s^a$
8	e^{-at}	$\frac{1}{s+a}$
9	te^{-at}	$\frac{1}{(s+a)^2}$
10	$t^n e^{-at}$, $n = 1, 2, 3, \ldots$	$\frac{n!}{(s+a)^{n+1}}$
11	$\frac{1}{b-a}(e^{-at}-e^{-bt})$, $a \neq b$	$\frac{1}{(s+a)(s+b)}$
12	$\frac{1}{a-b}(ae^{-at}-be^{-bt})$, $a \neq b$	$\frac{s}{(s+a)(s+b)}$
13	$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$
14	$\frac{1}{a^2}(-1+at+e^{-at})$	$\frac{1}{s^2(s+a)}$
15	$\frac{1}{a^2}(1-e^{-at}-ate^{-at})$	$\frac{1}{s(s+a)^2}$
16	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
17	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
18	$e^{-\sigma t}\sin\omega t$	$\frac{\omega}{(s+\sigma)^2+\omega^2}$
19	$e^{-\sigma t}\cos\omega t$	$\frac{s+\sigma}{(s+\sigma)^2+\omega^2}$
20	$1-\cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
21	$\omega t-\sin\omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
22	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
23	$\frac{1}{2\omega}t\sin\omega t$	$\frac{s}{(s^2+\omega^2)^2}$

(*continued*)

TABLE 2.2 (Continued)
Laplace Transform Pairs

No.	$f(t)$	$F(s)$
24	$\frac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)$	$\frac{1}{(s^2+\omega^2)^2}$
25	$\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$	$\frac{s^2}{(s^2+\omega^2)^2}$
26	$\frac{1}{\omega_2^2-\omega_1^2}\left[\frac{1}{\omega_2}\sin\omega_2 t - \frac{1}{\omega_1}\sin\omega_1 t\right],\ \omega_1^2 \neq \omega_2^2$	$\frac{1}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
27	$\frac{1}{\omega_2^2-\omega_1^2}(\cos\omega_1 t - \cos\omega_2 t),\ \omega_1^2 \neq \omega_2^2$	$\frac{s}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
28	$\sinh at$	$\frac{a}{s^2-a^2}$
29	$\cosh at$	$\frac{s}{s^2-a^2}$
30	$\frac{1}{a^2-b^2}\left[\frac{1}{a}\sinh at - \frac{1}{b}\sinh bt\right],\ a \neq b$	$\frac{1}{(s^2-a^2)(s^2-b^2)}$
31	$\frac{1}{a^2-b^2}[\cosh at - \cosh bt],\ a \neq b$	$\frac{s}{(s^2-a^2)(s^2-b^2)}$
32	$\frac{1}{3a^2}\left[e^{-at} + 2e^{\frac{1}{2}at}\sin\left(\frac{\sqrt{3}}{2}at - \frac{1}{6}\pi\right)\right]$	$\frac{1}{s^3+a^3}$
33	$\frac{1}{3a}\left[-e^{-at} + 2e^{\frac{1}{2}at}\sin\left(\frac{\sqrt{3}}{2}at + \frac{1}{6}\pi\right)\right]$	$\frac{s}{s^3+a^3}$
34	$\frac{1}{3a^2}\left[e^{-at} + 2e^{-\frac{1}{2}at}\sin\left(\frac{\sqrt{3}}{2}at + \frac{1}{6}\pi\right)\right]$	$\frac{1}{s^3-a^3}$
35	$\frac{1}{3a}\left[e^{-at} + 2e^{-\frac{1}{2}at}\sin\left(\frac{\sqrt{3}}{2}at - \frac{1}{6}\pi\right)\right]$	$\frac{s}{s^3-a^3}$
36	$\frac{1}{4a^3}[\cosh at\sin at - \sinh at\cos at]$	$\frac{1}{s^4+4a^4}$
37	$\frac{1}{2a^2}\sinh at\sin at$	$\frac{s}{s^4+4a^4}$
38	$\frac{1}{2a^3}(\sinh at - \sin at)$	$\frac{1}{s^4-a^4}$
39	$\frac{1}{2a^2}(\cosh at - \cos at)$	$\frac{s}{s^4-a^4}$

2.3.1 Linearity of Laplace and Inverse Laplace Transforms

The Laplace transform operator $\mathcal{L}$ is linear, that is, if the Laplace transforms of functions $x_1(t)$ and $x_2(t)$ exist, and c_1 and c_2 are constant scalars, then

$$\mathcal{L}\{c_1x_1(t)+c_2x_2(t)\} = \int_0^\infty e^{-st}[c_1x_1(t)+c_2x_2(t)]dt = c_1\int_0^\infty e^{-st}x_1(t)dt + c_2\int_0^\infty e^{-st}x_2(t)dt$$
$$= c_1\mathcal{L}\{x_1(t)\} + c_2\mathcal{L}\{x_2(t)\} = c_1X_1(s)+c_2X_2(s)$$

To establish the linearity of $\mathcal{L}^{-1}$, take the inverse Laplace transforms of the expressions on the far left and far right of the equation above to obtain

$$c_1x_1(t)+c_2x_2(t) = \mathcal{L}^{-1}\{c_1X_1(s)+c_2X_2(s)\}$$

Noting that $x_1(t) = \mathcal{L}^{-1}\{X_1(s)\}$ and $x_2(t) = \mathcal{L}^{-1}\{X_2(s)\}$, the result follows.

2.3.2 Differentiation and Integration of Laplace Transforms

We now consider two specific types of situations: $\mathcal{L}\{tg(t)\}$ and $\mathcal{L}\{g(t)/t\}$. In both cases, it is assumed that $G(s) = \mathcal{L}\{g(t)\}$ is either known directly from Table 2.2 or can be determined by other means. Either way, once $G(s)$ is available, the two above-mentioned transforms will be obtained in terms of the derivative and the integral of $G(s)$, respectively. We first make the following definition. If a transform function is in the form $G(s) = N(s)/D(s)$, then any value of s for which $D(s) = 0$ is called a pole of $G(s)$. A pole with a *multiplicity* (number of occurrences) of one is known as a simple pole.

2.3.2.1 Differentiation of Laplace Transforms

If $G(s) = \mathcal{L}\{g(t)\}$ exists, then at any point except at the poles of $G(s)$, we have

$$\mathcal{L}\{tg(t)\} = -\frac{dG(s)}{ds} \tag{2.16}$$

In the general case,

$$\mathcal{L}\{t^n g(t)\} = (-1)^n \frac{d^nG(s)}{ds^n}$$

Example 2.12: Derivative of Laplace Transforms

a. Find $\mathcal{L}\{te^{-2t}\}$.
b. Repeat (a) in MATLAB.

Solution

a. Comparison with Equation 2.16 reveals $g(t) = e^{-2t}$ so that $G(s) = \dfrac{1}{s+2}$. Then,

$$\mathcal{L}\{te^{-2t}\} = -\frac{d}{ds}\left(\frac{1}{s+2}\right) = \left(\frac{1}{s+2}\right)^2$$

b.

```
>> syms t
>> laplace(t*exp(-2*t))
ans =
1/(s + 2)^2
```

2.3.2.2 Integration of Laplace Transforms

If $\mathcal{L}\{g(t)/t\}$ exists and the order of integration can be interchanged, then

$$\mathcal{L}\left\{\frac{g(t)}{t}\right\} = \int_s^\infty G(\sigma)\,d\sigma \tag{2.17}$$

Example 2.13: Integral of Laplace Transforms

a. Find $\mathcal{L}\left\{\dfrac{\sin\omega t}{t}\right\}$.

b. Repeat in MATLAB.

Solution

a. Comparing with Equation 2.17, we find $g(t) = \sin\omega t$ so that $G(s) = \dfrac{\omega}{s^2+\omega^2}$. Consequently,

$$\mathcal{L}\left\{\frac{\sin\omega t}{t}\right\} = \int_s^\infty \frac{\omega}{\sigma^2+\omega^2}\,d\sigma = \int_s^\infty \frac{1}{1+(\sigma/\omega)^2}\frac{d\sigma}{\omega} = \left[\tan^{-1}\frac{\sigma}{\omega}\right]_{\sigma=s}^{\infty} = \frac{\pi}{2} - \tan^{-1}\frac{s}{\omega} = \tan^{-1}\frac{\omega}{s}$$

b.

```
>> syms w t
>> laplace(sin(w*t)/t)
ans =
atan(w/s)
```

2.3.3 Special Functions

Dynamic systems characteristics, such as stability and damping levels, can be largely determined by studying their responses to external excitations or disturbances, which are mathematically modeled as special functions. In this section, we introduce the step, ramp, pulse, and impulse functions, as well as their Laplace transforms.

2.3.3.1 Unit-Step Function

The unit-step function (Figure 2.8) is defined as

$$u(t) = \begin{cases} 1 & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t < 0 \\ \text{undefined (finite)} & \text{if} \quad t = 0 \end{cases}$$

Physically, this may be realized as a constant signal (of magnitude 1) suddenly applied to the system at time $t = 0$. Using the definition of the Laplace transform (Equation 2.15), we find

$$\mathcal{L}\{u(t)\} \overset{\text{Notation}}{=} U(s) = \int_0^\infty e^{-st} u(t)\,dt = \int_0^\infty e^{-st}\,dt = \frac{1}{s}$$

When the magnitude is A, the signal is called a step function, denoted by $Au(t)$ and

$$\mathcal{L}\{Au(t)\} = \int_0^\infty e^{-st} A\,dt = \frac{A}{s}$$

If the step occurs at $a \neq 0$ (Figure 2.9), it is then denoted by $Au(t-a)$, where

$$u(t-a) = \begin{cases} 1 & \text{if} \quad t > a \\ 0 & \text{if} \quad t < a \\ \text{undefined (finite)} & \text{if} \quad t = a \end{cases}$$

To find the Laplace transform of $Au(t-a)$, we need the following shifting property.

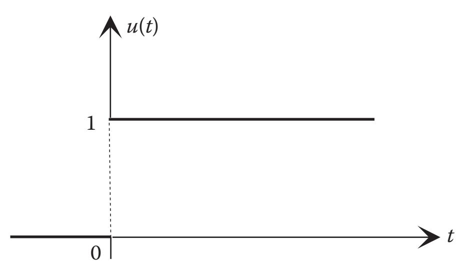


FIGURE 2.8 Unit-step at $t = 0$.

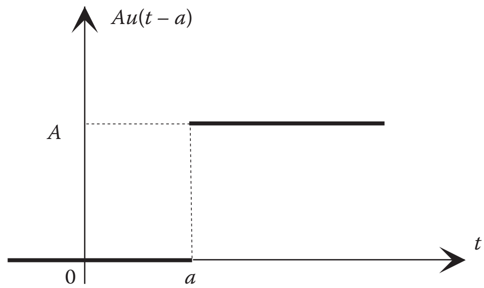


FIGURE 2.9 Time-delayed step function.

Shift on t axis: If $F(s) = \mathcal{L}\{f(t)\}$ exists and $a \geq 0$, then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \tag{2.18}$$

Comparing $\mathcal{L}\{Au(t-a)\}$ to the left side of Equation 2.18, we find $f(t-a) = A$ which in turn implies $f(t) = A$ and hence $F(s) = A/s$. Therefore,

$$\mathcal{L}\{Au(t-a)\} = \frac{Ae^{-as}}{s}$$

Example 2.14: Shift on *t* Axis

Consider the function $g(t)$ defined in Figure 2.10.

a. Express $g(t)$ in terms of unit-step functions.
b. Find $G(s)$ using the shift on t axis (Equation 2.18).
c. Confirm the result in MATLAB.

Solution

a. Following the definition of unit-step, it is readily seen that $g(t) = u(t-1) - u(t-2)$.
b. Applying Equation 2.18 to each term, yields

$$G(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} = \frac{e^{-s} - e^{-2s}}{s}$$

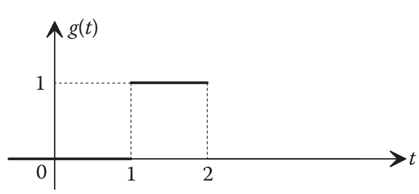


FIGURE 2.10 Example 2.14.

c.

```
>> syms t
>> g = heaviside(t-1) - heaviside(t-2);    % heaviside represents the unit-step
>> G = laplace(g)
G =
(exp(-s)-exp(-2*s))/s
```

2.3.3.2 Unit-Ramp Function

The unit-ramp function (Figure 2.11) is defined as

$$u_{\mathrm{r}}(t) = \begin{cases} t & \text{if} \quad t \ge 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

Physically, this models a signal that changes linearly with a unit rate. By Equation 2.15,

$$\mathcal{L}\{u_{\mathrm{r}}(t)\} = U_{\mathrm{r}}(s) = \int_0^\infty t e^{-st}\, dt = \frac{1}{s^2}$$

In general, when the rate is A, the signal is called a ramp function, denoted by $Au_{\mathrm{r}}(t)$ and

$$\mathcal{L}\{Au_{\mathrm{r}}(t)\} = \frac{A}{s^2}$$

2.3.3.3 Unit-Pulse Function

The unit-pulse function (Figure 2.12a) is defined as

$$u_{\mathrm{p}}(t) = \begin{cases} 1/t_1 & \text{if} \quad 0 < t < t_1 \\ 0 & \text{if} \quad t < 0 \;\; \text{or} \;\; t > t_1 \end{cases}$$

The word "unit" signifies the unit area that the signal occupies. It is readily verified that

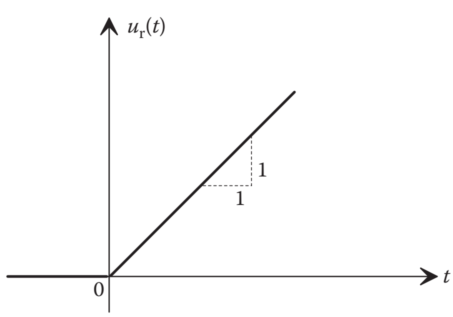


FIGURE 2.11 Unit-ramp.

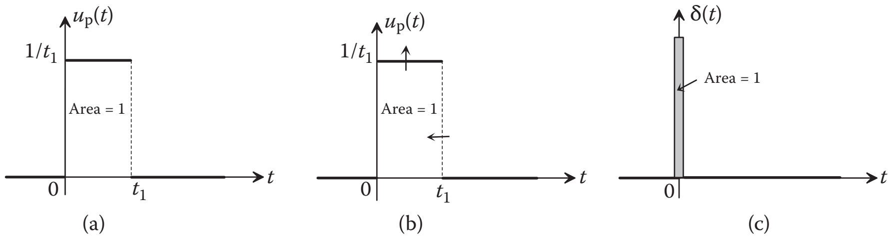


FIGURE 2.12 (a) Unit-pulse, (b) $t_1 \to 0$, and (c) unit-impulse.

$$u_p(t) = \frac{1}{t_1}[u(t) - u(t - t_1)]$$

so that

$$\mathcal{L}\{u_p(t)\} = \frac{1}{t_1}\left[\frac{1}{s} - \frac{e^{-t_1 s}}{s}\right] = \frac{1 - e^{-t_1 s}}{t_1 s}$$

In general, if the signal occupies an area A, it is called a pulse function and is denoted by $Au_p(t)$.

2.3.3.4 Unit-Impulse (Dirac Delta) Function

In the unit-pulse function of Figure 2.12a, let $t_1 \to 0$ (Figure 2.12b). In the limit, the rectangular-shaped signal occupies a region with an infinitesimally small width and an infinitely large height (Figure 2.12c). The area, however, remains unit throughout the process. This limiting signal is known as the unit-impulse (or Dirac delta) function, denoted by $\delta(t)$. In general, if the area is A, the signal is called an impulse, denoted by $A\delta(t)$. An impulse with zero duration and infinite magnitude is a mathematical fabrication and does not occur physically. However, if an external disturbance (such as an applied force, voltage, torque, etc.) is a pulse with a very large magnitude and is applied for a very short period, then it can be approximated as an impulse. Because $\delta(t)$ is the limit of $u_p(t)$ as $t_1 \to 0$, we have

$$\mathcal{L}\{\delta(t)\} \overset{\text{Notation}}{=} (s) = \lim_{t_1 \to 0}\left\{\frac{1 - e^{-st_1}}{st_1}\right\} \overset{\text{L'Hôpital's rule}}{=} \lim_{t_1 \to 0}\left\{\frac{se^{-st_1}}{s}\right\} = 1$$

If the unit-impulse occurs at $t = a$, it is represented by $\delta(t - a)$ and has the properties

$$\delta(t - a) = \begin{cases} 0 & \text{if} \quad t \neq a \\ \infty & \text{if} \quad t = a \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t - a)\, dt = 1$$

It can also be shown that

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

2.3.3.5 The Relation between Unit-Impulse and Unit-Step Functions

Noting that $t = a$ is a point of discontinuity of $u(t-a)$, the unit-impulse signal $\delta(t-a)$ can be regarded as the derivative of $u(t-a)$ at this point of discontinuity, that is,

$$\frac{d}{dt}u(t-a) = \delta(t-a)$$

Therefore, the idea of the impulse function allows for the differentiation of time-variant functions with discontinuities.

2.3.3.6 Periodic Functions

Physical systems are often subjected to excitations that exhibit repeated behavior over long periods. To determine a system's response to this type of input, the Laplace transform of the input must be identified properly. A function $f(t)$ is periodic with period $P > 0$ if it is defined for all $t > 0$, and $f(t+P) = f(t)$ for all $t > 0$. We assume that $f(t)$ is also piecewise continuous, as basically all signals of physical interest are. By Equation 2.15,

$$F(s) = \int_0^\infty e^{-st} f(t)dt = \int_0^P e^{-st} f(t)dt + \int_P^{2P} e^{-st} f(t)dt + \quad \cdots \quad = \sum_{k=0}^{\infty}\left\{\int_{kP}^{(k+1)P} e^{-st} f(t)dt\right\}$$

To make the lower- and upper-limits of the integral independent of P, we introduce the dummy variable $\tau = t - kP$. Then,

$$F(s) = \sum_{k=0}^{\infty}\left\{\int_0^P e^{-s(\tau+kP)} f(\tau)d\tau\right\} = \sum_{k=0}^{\infty} e^{-skP}\left\{\int_0^P e^{-s\tau} f(\tau)d\tau\right\}$$

Note that the summation only affects the exponential term because the integral term is independent of the summation index k. But the exponential term is a geometric series so that

$$\sum_{k=0}^{\infty} e^{-skP} = \sum_{k=0}^{\infty}\left(e^{-sP}\right)^k \overset{\text{Geometric series}}{=} \frac{1}{1-e^{-sP}}$$

Using this information in the expression of $F(s)$, we find

$$F(s) = \left\{\int_0^P e^{-s\tau} f(\tau)d\tau\right\}\sum_{k=0}^{\infty} e^{-skP} = \left\{\int_0^P e^{-s\tau} f(\tau)d\tau\right\}\frac{1}{1-e^{-sP}}$$

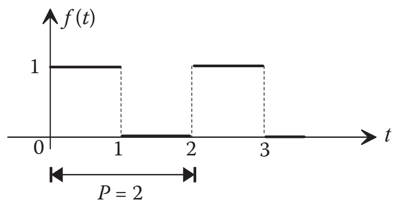


FIGURE 2.13 Periodic signal in Example 2.15.

Therefore, the Laplace transform of the periodic function is determined as

$$F(s) = \frac{1}{1-e^{-Ps}} \int_0^P e^{-st} f(t)dt \tag{2.19}$$

Example 2.15: Periodic Signal

Find the Laplace transform of the periodic function $f(t)$ defined in Figure 2.13.

Solution

It is clear that the period is $P = 2$. Noting that $f(t) = 1$ for $0 < t < 1$ and 0 otherwise, the integral in Equation 2.19 is evaluated as,

$$\int_0^2 e^{-st} f(t)dt = \int_0^1 e^{-st}dt = \frac{1-e^{-s}}{s}$$

Then, by Equation 2.19,

$$F(s) = \frac{1-e^{-s}}{s(1-e^{-2s})} \quad \xrightarrow{1-e^{-2s}=1-(e^{-s})^2=(1-e^{-s})(1+e^{-s})} \quad F(s) = \frac{1}{s(1+e^{-s})}$$

2.3.4 Laplace Transforms of Derivatives and Integrals

Mathematical models of dynamic systems generally involve differential equations of various orders. In other occasions, such as electrical circuits involving capacitors, the system may be described by an equation that contains derivatives, as well as integrals of a function. Consequently, using the Laplace transform approach to solve such equations requires knowledge of the Laplace transform of derivatives and integrals.

2.3.4.1 Laplace Transforms of Derivatives

The Laplace transforms of the first and second derivatives of a function $x(t)$ are defined by

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0)$$

$$\mathcal{L}\{\ddot{x}(t)\} = s^2X(s) - sx(0) - \dot{x}(0) \tag{2.20}$$

In general,

$$\mathcal{L}\{x^{(n)}(t)\} = s^n X(s) - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - \ \dots \ - x^{(n-1)}(0)$$

2.3.4.2 Laplace Transforms of Integrals

The Laplace transform of the integral of a function $x(t)$ is

$$\mathcal{L}\left\{\int_0^t x(t)\,dt\right\} = \frac{1}{s}X(s) \tag{2.21}$$

Equivalently,

$$\mathcal{L}^{-1}\left\{\frac{1}{s}X(s)\right\} = \int_0^t x(t)\,dt \tag{2.22}$$

Example 2.16: Inverse Laplace

a. Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\}$.
b. Confirm in MATLAB.

Solution

a. Comparing with the left side of Equation 2.22, $X(s) = 1/(s + 2)$ so that $x(t) = e^{-2t}$. Then,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = \int_0^t e^{-2t}dt = \frac{1}{2}(1 - e^{-2t})$$

b.

```
>> syms s
>> ilaplace(1/s/(s + 2))
ans =
1/2-1/2*exp(-2*t)
% Alternatively, performing the integration above yields the same result.
>> int(exp(-2*t),0,t)
ans =
1/2-1/2*exp(-2*t)   % Result confirmed
```

2.3.5 Inverse Laplace Transformation

As indicated in Figure 2.7, the final step in solving an IVP involves inverse Laplace transform $\mathcal{L}^{-1}\{X(s)\}$. This can be done in many ways, but the simplest involves either the direct use of the table of Laplace transforms (Table 2.2), if possible, or the partial-fraction expansion method.

2.3.5.1 Partial-Fraction Expansion Method

In system dynamics, the transform function $X(s)$ is normally in the form

$$X(s) = \frac{N(s)}{D(s)} = \frac{\text{Polynomial of degree } m}{\text{Polynomial of degree } n}, \quad m < n \tag{2.23}$$

The idea behind the partial-fraction expansion method is to express $X(s)$ as a sum of suitable fractions, find the inverse Laplace transform of each fraction, and ultimately, through the linearity of $\mathcal{L}^{-1}$, the sum of the resulting time functions yields $x(t)$. How these partial fractions are formed depends on the nature of the poles of $X(s)$, that is, the roots of $D(s)$, which can be real or complex.

- All roots of $D(s)$ are real
  Suppose $D(s) = s^3 + 3s^2 + 2s$ so that its roots are $s = 0,-1,-2$ (all real). In this case, we express $D(s)$ as a product of linear factors, as $D(s) = s(s+1)(s+2)$.

- $D(s)$ has real and complex roots
  Suppose $D(s) = s^3 + 2s^2 + 2s$ so that its roots are $s = 0,-1 \pm j$. In this case, instead of writing $D(s) = s(s+1+j)(s+1-j)$, we write $D(s) = s(s^2+2s+2)$ with the second-degree polynomial with complex roots remaining intact. Any second-degree polynomial with complex roots is called an irreducible polynomial.

Regardless of whether poles of $X(s)$ are real or complex, four possible cases could arise.

Case (1) Linear factor $s - p_i$

If p_i is a simple pole of $X(s)$, then $D(s)$ contains the factor $s - p_i$. This factor is associated with a fraction in the form of

$$\frac{A}{s - p_i}$$

where A = const is called a residue and is to be determined appropriately.

Example 2.17: Linear Factors

a. Find $\mathcal{L}^{-1}\{X(s)\}$ where $X(s) = \dfrac{s+3}{(s+1)(s+2)}$.
b. Verify in MATLAB.

Solution

a. $D(s)$ contains two linear factors because the poles of $X(s)$ are real and distinct: $-1,-2$. Each linear term is associated with a simple fraction as mentioned above. Thus,

$$X(s) = \frac{s+3}{(s+1)(s+2)} = \boxed{\frac{A}{s+1}} + \boxed{\frac{B}{s+2}} = \frac{A(s+2)+B(s+1)}{(s+1)(s+2)} = \frac{(A+B)s+2A+B}{(s+1)(s+2)}$$

The denominators of the original and the final fractions are identical (by design), so we force their respective numerators to be identical, that is,

$$s+3 \equiv (A+B)s + 2A + B$$

However, this identity holds only if the coefficients of like powers of s on both sides are the same. So, we have

$$\begin{array}{lll} \text{Coefficient of } s\text{:} \quad 1 = A + B & \xrightarrow{\text{Solve}} & A = 2 \\ \text{Constant term:} \quad 3 = 2A + B & & B = -1 \end{array}$$

Insert the two residues into the partial fractions, and perform term-by-term inverse Laplace transformation to obtain

$$X(s) = \frac{2}{s+1} - \frac{1}{s+2} \quad \xrightarrow{\mathcal{L}^{-1}} \quad x(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = 2e^{-t} - e^{-2t}$$

b.

```
>> syms s
>> x = ilaplace((s+3)/(s+1)/(s+2))
x =
2*exp(-t)-exp(-2*t)
```

Case (2) Repeated linear factor $(s - p_i)^k$

If p_i is a pole of $X(s)$ with multiplicity k, then $D(s)$ contains the factor $(s - p_i)^k$. This factor is then associated with partial fractions

$$\frac{A_k}{(s-p_i)^k} + \frac{A_{k-1}}{(s-p_i)^{k-1}} + \ldots + \frac{A_2}{(s-p_i)^2} + \frac{A_1}{s-p_i}$$

where the residues $A_k, \ldots, A_1$ are determined as in Case (1).

Case (3) Irreducible polynomial $s^2 + as + b$

Each irreducible polynomial is associated with a single fraction in the form of

$$\frac{Bs+C}{s^2+as+b}$$

with constants B and C to be determined. Before taking the inverse Laplace transform, we complete the square in the irreducible polynomial, that is, $s^2 + as + b = (s + \sigma)^2 + \omega^2$. For example, $s^2 + 2s + 2 = (s + 1)^2 + 1^2$. Then, at some point, we need to determine

$$\mathcal{L}^{-1}\left\{\frac{Bs+C}{(s+\sigma)^2+\omega^2}\right\}$$

The key is to split the fraction in terms of the two expressions

$$\frac{\omega}{(s+\sigma)^2+\omega^2}, \quad \frac{s+\sigma}{(s+\sigma)^2+\omega^2}$$

so that we can ultimately use the relations (see Table 2.2)

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+\sigma)^2+\omega^2}\right\} = e^{-\sigma t}\sin\omega t, \quad \mathcal{L}^{-1}\left\{\frac{s+\sigma}{(s+\sigma)^2+\omega^2}\right\} = e^{-\sigma t}\cos\omega t$$

Example 2.18: Real and Complex Poles

Find $\mathcal{L}^{-1}\{X(s)\}$ where $X(s) = \dfrac{2}{(s+3)(s^2+2s+5)}$.

Solution

The term $s^2 + 2s + 5 = (s + 1)^2 + 2^2$ is irreducible, whereas $s + 3$ is a linear factor. Therefore,

$$X(s) = \frac{2}{(s+3)(s^2+2s+5)} = \boxed{\frac{A}{s+3}} + \boxed{\frac{Bs+C}{s^2+2s+5}} = \frac{A(s^2+2s+5)+(Bs+C)(s+3)}{(s+3)(s^2+2s+5)}$$

$$= \frac{(A+B)s^2+(2A+3B+C)s+5A+3C}{(s+3)(s^2+2s+5)}$$

Proceeding as before,

$$\begin{aligned} A+B&=0 \qquad & A&=\tfrac{1}{4} \\ 2A+3B+C&=0 \qquad & B&=-\tfrac{1}{4} \\ 5A+3C&=2 \qquad & C&=\tfrac{1}{4} \end{aligned}$$

Substituting into the partial fractions, we arrive at

$$X(s) = \frac{1}{4}\left[\frac{1}{s+3} - \frac{s-1}{s^2+2s+5}\right] = \frac{1}{4}\left[\frac{1}{s+3} - \frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}\right]$$

Term-by-term Laplace inversion yields $x(t) = \frac{1}{4}[e^{-3t} - e^{-t}\cos 2t + e^{-t}\sin 2t]$.

Case (4) Repeated irreducible polynomial $(s^2 + as + b)^k$

Each factor $(s^2 + as + b)^k$ in $D(s)$ is associated with partial fractions

$$\frac{B_k s + C_k}{(s^2+as+b)^k} + \ldots + \frac{B_2 s + C_2}{(s^2+as+b)^2} + \frac{B_1 s + C_1}{s^2+as+b}$$

2.3.5.2 Performing Partial Fractions in MATLAB

As we saw in Example 2.17, the command `ilaplace` returns the inverse Laplace transform but not the actual partial fractions. For that purpose, the `residue` command is used instead. This command is concerned with the partial-fraction expansion of the ratio of two polynomials $B(s)/A(s)$ in which, unlike the restriction cited in Equation 2.23, the degree of $B(s)$ can be higher than that of $A(s)$. When $\deg B(s) \geq \deg A(s)$, polynomial division results in

$$\frac{B(s)}{A(s)} = K(s) + \frac{R(1)}{s-P(1)} + \frac{R(2)}{s-P(2)} + \ldots + \frac{R(n)}{s-P(n)} \tag{2.24}$$

where $K(s)$ is called the direct term and $R(1), \ldots, R(n)$ are the residues. Note that the assumption here is that there are no multiple poles. For further information, type `help residue` at the MATLAB command prompt. The structure `[R,P,K] = residue(B,A)` indicates that there are two input arguments `(B,A)` and three outputs `[R,P,K]`. Here, `B` and `A` are vectors of the coefficients of the numerator and denominator, respectively. The residues are returned in the column vector `R`, the pole locations in column vector `P`, and the direct terms in the row vector `K`.

Example 2.19: Revisiting Example 2.17

Find the partial-fraction expansion for $X(s) = \dfrac{s+3}{(s+1)(s+2)}$.

Solution

```
>> A = [1 3 2];   % denominator s^2+3*s+2
>> B = [1 3];   % numerator s+3
>> [R,P,K] = residue(B,A)
R =
   -1
    2        % residues are -1 and 2
P =
   -2
   -1        % poles (corresponding to residues) are -2 and -1
K =
   []        % empty set because deg B(s)<deg A(s)
```

Following the standard form in Equation 2.24, we find

$$X(s) = \frac{-1}{s-(-2)} + \frac{2}{s-(-1)} = \frac{-1}{s+2} + \frac{2}{s+1}$$

which completely agrees with the result of Example 2.17.

Example 2.20: Revisiting Example 2.18

Find the partial-fraction expansion for $X(s) = \dfrac{2}{(s+3)(s^2+2s+5)}$.

Solution

```
>> B = 2;
>> A = conv([1 3],[1 2 5])          % "conv" gives the product
A =
     1    5    11    15
>> [R,P,K] = residue(B,A)
```

```
R =

    0.2500
   -0.1250 - 0.1250i
   -0.1250 + 0.1250i

P =

   -3.0000
   -1.0000 + 2.0000i
   -1.0000 - 2.0000i

K =

   []
```

Based on these findings, the partial fractions are formed as

$$X(s) = \frac{2}{(s+3)(s^2+2s+5)} = \frac{\frac{1}{4}}{s-(-3)} + \frac{-\frac{1}{8}-\frac{1}{8}j}{s-(-1+2j)} + \frac{-\frac{1}{8}+\frac{1}{8}j}{s-(-1-2j)}$$

Inspection confirms that the above is equivalent to the expansion obtained in Example 2.18.

2.3.5.3 Convolution Method

In addition to partial-fraction expansion, there is a second method of practical importance known as the convolution method. In systems analysis, the problem of determining the time history of a function often boils down to finding $\mathcal{L}^{-1}\{F(s)\}$ where $F(s) = G(s)H(s)$ and $g(t)$ and $h(t)$ are available. Then,

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \underbrace{(g*h)(t)}_{\text{convolution of } g \text{ and } h} = \int_0^t g(\tau)h(t-\tau)\,d\tau \overset{\text{symmetry}}{=} \int_0^t h(\tau)g(t-\tau)\,d\tau = (h*g)(t)$$

To prove this, it suffices to show

$$\mathcal{L}\left\{\int_0^t g(\tau)h(t-\tau)\,d\tau\right\} = G(s)H(s)$$

First, using the fact that $u(t-\tau) = 0$ for all $\tau > t$, the left side is rewritten as

$$\mathcal{L}\left\{\int_0^t g(\tau)h(t-\tau)\,d\tau\right\} = \mathcal{L}\left\{\int_0^\infty g(\tau)h(t-\tau)u(t-\tau)\,d\tau\right\}$$

Next, using Equation 2.15 and switching the order of integration, the above reduces to

$$\begin{aligned}\mathcal{L}\left\{\int_0^\infty g(\tau)h(t-\tau)u(t-\tau)\,d\tau\right\} &= \int_0^\infty e^{-st}\left[\int_0^\infty g(\tau)h(t-\tau)u(t-\tau)\,d\tau\right]dt \\ &= \int_0^\infty e^{-st}h(t-\tau)u(t-\tau)\,dt \int_0^\infty g(\tau)\,d\tau\end{aligned}$$

This is allowed because the Laplace transforms of g and h exist. Once again, because $u(t-\tau)=0$ for all $\tau > t$, the first integral can be rewritten to give

$$\mathcal{L}\left\{\int_0^\infty g(\tau)h(t-\tau)u(t-\tau)\,d\tau\right\} = \int_\tau^\infty e^{-st}h(t-\tau)\,dt \int_0^\infty g(\tau)\,d\tau$$

Introducing the change of variables $\xi = t - \tau$, the above becomes

$$\begin{aligned}\mathcal{L}\left\{\int_0^\infty g(\tau)h(t-\tau)u(t-\tau)\,d\tau\right\} &= \int_0^\infty e^{-s(\xi+\tau)}h(\xi)\,d\xi \int_0^\infty g(\tau)\,d\tau \\ &= \underbrace{\int_0^\infty e^{-s\xi}h(\xi)\,d\xi}_{H(s)} \underbrace{\int_0^\infty e^{-s\tau}g(\tau)\,d\tau}_{G(s)} = H(s)G(s)\end{aligned}$$

Example 2.21: Convolution

Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s+1)}\right\}$.

Solution

Let $F(s) = \dfrac{1}{s^2(s+1)} = G(s)H(s)$ with $G(s) = \dfrac{1}{s^2}$, $H(s) = \dfrac{1}{s+1}$. Noting $g(t) = t$ and $h(t) = e^{-t}$,

$$f(t) = (g * h)(t) = \int_0^t \tau \cdot e^{-(t-\tau)}\,d\tau \overset{\text{Integration by parts}}{=} \left[\tau e^{-(t-\tau)}\right]_{\tau=0}^{t} - \int_0^t e^{-(t-\tau)}d\tau = t - 1 + e^{-t}$$

2.3.5.3.1 Solving IVPs

The foregoing analysis has provided us with the necessary tools to complete the procedure of solving IVPs as depicted in Figure 2.7. This will be illustrated by the following examples.

Example 2.22: Initial-Value Problem

a. Solve $\ddot{x} + 4x = 2u(t)$, $x(0) = 0$, $\dot{x}(0) = 0$, where $u(t)$ is the unit-step function.
b. Repeat in MATLAB.

Solution

a. Laplace transformation of the ODE (using Equation 2.20), taking into account the zero initial conditions, we arrive at

$$(s^2+4)X(s) = \frac{2}{s} \qquad X(s) = \frac{2}{s(s^2+4)} \overset{\text{Partial fractions}}{=} \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{(A+B)s^2 + Cs + 4A}{s(s^2+4)}$$

Equating the coefficients of like powers of s on both sides yields $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = 0$. Then,

$$X(s) = \frac{2}{s(s^2+4)} = \frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}s}{s^2+4} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Inverse Laplace transformation gives $x(t) = \frac{1}{2}(1-\cos 2t)$, $t \geq 0$. Of course, the convolution method would have led to the same result.

b.

```
>> x = simple(dsolve('D2x + 4*x = 2*heaviside(t)','x(0) = 0, Dx(0) = 0'))
x =
heaviside(t)*sin(t)^2
```

Noting that $\sin^2(t) = \frac{1}{2}(1-\cos 2t)$ and `heaviside` represents the unit-step function, the result agrees with that in Part (a).

Example 2.23: Initial-Value Problem

Solve $\ddot{x} + 3\dot{x} + 2x = 0$, $x(0) = 0$, $\dot{x}(0) = 1$.

Solution

Taking the Laplace transform,

$$s^2X(s) - sx(0) - \dot{x}(0) + 3[sX(s) - x(0)] + 2X(s) = 0$$

Using the given initial conditions and solving for $X(s)$, yields

$$X(s) = \frac{1}{(s+1)(s+2)} \xrightarrow[\text{or convolution}]{\mathcal{L}^{-1} \text{ using partial-fraction expansion}} x(t) = e^{-t} - e^{-2t}$$

2.3.6 Final-Value Theorem and Initial-Value Theorem

Suppose that a function $x(t)$ attains a finite limit as $t \to \infty$, that is, it settles down after a sufficiently long time. This finite limit is the steady-state value (or final value) of $x(t)$, denoted by x_{ss}. However, there are many situations in which the time history $x(t)$ is not available, but instead $X(s)$ is. The final-value theorem (FVT) allows us to find x_{ss}, if it exists, by directly using $X(s)$ without knowledge of $x(t)$.

2.3.6.1 Final-Value Theorem

Suppose $X(s)$ has no poles in the right half-plane (RHP) or on the imaginary axis, except possibly a simple pole (multiplicity of 1) at the origin. Then, $x(t)$ has a definite steady-state value given by

$$x_{ss} = \lim_{s \to 0}\{sX(s)\} \tag{2.25}$$

The FVT must be used only when it is applicable. Using the FVT when the required conditions do not hold may yield incorrect results.

Example 2.24: Final-Value Theorem

a. Find x_{ss} if $X(s) = \dfrac{s+1}{s(s^2+2s+2)}$.
b. Confirm in MATLAB.

Solution

a. The poles of $X(s)$ are at 0 and $-1 \pm j$. The complex conjugate pair lie in the left half-plane, and 0 is a simple pole (at the origin), all allowed by the FVT. Therefore,

$$x_{ss} = \lim_{s\to 0}\{sX(s)\} = \lim_{s\to 0}\left\{\frac{s+1}{s^2+2s+2}\right\} = \frac{1}{2}$$

b.

```
>> syms s
>> X = (s+1)/s/(s^2+2*s+2);
>> xss = limit(s*X,s,0)
xss =
1/2
% Re-confirm the result by inspecting the limit of x(t) as t goes to infinity
>> xt = ilaplace(X)
xt =
1/2+1/2*(-cos(t)+sin(t))*exp(-t)        % time history x(t)
>> syms t
>> xss = limit(xt,t,inf)
xss =
1/2
```

Example 2.25: FVT Not Applicable

Let $X(s) = \dfrac{1}{s^2+4}$ so that its poles are at $\pm 2j$, on the imaginary axis, not permitted by the FVT. Therefore, FVT is not applicable and should not be applied! If it were to be applied, it would yield

$$x_{ss} = \lim_{s\to 0}\{sX(s)\} = \lim_{s\to 0}\left\{\frac{s}{s^2+4}\right\} = 0$$

which is obviously false. To explain this, we first find $x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2}\sin 2t$. Then, it is clear that $\lim_{t\to\infty} x(t)$ does not exist because $x(t)$ is oscillatory and there is no steady-state value.

2.3.6.2 Initial-Value Theorem

If $\lim_{s\to\infty}\{sX(s)\}$ exists, then the initial value (see Section 2.2) of $x(t)$ is given by

$$x(0^+) = \lim_{s\to\infty}\{sX(s)\} \tag{2.26}$$

Note that in the case of the initial-value theorem (IVT), the poles of $X(s)$ are not limited to specific regions in the complex plane as they were with the FVT.

Example 2.26: Initial-Value Theorem

If $X(s) = \dfrac{s^2+3}{s^2(2s+1)}$, find $x(0^+)$.

Solution

By Equation 2.26,

$$x(0^+) = \lim_{s\to\infty}\{sX(s)\} = \lim_{s\to\infty}\left\{\frac{s^2+3}{s(2s+1)}\right\} = \frac{1}{2}$$

Example 2.27: Initial Condition ≠ Initial Value

Consider $\ddot{x}+\dot{x}+2x = \delta(t)$, $x(0^-) = 0$, $\dot{x}(0^-) = 0$ where $\delta(t)$ denotes the unit-impulse. Recall that 0^- refers to the time immediately prior to $t = 0$ and $x(0^-)$ is the initial condition of x. Determine the initial values of x and $\dot{x}$, that is, $x(0^+)$ and $\dot{x}(0^+)$.

Solution

Taking the Laplace transform of the ODE and using the zero initial conditions, we find

$$X(s) = \frac{1}{s^2+s+2}$$

By the IVT,

$$x(0^+) = \lim_{s\to\infty}\{sX(s)\} = \lim_{s\to\infty}\left\{\frac{s}{s^2+s+2}\right\} = 0$$

Thus, $x(0^+) = x(0^-)$. To evaluate $\dot{x}(0^+)$, we apply the IVT while x is replaced with $\dot{x}$:

$$\dot{x}(0^+) = \lim_{s\to\infty}\{s\mathcal{L}\{\dot{x}\}\} = \lim_{s\to\infty}\{s[sX(s)]\} = \lim_{s\to\infty}\left\{\frac{s^2}{s^2+s+2}\right\} = 1$$

Therefore, $\dot{x}(0^+) \neq \dot{x}(0^-)$. This clearly shows that when impulsive forces are present in the system, initial values and initial conditions are indeed different.

PROBLEM SET 2.3

In Problems 1 through 8,

a. Find the Laplace transform of the given function (use Table 2.2 when applicable).
b. Confirm the result in MATLAB.

1. e^{at-b}, $a, b = \text{const}$
2. $\frac{2}{3}t^2 - 1$
3. $\sin(\omega t + \phi)$, $\omega, \phi = \text{const}$
4. $\cos(\omega t - \phi)$, $\omega, \phi = \text{const}$
5. $\cos^2 t$
6. $t\cos t$
7. $t^2 \sin\omega t$
8. $t\sinh t$

In Problems 9 through 12,

a. Express the signal in terms of unit-step functions.
b. Find the Laplace transform of the expression in Part (a) using the shift on t axis.

9. $g(t) = \begin{cases} -1 & \text{if } 0 < t < 1 \\ 1 & \text{if } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$

10. $g(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$

11. $g(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 < t < 1 \\ 1 & \text{if } t > 1 \end{cases}$

12. $g(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - t & \text{if } 0 < t < 1 \\ 0 & \text{if } t > 1 \end{cases}$

In Problems 13 through 16, find the Laplace transform of each periodic function whose definition in one period is given.

13. $f(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ -1 & \text{if } 1 < t < 2 \end{cases}$

14. $f(t) = 2(1 - t)$, $0 < t < 1$

15. $f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 1 - t & \text{if } 1 < t < 2 \end{cases}$

16. $f(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 2 - t & \text{if } 1 < t < 2 \end{cases}$

17. Find the Laplace transform of the periodic function $f(t)$ in Figure 2.14.
18. Find the Laplace transform of the periodic function $f(t)$ in Figure 2.15.

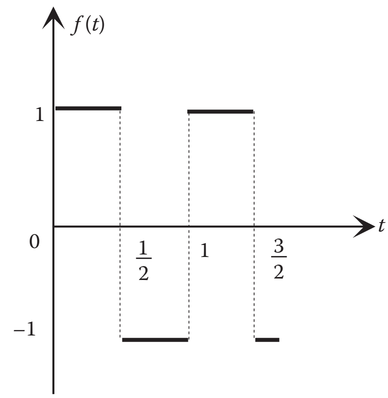


FIGURE 2.14 Periodic function in Problem 17.

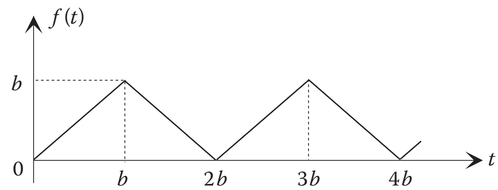


FIGURE 2.15 Periodic function in Problem 18.

In Problems 19 through 24,

a. Find the inverse Laplace transform using the partial-fraction expansion method.
b. Repeat in MATLAB.

19. $\dfrac{3s+4}{s(s+1)}$
20. $\dfrac{3s^2+2s+2}{(s^2+1)(s+2)}$
21. $\dfrac{s+10}{s(s^2+2s+5)}$
22. $\dfrac{4s+5}{s^2(s^2+4s+5)}$
23. $\dfrac{s-8}{s(s+2)^2}$
24. $\dfrac{s^2+s-1}{(s+3)(s^2+2s+2)}$

In Problems 25 through 30,

a. Solve the IVP.
b. Confirm the result in MATLAB.

25. $\dot{x}+2x=2u(t)-u(t-1),\quad x(0)=0$
26. $\ddot{x}+2\dot{x}+x=g(t),\quad x(0)=0,\quad \dot{x}(0)=0,\quad g(t)=\begin{cases}1 & \text{if} \quad 1<t<2\\ 0 & \text{otherwise}\end{cases}$

27. $3\ddot{x} + \dot{x} = e^{-t}, \;\; x(0) = 0, \;\; \dot{x}(0) = \frac{1}{3}$
28. $\ddot{x} + 9x = \sin t, \;\; x(0) = 1, \;\; \dot{x}(0) = 0$
29. $\ddot{x} + \dot{x} - 2x = e^{t}, \;\; x(0) = 0, \;\; \dot{x}(0) = 1$
30. $\ddot{x} + 3\dot{x} = 1, \;\; x(0) = 2, \;\; \dot{x}(0) = 0$

In Problems 31 through 36, decide whether the FVT is applicable, and if so, find x_{ss}.

31. $X(s) = \dfrac{1}{2s(s+3)}$
32. $X(s) = \dfrac{s+2}{(s+4)(s^2+4s+5)}$
33. $X(s) = \dfrac{s+1}{s^2(s+3)(s+2)}$
34. $X(s) = \dfrac{s+3}{(s+2)^2(s+1)}$
35. $X(s) = \dfrac{s^2+1}{s(s+1)^2}$
36. $X(s) = \dfrac{s+\frac{1}{2}}{s(s^2+s+1)}$

In Problems 37 through 40, evaluate $x(0^+)$ using the IVT.

37. $X(s) = \dfrac{s^2+1}{s(2s^2+s+3)}$
38. $X(s) = \dfrac{3s+2}{(s+1)(s+2)^2}$
39. $X(s) = \dfrac{s(s+4)}{(s+1)(s+2)(s+3)}$
40. $X(s) = \dfrac{s^2+2}{(3s+1)(s^2+9)}$

2.4 SUMMARY

The rectangular form of a complex number z is $z = x + jy$, where x and y are the real and imaginary parts of z, respectively. The magnitude of z is $|z| = \sqrt{x^2 + y^2}$. The distance between two complex numbers z_1 and z_2 is $|z_1 - z_2|$. The complex conjugate of z, denoted by $\bar{z}$, is defined as $\bar{z} = x - jy$, and $z\bar{z} = x^2 + y^2 = |z|^2$. The polar form of z is $z = re^{j\theta}$ where $r = |z|$ and θ is measured from the positive real axis, and regarded as positive in the counterclockwise direction. Given a complex number $z = re^{j\theta} \neq 0$, and a positive integer n, the nth root of z is multivalued and defined by

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + j\sin\frac{\theta + 2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1$$

An nth-order ODE

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \;\ldots\; + a_1\dot{x} + a_0 x = F(t)$$

is linear if coefficients a_0, $a_1, \ldots, a_n$ are either constants or functions of the independent variable t. If $F(t) \equiv 0$, the ODE is called homogeneous. Otherwise, it is nonhomogeneous. A linear, first-order ODE

$$\dot{x} + g(t)x = f(t)$$

has a general solution

$$x(t) = e^{-h}\left[\int e^{h} f(t)\, dt + c\right], \quad h = \int g(t)dt, \quad c = \text{const}$$

If a function $x(t)$ is defined for all $t \geq 0$, its Laplace transform is defined by

$$X(s) \overset{\text{Notation}}{=} \mathcal{L}\{x(t)\} \overset{\text{Definition}}{=} \int_0^\infty e^{-st} x(t)dt$$

provided that the integral exists. If $F(s) = \mathcal{L}\{f(t)\}$ exists and $a \geq 0$, then

$$\mathcal{L}\{f(t-a)\, u(t-a)\} = e^{-as}F(s)$$

If $f(t)$ is periodic with period P, then

$$F(s) = \frac{1}{1-e^{-Ps}} \int_0^P e^{-st} f(t)dt$$

The Laplace transforms of the first and second derivatives of a function $x(t)$ are defined by

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0)$$

$$\mathcal{L}\{\ddot{x}(t)\} = s^2X(s) - sx(0) - \dot{x}(0)$$

The Laplace transform of the integral of a function $x(t)$ is

$$\mathcal{L}\left\{\int_0^t x(t)\, dt\right\} = \frac{1}{s} X(s)$$

Inverse Laplace transformation can be performed by either partial-fraction expansion or the convolution method, which states

$$\mathcal{L}^{-1}\{G(s)H(s)\} = \underbrace{(g*h)(t)}_{\text{convolution of } g \text{ and } h} = \int_0^t g(\tau)h(t-\tau)d\tau \overset{\text{symmetry}}{=} \int_0^t h(\tau)g(t-\tau)d\tau = (h*g)(t)$$

FVT: If $X(s)$ has no poles in the RHP or on the imaginary axis, except possibly a simple pole at the origin, then $x(t)$ has a definite steady-state value given by

$$x_{ss} = \lim_{s \to 0}\{sX(s)\}$$

IVT: If $\lim_{s \to \infty}\{sX(s)\}$ exists, then

$$x(0^+) = \lim_{s \to \infty}\{sX(s)\}$$

REVIEW PROBLEMS

In Problems 1 through 4, perform the operations and express the result in rectangular form.

1. $\dfrac{(1-3j)^2}{2+j}$
2. $\dfrac{1+\frac{1}{3}j}{2j(3-2j)}$
3. $(0.6 - 0.8j)^5$
4. $\dfrac{(1+3j)^4}{(3+j)^3}$
5. Find all possible values of $\left(-3-\frac{3}{2}j\right)^{1/3}$.
6. Find all possible values of $\sqrt{1-0.3j}$.
7. Solve the IVP $3\dot{y} + y = 2t$, $y(0) = -4$.
8. Express $\cos t - 2\sin t$ in the form $D\cos(t + \phi)$ for suitable amplitude D and phase ϕ.
9. Solve $\ddot{x} + 3\dot{x} = 3e^{-3t}$, $x(0) = 0$, $\dot{x}(0) = 2$ using
   a. The method of undetermined coefficients.
   b. Laplace transformation.
10. Solve the IVP $\ddot{x} + 2\dot{x} = \delta(t-1)$, $x(0) = 0$, $\dot{x}(0) = 1$.
11. Find the Laplace transform of the periodic function in Figure 2.16.
12. Find the Laplace transform of the periodic function whose definition in one period is

$$f(t) = t^2, \quad 0 < t < 1.$$

13. Evaluate the convolution $u(t-a)*t$.
14. Find the convolution $u(t-a)*e^{-t}$.

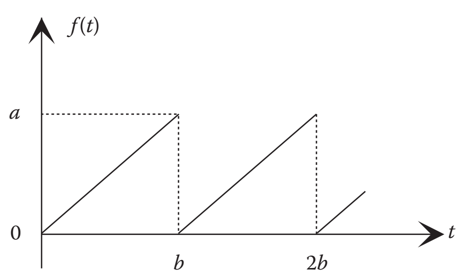


FIGURE 2.16 Problem 11.

15. Using partial-fraction expansion, find

$$\mathcal{L}^{-1}\left\{\frac{2s^2+1}{s^2(4s^2+1)}\right\}$$

16. Using convolution, find

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s+1)(s^2+1)}\right\}$$

17. Consider

$$X(s)=\frac{1}{s(s+1)^2}$$

  a. Using the FVT, if applicable, evaluate x_{ss}.
  b. Confirm the result of Part (a) by evaluating $\lim_{t\to\infty}\{x(t)\}$.

18. Repeat Problem 17 for $X(s)=\dfrac{s+0.1}{s(s^2+0.2s+25.01)}$.
19. Consider

$$X(s)=\frac{3s}{2(s^2+0.4s+1.04)}$$

  a. Using the IVT, evaluate $x(0^+)$.
  b. Confirm the result of Part (a) by evaluating $\lim_{t\to 0^+}\{x(t)\}$.

20. Assuming $X(s)=\dfrac{0.4s+0.3}{s(3s^2+1)}$, evaluate $\dot{x}(0^+)$ using the IVT.

3 Matrix Analysis

The fundamentals of matrix analysis, including vector and matrix operations and properties, as well as matrix characteristics such as the rank, the determinant, and the inverse are presented in this chapter. The methods of matrix analysis are mainly useful in the treatment of systems of algebraic equations that are heavily coupled, that is, when several unknown variables are involved in several equations within the system. Also discussed in this chapter is the matrix eigenvalue problem, which plays a key role in the determination of a system's natural frequencies, as well as the solution process for a system of differential equations. We focus on algebraic systems first, and then extend the ideas to systems of differential equations and the matrix eigenvalues problem.

3.1 VECTORS AND MATRICES

An n-dimensional vector $\mathbf{v}$ is an ordered set of n scalars, written as

$$\mathbf{v} = \begin{Bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{Bmatrix}$$

where each v_i ($i = 1,2, \dots , n$) is called a component of vector $\mathbf{v}$. A matrix is a collection of numbers (real or complex) or possibly functions, arranged in a rectangular array and enclosed by square brackets. Each of the elements in a matrix is called an entry of the matrix. The horizontal and vertical lines of entries are the rows and columns of the matrix, respectively. The number of rows and columns of a matrix determine the size of that matrix. If a matrix $\mathbf{A}$ has m rows and n columns, then it is said to be of size $m \times n$. If a matrix has the same number of rows as columns, it is called a square matrix. Otherwise, it is called rectangular. It is customary to denote matrices by bold-faced capital letters, such as $\mathbf{A}$. The abbreviated form of an $m \times n$ matrix is

$$\mathbf{A} = [a_{ij}]_{m\times n}, \qquad i = 1,2, \dots, m, \qquad j = 1,2, \dots, n$$

where a_{ij} is known as the (i,j) entry of $\mathbf{A}$, located at the intersection of the ith row and the jth column of $\mathbf{A}$. Therefore, a_{12}, for instance, occupies the entry at the intersection of the first row and the second column. If $\mathbf{A}$ is a square matrix ($m = n$), the elements $a_{11}, a_{22}, \dots, a_{nn}$ are the diagonal entries of $\mathbf{A}$. These diagonal elements form the main diagonal of $\mathbf{A}$. Two matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are equal if they have the same size and the same entries in the respective locations. If any rows or columns (or possibly both) of $\mathbf{A}$ are deleted, a submatrix of $\mathbf{A}$ is generated. The sum of $\mathbf{A} = [a_{ij}]_{m\times n}$ and $\mathbf{B} = [b_{ij}]_{m\times n}$ is $\mathbf{C} = [c_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$. The product of a scalar k and matrix $\mathbf{A} = [a_{ij}]_{m\times n}$ is $k\mathbf{A} = [ka_{ij}]_{m\times n}$. Consider $\mathbf{A} = [a_{ij}]_{m\times n}$ and $\mathbf{B} = [b_{ij}]_{n\times p}$ so that the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$. Then, their product $\mathbf{C} = \mathbf{AB}$ is $m \times p$ whose entries are obtained as

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad i = 1,2, \dots, m, \quad j = 1,2, \dots, p$$

$$
\begin{array}{c c c c c}
 & & j\text{th column} & & \\
\begin{array}{r} \\ \\ i\text{th row} \\ \\ \\ \end{array} &
\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \boxed{a_{i1}} & \boxed{a_{i2}} & \cdots & \cdots & \boxed{a_{in}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}_{m\times n}
&
\begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}_{n\times p}
& = &
\begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i1} & \cdots & \boxed{c_{ij}} & \cdots & c_{ip} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}_{m\times p}
\end{array}
$$

FIGURE 3.1 Construction of the matrix product.

This is schematically shown in Figure 3.1. If the number of columns of **A** does not match the number of rows of **B**, the product is undefined. If the product is defined, then the (i,j) entry of **C** is simply the dot (inner) product of the ith row of **A** and the jth column of **B**.

Given an $m \times n$ matrix **A**, its transpose, denoted by $\mathbf{A}^T$, is an $n \times m$ matrix with the property that its first row is the first column of **A**, its second row is the second column of **A**, and so on. Given that all matrix operations are valid,

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$(k\mathbf{A})^T = k\mathbf{A}^T, \quad k = \text{scalar}$$

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

Example 3.1: Transpose

Consider

$$\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

a. Verify $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.
b. Confirm the result in MATLAB®.

Solution

a. Performing the operations on the left side of the identity, we find

$$\mathbf{AB} = \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -5 & 10 \\ 3 & -6 \end{bmatrix} \quad \overset{\text{Transpose}}{} \quad (\mathbf{AB})^T = \begin{bmatrix} -5 & 3 \\ 10 & -6 \end{bmatrix}$$

Performing the operations on the right side, we have

$$\mathbf{B}^T\mathbf{A}^T = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 10 & -6 \end{bmatrix}$$

b.

```
>> A = [4 -1;0 3]; B = [-1 2;1 -2];
>> Left_side = (A*B).'   % .' returns the non-conjugate transpose
Left_side =
     -5    3
     10   -6
>> Right_side = B.'*A.'
Right_side =
     -5    3
     10   -6
```

3.1.1 Special Matrices

A square matrix $\mathbf{A}$ is symmetric if $\mathbf{A}^T = \mathbf{A}$ and skew-symmetric if $\mathbf{A}^T = -\mathbf{A}$. A square matrix $\mathbf{A}_{n\times n} = [a_{ij}]$ is upper-triangular if $a_{ij} = 0$ for all $i > j$, that is, all entries below the main diagonal are zeros; lower-triangular if $a_{ij} = 0$ for all $i < j$, that is, all elements above the main diagonal are zeros; and diagonal if $a_{ij} = 0$ for all $i \neq j$. In the upper-triangular and lower-triangular matrices, the diagonal elements may be all zeros. However, in a diagonal matrix, at least one diagonal entry must be nonzero. The $n \times n$ identity matrix, denoted by $\mathbf{I}$, is a diagonal matrix whose diagonal entries are all equal to 1.

3.1.2 Elementary Row Operations

In matrix analysis, we often encounter matrices that do not appear in a special form such as triangular or diagonal. Because special matrices are so much easier to work with, it is desirable to transform (or reduce) a general matrix into one of the special forms mentioned earlier. This is done with the aid of elementary row operations (EROs). There are three types of EROs:

- ERO_1: Multiply a row by a nonzero constant.
- ERO_2: Interchange two rows.
- ERO_3: Multiply row i by $\alpha = \text{const} \neq 0$, add the result to row k, then replace row k with the outcome. In this process, row i is called the pivot row.

It is important to note that a matrix and its transformed form are not the same, but depending on the transformation, certain characteristics of the original matrix may be preserved. These EROs are often used to transform a matrix $\mathbf{A}$ into an upper-triangular form. The form of matrix $\mathbf{A}$ in the final step of this process is called the row–echelon form of $\mathbf{A}$, denoted by REF($\mathbf{A}$). We reiterate that the original matrix and any subsequent one generated by an ERO are completely different matrices. Furthermore, the row–echelon form of a matrix is not unique.

Example 3.2: Row–Echelon Form

Find the row–echelon form of

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & 3 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\underset{\substack{\text{ERO}_3\\ 1 \quad -2}}{}\begin{bmatrix}1 & -2 & 1\\ 2 & 0 & 3\\ -1 & 1 & 2\end{bmatrix} \longrightarrow \underset{\text{ERO}_2}{}\begin{bmatrix}1 & -2 & 1\\ 0 & 4 & 1\\ 0 & -1 & 3\end{bmatrix} \longrightarrow \underset{\substack{\text{ERO}_3\\ 4}}{}\begin{bmatrix}1 & -2 & 1\\ 0 & -1 & 3\\ 0 & 4 & 1\end{bmatrix} \longrightarrow \underset{\text{REF}}{\begin{bmatrix}1 & -2 & 1\\ 0 & -1 & 3\\ 0 & 0 & 13\end{bmatrix}}$$

FIGURE 3.2 Steps in construction of the row–echelon form.

Solution

The details are depicted in Figure 3.2. In the first segment, the first row is the pivot row and used to generate zeros beneath the (1,1) entry. In the second segment, the lower two rows are switched to replace the entry of 4 with –1 to avoid fractions. In the third segment, the new second row is the pivot row, which is used to create a zero under the (2,2) entry. The final matrix represents REF(**A**). The row–echelon form is clearly not unique, and depends on the manner in which the ultimate upper-triangular matrix is achieved.

3.1.3 Rank of a Matrix

The rank of a matrix **A**, denoted by rank(**A**), is the number of nonzero rows in the row–echelon form of **A**. This number is independent of the sequence of EROs used. The rank is sometimes called the row rank. If **A** is $m \times n$, then rank(**A**) can at most be equal to m or n, whichever is smaller. For instance, the rank of a 3×4 matrix can at most be 3. If elementary column operations are employed to find the column–echelon form, then the number of nonzero columns is the column rank of the matrix. It turns out that the row rank and the column rank of a matrix are the same, and consequently,

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$$

Example 3.3: Rank

For the matrix given below,

a. Find the row–echelon form and rank.
b. Repeat in MATLAB.

$$\mathbf{A} = \begin{bmatrix}-1 & 2 & 4\\ 3 & 1 & -2\\ 2 & 3 & 2\end{bmatrix}$$

Solution

a. The details are shown in Figure 3.3.
   Because there are two nonzero rows in the row–echelon form of **A**, we have rank(**A**) = 2.

$$\underset{\substack{\text{ERO}_3\\ 2 \quad 3}}{}\begin{bmatrix}-1 & 2 & 4\\ 3 & 1 & -2\\ 2 & 3 & 2\end{bmatrix} \longrightarrow \underset{\substack{\text{ERO}_3\\ -1}}{}\begin{bmatrix}-1 & 2 & 4\\ 0 & 7 & 10\\ 0 & 7 & 10\end{bmatrix} \longrightarrow \begin{bmatrix}-1 & 2 & 4\\ 0 & 7 & 10\\ 0 & 0 & 0\end{bmatrix}$$

FIGURE 3.3 Construction of REF in Example 3.3.

b. The `rref` command in MATLAB produces the *reduced* row–echelon form by trying to create ones in the diagonal entries and zeros elsewhere for a square matrix. For non-square matrices, the outcome will contain either an additional row or an additional column depending on the size of the matrix.

```
>> A = [-1 2 4;3 1 -2;2 3 2];
>> rref(A)
ans =
    1.0000         0   -1.1429
         0    1.0000    1.4286
         0         0         0
>> rank(A)
ans =
     2
```

3.1.4 Determinant of a Matrix

The determinant of a square matrix $\mathbf{A} = [a_{ij}]_{n\times n}$ is a real scalar denoted by $|\mathbf{A}|$ or det($\mathbf{A}$). The most trivial case is $\mathbf{A} = [a_{11}]_{1\times 1}$, for which the determinant is simply $|\mathbf{A}| = a_{11}$. For $n \geq 2$, the determinant may be calculated using any row—with preference given to the row with the most number of zeros—as

$$|\mathbf{A}| = \sum_{k=1}^{n} a_{ik}(-1)^{i+k} M_{ik}, \quad i = 1, 2, \ldots, n \tag{3.1}$$

In Equation 3.1, M_{ik} is called the minor of the entry a_{ik}, defined as the determinant of the $(n-1)\times(n-1)$ submatrix of $\mathbf{A}$ obtained by deleting the ith row and the kth column of $\mathbf{A}$. The quantity $(-1)^{i+k}M_{ik}$ is the cofactor of a_{ik} and is denoted by C_{ik}. Also note that $(-1)^{i+k}$ is responsible for whether a term is multiplied by +1 or −1. A square matrix with a nonzero determinant is called nonsingular. Otherwise, it is called singular.

Example 3.4: Determinant

a. Calculate the determinant of

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 & 3 \\ 1 & 2 & 4 & -1 \\ 3 & -1 & 2 & 4 \\ 1 & 5 & 6 & -2 \end{bmatrix}$$

b. Repeat in MATLAB.

Solution

a. We will use the first row because it contains the smallest (in magnitude) entries.

$$\begin{vmatrix} -2 & 1 & 1 & 3 \\ 1 & 2 & 4 & -1 \\ 3 & -1 & 2 & 4 \\ 1 & 5 & 6 & -2 \end{vmatrix} \overset{\text{Using the first row}}{=} -2\begin{vmatrix} 2 & 4 & -1 \\ -1 & 2 & 4 \\ 5 & 6 & -2 \end{vmatrix} - \begin{vmatrix} 1 & 4 & -1 \\ 3 & 2 & 4 \\ 1 & 6 & -2 \end{vmatrix}$$

$$+\begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 4 \\ 1 & 5 & -2 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 & 4 \\ 3 & -1 & 2 \\ 1 & 5 & 6 \end{vmatrix}$$

Each of the four 3 × 3 determinants is computed via Equation 3.1, resulting in

$$\left|\mathbf{A}\right| = -2(32) - (-4) + (-14) - 3(16) = -122$$

b.

```
>> A = [-2 1 1 3;1 2 4 -1;3 -1 2 4;1 5 6 -2];
>> det(A)
ans =
   -122
```

3.1.4.1 Properties of Determinant

The following properties are associated with the determinant:

- If an entire row (or column) of **A** is zero, then $\left|\mathbf{A}\right| = 0$, that is, **A** is singular.
- If any rows (or columns) of **A** are linearly dependent, then **A** is singular.
- If **A** is $n \times n$ and rank(**A**) < n, then **A** is singular.
- $\left|\mathbf{A}^T\right| = \left|\mathbf{A}\right|$
- $\left|\mathbf{AB}\right| = \left|\mathbf{A}\right|\left|\mathbf{B}\right|$
- The determinant of a lower triangular, upper triangular, or diagonal matrix is the product of the diagonal entries.
- If k is a scalar and **A** is $n \times n$, then $\left|k\mathbf{A}\right| = k^n\left|\mathbf{A}\right|$.
- If any two rows of **A** are interchanged (ERO_2), then the determinant of the resulting matrix is $-\left|\mathbf{A}\right|$.
- $\left|\mathbf{A}\right|$ is preserved under ERO_3.

3.1.4.2 Rank in Terms of Determinant

The rank of any matrix **A** is the size of the largest nonsingular submatrix of **A**. In other words, rank(**A**) = r if there exists an $r \times r$ submatrix of **A** with nonzero determinant and any other $p \times p$ (with $p > r$) submatrix of **A** is singular. If $\mathbf{A}_{n\times n}$ is nonsingular, then rank(**A**) = n. If $\left|\mathbf{A}\right| = 0$, then rank(**A**) $< n$.

Example 3.5: Rank via Determinant

a. Find the rank of

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & 5 \end{bmatrix}$$

b. Confirm in MATLAB.

Solution

a. Because $\mathbf{A}$ is 4×3, rank($\mathbf{A}$) can at most be 3. Inspection of all possible 3×3 submatrices of $\mathbf{A}$ yields

$$\begin{vmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \\ 1 & 0 & 5 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 3 \\ 1 & 0 & 5 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & -1 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & 5 \end{vmatrix} = 0$$

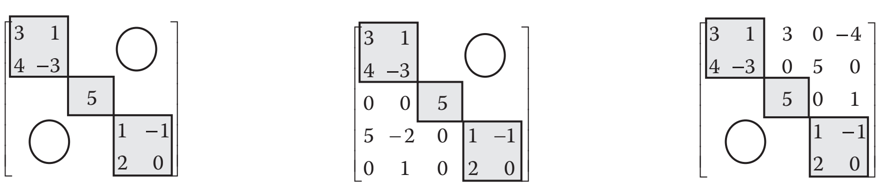
This implies rank($\mathbf{A}$) < 3, hence it may be either 2 or 1. Because $\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = -2 \neq 0$, we conclude that rank($\mathbf{A}$) = 2.

b.

```
>> A = [2 3 4;0 -1 2;1 1 3;1 0 5];

>> rank(A)

ans =

     2
```

3.1.4.3 Block Diagonal and Block Triangular Matrices

A block diagonal matrix is defined as a square matrix partitioned such that its diagonal elements are square matrices, whereas all other elements are zeros (Figure 3.4a). Similarly, we define a block triangular matrix as a square matrix partitioned in such a way that its diagonal elements are square blocks, whereas all entries either above or below this main block diagonal are zeros (Figure 3.4b and c). It is interesting to note that many properties of these special block matrices essentially generalize those of diagonal and triangular matrices. In particular, the determinant of each of these matrices is equal to the product of the individual determinants of the blocks along the main diagonal. Consequently, a block diagonal (or triangular) matrix is singular if and only if one of the blocks



(a) (b) (c)

FIGURE 3.4 (a) Block diagonal, (b) block lower-triangular, (c) block upper-triangular.

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & \boxed{5} & 0 \\ 0 & 0 & 0 & \boxed{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(a) (b)

FIGURE 3.5 Block diagonal in Example 3.6: (a) three blocks, (b) two blocks.

along the main diagonal is singular. We also mention that the rank of a block triangular matrix is at least equal to the sum of the ranks of the individual diagonal blocks.

Example 3.6: Block Diagonal Matrix

Evaluate det(**A**) and rank(**A**), where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Solution

This matrix can be thought of as a block diagonal matrix in two ways, as listed in Figure 3.5. In Figure 3.5a, the 2×2 block in the upper-left corner has a determinant of 10 and a rank of 2, whereas the other two blocks (single entries) are 5 and 2, each with a rank of 1. Therefore, $|\mathbf{A}| = (10)(5)(2) = 100$ and rank(**A**) = 2 + 1 + 1 = 4. In Figure 3.5b, the two 2×2 blocks each have a determinant of 10 and a rank of 2, thus the determinant of the matrix is (10)(10) = 100 and its rank is 4, as asserted.

3.1.5 Inverse of a Matrix

The inverse of a square matrix $\mathbf{A}_{n \times n}$ is denoted by $\mathbf{A}^{-1}$ with the property $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$ where $\mathbf{I}$ is the $n \times n$ identity matrix. The inverse of **A** exists only if $|\mathbf{A}| \neq 0$ and is obtained using the adjoint matrix of **A**, denoted by adj(**A**).

3.1.5.1 Adjoint Matrix

If $\mathbf{A} = [a_{ij}]_{n \times n}$, then the adjoint of **A** is defined as

$$\mathrm{adj}(\mathbf{A}) = \begin{bmatrix} (-1)^{1+1} M_{11} & (-1)^{2+1} M_{21} & \dots & (-1)^{n+1} M_{n1} \\ (-1)^{1+2} M_{12} & (-1)^{2+2} M_{22} & \dots & (-1)^{n+2} M_{n2} \\ \dots & \dots & \dots & \dots \\ (-1)^{1+n} M_{1n} & (-1)^{2+n} M_{2n} & \dots & (-1)^{n+n} M_{nn} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix} \tag{3.2}$$

where M_{ij} is the minor of a_{ij} and $C_{ij} = (-1)^{i+j} M_{ij}$ is the cofactor of a_{ij}. Note that each minor M_{ij} (or cofactor C_{ij}) occupies the (j,i) position in the adjoint matrix, the opposite of what one would normally expect. Then,

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\,\mathrm{adj}(\mathbf{A}) \tag{3.3}$$

Equation 3.3 clearly indicates that $\mathbf{A}$ must be nonsingular for $\mathbf{A}^{-1}$ to exist.

Example 3.7: Inverse

a. Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

b. Confirm in MATLAB.

Solution

a. Noting that $|\mathbf{A}| = -8 \neq 0$, the inverse exists. The first few minors and cofactors are computed as

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} = -3, \quad C_{11} = (-1)^{1+1} M_{11} = -3$$

$$M_{12} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1, \quad C_{12} = (-1)^{1+2} M_{12} = 1$$

$$M_{21} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{21} = (-1)^{2+1} M_{21} = -1$$

Continuing this process, using Equation 3.3, we find

$$\mathrm{adj}(\mathbf{A}) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} -3 & -1 & 2 \\ 1 & 3 & -6 \\ 2 & -2 & -4 \end{bmatrix}$$

Finally,

$$\mathbf{A}^{-1} = \frac{1}{-8}\begin{bmatrix} -3 & -1 & 2 \\ 1 & 3 & -6 \\ 2 & -2 & -4 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} & -\frac{1}{4} \\ -\frac{1}{8} & -\frac{3}{8} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

b.

```
>> A = [3 1 0;1 -1 2;1 1 1];
>> inv(A)
ans =
    0.3750    0.1250   -0.2500
   -0.1250   -0.3750    0.7500
   -0.2500   -0.2500    0.5000
```

Example 3.8: Inverse of a Symbolic Matrix

a. Determine $(s\mathbf{I} - \mathbf{A})^{-1}$ where s is the Laplace variable, $\mathbf{I}$ is the 2×2 identity matrix, and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$$

b. Repeat in MATLAB.

Solution

a. First,

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 4 & s+5 \end{bmatrix} \qquad \left| s\mathbf{I} - \mathbf{A} \right| = s^2 + 5s + 4 = (s+1)(s+4)$$

Then, by Equation 3.3,

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\mathrm{adj}(s\mathbf{I} - \mathbf{A})}{\left| s\mathbf{I} - \mathbf{A} \right|} = \frac{1}{(s+1)(s+4)}\begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix} = \begin{bmatrix} \frac{s+5}{(s+1)(s+4)} & \frac{1}{(s+1)(s+4)} \\ \frac{-4}{(s+1)(s+4)} & \frac{s}{(s+1)(s+4)} \end{bmatrix}$$

b.

```
>> syms s
>> A = [0 1;-4 -5]; inv(s*eye(2) - A)
```

```
ans =

 [(s+5)/(s^2+5*s+4),   1/(s^2+5*s+4)]

[ -4/(s^2+5*s+4),    s/(s^2+5*s+4)]
```

The following properties are associated with the inverse of a matrix:

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $(\mathbf{A}^{-1})^{p} = (\mathbf{A}^{p})^{-1}$, p = integer > 0
- $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$
- $(\mathbf{A}^{-1})^{T} = (\mathbf{A}^{T})^{-1}$
- Inverse of a (nonsingular) symmetric matrix is symmetric.
- Inverse of a diagonal matrix is diagonal whose entries are the reciprocals of the entries of the original matrix.

PROBLEM SET 3.1

In Problems 1 through 8, perform the indicated operations, if defined, for the vectors and matrices below.

$$\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 5 & -4 \end{bmatrix}, \quad \mathbf{v} = \begin{Bmatrix} -1 \\ 2 \end{Bmatrix}, \quad \mathbf{w} = \begin{Bmatrix} 0 \\ 1 \\ -3 \end{Bmatrix}$$

1. $\mathbf{B} + \mathbf{v}\mathbf{w}^{T}$
2. $\mathbf{v}^{T}\mathbf{B} + \mathbf{w}$
3. $\mathbf{w}^{T}\mathbf{w} + \mathbf{v}^{T}\mathbf{v}$
4. $(\mathbf{v}\mathbf{w}^{T})^{T} + \mathbf{B}^{T}$
5. $\mathbf{A}\mathbf{v}\mathbf{w}^{T} - \mathbf{B}^{T}\mathbf{A}$
6. $\mathbf{B}\mathbf{B}^{T} + \mathbf{A}\mathbf{v}\mathbf{v}^{T}$
7. $\mathbf{A}\mathbf{v} - 2\mathbf{B}\mathbf{w}$
8. $\mathbf{w}^{T}\mathbf{B}^{T}\mathbf{v}$

In Problems 9 through 14,

a. Find the row–echelon form and use it to determine the rank.
b. Confirm using MATLAB.

9. $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$

10. $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & -2 \\ 1 & 4 & -5 \end{bmatrix}$

11. $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 4 & -2 & 0 \end{bmatrix}$

12. $\mathbf{A} = \begin{bmatrix} -3 & -1 & -2 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$

13. $\mathbf{A} = \begin{bmatrix} 3 & -2 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 0 & 2 & 4 & 1 \\ 1 & 1 & 3 & 1 \end{bmatrix}$

14. $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & -2 & 1 \\ 4 & 1 & 2 & 1 \end{bmatrix}$

In Problems 15 through 20 evaluate the determinant.

15. $\begin{vmatrix} -1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$

16. $\begin{vmatrix} a & -1 & a & 2 \\ 0 & b & 0 & 1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 2a \end{vmatrix}, \quad a, b = \text{parameters}$

17. $\begin{vmatrix} s+1 & 1 & -1 \\ 0 & s+2 & 2 \\ -1 & 2 & s \end{vmatrix}, \quad s = \text{parameter}$

18. $\begin{vmatrix} 1 & 2 & 3 & 5 \\ -2 & 2 & 1 & 4 \\ 3 & 1 & 0 & -1 \\ 2 & 1 & -3 & 1 \end{vmatrix}$

19. $\begin{vmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix}$

20. $\begin{vmatrix} 1 & 2 & 0 & -6 & 0 \\ -2 & 3 & 0 & 0 & 10 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 & 5 \end{vmatrix}$

In Problems 21 through 28, find the inverse of the matrix.

21. $\mathbf{A} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad a \neq b$

22. $\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

23. $\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & -3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

24. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

25. $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$

26. $\mathbf{A} = \begin{bmatrix} a & 0 & -1 \\ 0 & a+1 & 2 \\ 1 & 0 & a+2 \end{bmatrix}, \quad a = \text{parameter}$

27. $\mathbf{A} = \begin{bmatrix} 3a & 0 & 0 \\ 0 & a+1 & 0 \\ 0 & 0 & 3(a+1) \end{bmatrix}, \quad a = \text{parameter}$

28. $\mathbf{A} = \begin{bmatrix} -L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) & -L_2\sin(\theta_1+\theta_2) \\ L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) & L_2\cos(\theta_1+\theta_2) \end{bmatrix}, \quad L_1, L_2, \theta_1, \theta_2 = \text{parameters}$

29. Given $\mathbf{A}_{n\times n}$ and scalar α, show that $(\alpha\mathbf{A})^{-1} = \frac{1}{\alpha}\mathbf{A}^{-1}$.
30. Show that the inverse of a (nonsingular) symmetric matrix is symmetric.

3.2 SOLUTION OF LINEAR SYSTEMS OF EQUATIONS

A linear system of n algebraic equations in n unknowns appears in the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ \dots \ + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ \dots \ + a_{2n}x_n = b_2 \\ \qquad\qquad \dots \\ a_{n1}x_1 + a_{n2}x_2 + \ \dots \ + a_{nn}x_n = b_n \end{cases} \tag{3.4}$$

where a_{ij} $(i, j = 1,2, \dots, n)$ and b_k $(k = 1,2, \dots, n)$ are known constants, and a_{ij}'s are the coefficients. Equation 3.4 can be expressed in matrix form, as

$$\mathbf{Ax} = \mathbf{b} \tag{3.5}$$

with

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n\times n}, \quad \mathbf{x}=\begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{Bmatrix}_{n\times 1}, \quad \mathbf{b}=\begin{Bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{Bmatrix}_{n\times 1}$$

where **A** is the coefficient matrix. A set of values for $x_1, x_2, \dots, x_n$ satisfying Equation 3.4 forms a solution of the system. The vector **x** with components $x_1, x_2, \dots, x_n$ is the solution of Equation 3.5. If $x_1 = 0 = x_2 = \dots = x_n$, the solution $\mathbf{x} = \mathbf{0}_{n\times 1}$ is called the trivial solution. The augmented matrix for Equation 3.5 is defined as

$$[\mathbf{A}|\,\mathbf{b}]=\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right]_{n\times(n+1)} \tag{3.6}$$

3.2.1 Gauss Elimination Method

Gauss elimination is a basic method to solve a linear system in the form of Equation 3.5, when n is not large, and comprises two main steps: (1) Use the EROs to transform the augmented matrix $[\mathbf{A}|\mathbf{b}]$ into upper triangular form, and (2) find the unknowns through back-substitution.

Example 3.9: Gauss Elimination Method

Using Gauss elimination, solve

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & 3 \\ -1 & 1 & 2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} -4 \\ 1 \\ 5 \end{Bmatrix}$$

Solution

Figure 3.6 shows the steps involved in transforming the augmented matrix into the upper-triangular form. The third row yields $13x_3 = 13$, which implies $x_3 = 1$. Using this in the second row, we have $-x_2 + 3 = 1$, hence $x_2 = 2$. Finally, using the first row, we find $x_1 = -1$.

$$1 \quad -2 \quad + \quad + \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 2 & 0 & 3 & 1 \\ -1 & 1 & 2 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & 4 & 1 & 9 \\ 0 & -1 & 3 & 1 \end{array}\right] \longrightarrow 4 \quad + \left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & -1 & 3 & 1 \\ 0 & 4 & 1 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 13 & 13 \end{array}\right]$$

FIGURE 3.6 Gauss elimination method.

3.2.2 Using the Inverse of the Coefficient Matrix

If the coefficient matrix **A** in Equation 3.5 is nonsingular, then the vector solution is readily obtained as $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Example 3.10: Coefficient Matrix

Solve the system in Example 3.9 using

a. The inverse of the coefficient matrix.
b. The "\" operator in MATLAB.

Solution

a. The inverse is calculated as

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & 3 \\ -1 & 1 & 2 \end{bmatrix}^{-1} = \frac{1}{13}\begin{bmatrix} -3 & 5 & -6 \\ -7 & 3 & -1 \\ 2 & 1 & 4 \end{bmatrix}$$

Therefore,

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{1}{13}\begin{bmatrix} -3 & 5 & -6 \\ -7 & 3 & -1 \\ 2 & 1 & 4 \end{bmatrix}\begin{Bmatrix} -4 \\ 1 \\ 5 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 2 \\ 1 \end{Bmatrix}$$

b.

```
>> A = [1 -2 1;2 0 3;-1 1 2]; b = [-4;1;5];
>> x = A\b    % More efficient than inv(A)*b
x =
    -1
     2
     1
```

3.2.3 Cramer's Rule

Consider the linear system in Equation 3.5. Assuming the coefficient matrix is nonsingular, each unknown x_i (i = 1,2, … , n) is uniquely determined via

$$x_i = \frac{D_i}{D}$$

where determinants D and D_i are described as

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}, \quad D_i = \begin{array}{c} \qquad\qquad i\text{th column of } D \\ \begin{vmatrix} a_{11} & \dots & b_1 & \dots & a_{1n} \\ a_{21} & \dots & b_2 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & b_n & \dots & a_{nn} \end{vmatrix} \end{array}$$

Example 3.11: Cramer's Rule

Consider the system in Example 3.9.

a. Solve using Cramer's rule.
b. Repeat in MATLAB.

Solution

a. The determinant of the coefficient matrix is computed first,

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 0 & 3 \\ -1 & 1 & 2 \end{vmatrix} = 13 \neq 0$$

Subsequently, the three determinants are calculated as

$$D_1 = \begin{vmatrix} -4 & -2 & 1 \\ 1 & 0 & 3 \\ 5 & 1 & 2 \end{vmatrix} = -13, \quad D_2 = \begin{vmatrix} 1 & -4 & 1 \\ 2 & 1 & 3 \\ -1 & 5 & 2 \end{vmatrix} = 26, \quad D_3 = \begin{vmatrix} 1 & -2 & -4 \\ 2 & 0 & 1 \\ -1 & 1 & 5 \end{vmatrix} = 13$$

The solutions are therefore obtained as

$$x_1 = \frac{D_1}{D} = -1, \quad x_2 = \frac{D_2}{D} = 2, \quad x_3 = \frac{D_3}{D} = 1$$

b.

```
>> A = [1 -2 1;2 0 3;-1 1 2]; b = [-4;1;5]; d = det(A);
>> A1=A; A1(:,1)=b; d1=det(A1);
>> A2=A; A2(:,2)=b; d2=det(A2);
>> A3=A; A3(:,3)=b; d3=det(A3);
>> x1=d1/d; x2=d2/d; x3=d3/d; x=[x1;x2;x3]

x =

    -1
     2
     1
```

3.2.4 Homogeneous Systems

Consider a homogeneous system of n equations $\mathbf{Ax} = \mathbf{0}$ and let $\mathbf{A}$ be nonsingular. Because $\mathbf{b} = \mathbf{0}_{n\times1}$, every determinant D_i defined in Cramer's rule contains a column of all zero entries, and hence is zero. Consequently, $x_i = D_i/D = 0$ for $i = 1,2, \ldots, n$ and the only possible solution is the trivial solution $\mathbf{x} = \mathbf{0}_{n\times1}$.

For $\mathbf{Ax} = \mathbf{0}$ to have a nontrivial solution, the coefficient matrix must be singular, $|\mathbf{A}| = 0$. This assures that the equations are linearly dependent so that there is at least one free variable that can ultimately generate infinitely many solutions.

PROBLEM SET 3.2

In Problems 1 through 6, solve the linear system $\mathbf{Ax} = \mathbf{b}$ using Gauss elimination.

1. $\mathbf{A} = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 5 & -1 \\ -1 & -6 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} -8 \\ 5 \\ -2 \end{Bmatrix}$

2. $\mathbf{A} = \begin{bmatrix} 1 & 5 & 2 \\ 3 & -1 & 3 \\ -4 & 2 & -5 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} -1 & 2 & -1 \\ 3 & 0 & 2 \\ 2 & 1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 1 \\ 4 \end{Bmatrix}$

4. $\mathbf{A} = \begin{bmatrix} 2 & 3 & 2 \\ -1 & 4 & 3 \\ 3 & 0 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 4 \\ -1 \\ 0 \end{Bmatrix}$

5. $\mathbf{A} = \begin{bmatrix} -2 & 4 & 3 & 1 \\ 3 & 2 & -6 & -1 \\ 1 & -2 & -3 & 2 \\ 4 & 8 & 9 & -3 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 7 \\ -6 \\ 1 \\ -3 \end{Bmatrix}$

6. $\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 3 & 1 & -4 \\ 2 & 0 & 3 & 5 \\ -1 & -2 & 4 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 10 \\ -14 \\ 9 \\ 11 \end{Bmatrix}$

In Problems 7 through 10, solve the linear system $\mathbf{Ax} = \mathbf{b}$ using

a. $\mathbf{A}^{-1}$.
b. the "\" command in MATLAB.

7. $\mathbf{A} = \begin{bmatrix} -1 & 2 & -1 \\ 3 & 0 & 2 \\ 2 & 1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ b \\ 4b \end{Bmatrix}, \quad b = \text{parameter}$

8. $\mathbf{A} = \begin{bmatrix} a & 1 & -2 \\ -1 & 2a & 1 \\ 0 & 1 & 3a \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} a \\ 4a \\ 3a+2 \end{Bmatrix}, \quad a = \text{parameter}$

9. $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 2 \\ -2 \\ 8 \end{Bmatrix}$

10. $\mathbf{A} = \begin{bmatrix} 2 & 4 & -1 \\ 1 & 3 & 1 \\ -1 & 2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 2 \\ 1 \end{Bmatrix}$

In Problems 11 through 14,

a. Solve the linear system $\mathbf{Ax} = \mathbf{b}$ using Cramer's rule.
b. Repeat in MATLAB.

11. $\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 1 \\ 8 \\ -13 \end{Bmatrix}$

12. $\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & -2 \\ 3 & -1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 3 \\ 3 \\ 1 \end{Bmatrix}$

13. $\mathbf{A} = \begin{bmatrix} s+1 & -1 \\ -2 & s+2 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 1/s \end{Bmatrix}, \quad s = \text{parameter}$

14. $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 1 & 0 \\ 1 & 4 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 3 \\ 5 \\ 4 \end{Bmatrix}$

In Problems 15 through 18, solve the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ for which the coefficient matrix is provided. The components of $\mathbf{x}$ are $x_1, x_2, \ldots, x_n$ where n is the system size.

15. $\mathbf{A} = \begin{bmatrix} -1 & 3 & 4 \\ 2 & -2 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

16. $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 3 \\ 2 & -2 & 3 \end{bmatrix}$

17. $\mathbf{A} = \begin{bmatrix} 5 & -3 & 2 \\ 0 & 2 & 4 \\ -3 & 2 & 6 \end{bmatrix}$

18. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

3.3 MATRIX EIGENVALUE PROBLEM

The matrix eigenvalue problem associated with a square matrix $\mathbf{A}_{n\times n}$ is formulated as

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}, \quad \mathbf{v} \neq \mathbf{0}_{n\times 1}, \quad \lambda = \text{scalar} \tag{3.7}$$

A number λ, complex in general, for which Equation 3.7 has a nontrivial solution ($\mathbf{v} \neq \mathbf{0}_{n\times 1}$) is called an eigenvalue or characteristic value of matrix $\mathbf{A}$. The corresponding solution ($\mathbf{v} \neq \mathbf{0}$) of Equation 3.7 is the eigenvector or characteristic vector of $\mathbf{A}$ corresponding to λ. The set of all eigenvalues of $\mathbf{A}$ is the spectrum of $\mathbf{A}$, denoted by $\lambda(\mathbf{A})$. The largest eigenvalue of $\mathbf{A}$, in absolute value, is the spectral radius of $\mathbf{A}$.

3.3.1 Solving the Eigenvalue Problem

Rewrite Equation 3.7 as

$$\mathbf{A}\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}_{n\times 1}$$

Note that every single term here is an $n \times 1$ vector. On the left-hand side, both terms contain vector $\mathbf{v}$. Although in the second term, λ and $\mathbf{v}$ commute, the same is not true with $\mathbf{A}$ and $\mathbf{v}$ in the first term. Therefore, we can only factor out $\mathbf{v}$ from the right to obtain

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \tag{3.8}$$

The identity matrix $\mathbf{I}$ has been inserted so that the two terms in parentheses are compatible. Equation 3.8 has a nontrivial solution ($\mathbf{v} \neq \mathbf{0}$) if and only if the coefficient matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular (see Section 3.2). Thus,

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$$

This is called the characteristic equation of $\mathbf{A}$. The determinant $\left|\mathbf{A} - \lambda\mathbf{I}\right|$ is an nth-degree polynomial in λ and is known as the characteristic polynomial of $\mathbf{A}$ whose roots are precisely the eigenvalues of $\mathbf{A}$. For each eigenvalue, we find the corresponding eigenvector by solving Equation 3.8. Because $\mathbf{A} - \lambda\mathbf{I}$ is singular, it has at least one row dependent on the other rows, which means the row–echelon form of $\mathbf{A} - \lambda\mathbf{I}$ will have at least one zero row. Therefore, for each fixed λ, Equation 3.8 has infinitely many solutions. A basis of solutions then represents all eigenvectors associated with each λ.

Eigenvalue Properties of Matrices

- Eigenvalues of upper-triangular and lower-triangular and diagonal matrices are the entries along the main diagonal of the matrix.
- The determinant of a matrix is the product of its eigenvalues.
- All eigenvalues of a symmetric matrix are real.
- Eigenvalues of a skew-symmetric matrix are either zero or pure imaginary.
- A matrix $\mathbf{A}$ is orthogonal if $\mathbf{A}^T = \mathbf{A}^{-1}$. All eigenvalues of an orthogonal matrix have absolute values of 1.
- The eigenvalues of block diagonal and block triangular matrices are the eigenvalues of the block matrices along the diagonal.

Example 3.12: Matrix Eigenvalue Problem

a. Find all eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

b. Repeat in MATLAB.

Solution

a. The characteristic equation yields

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - (1-\lambda) = \lambda(\lambda-1)(\lambda-2) = 0 \qquad \lambda_{1,2,3} = 0, 1, 2$$

For $\lambda_1 = 0$, we solve

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = \mathbf{0} \qquad \mathbf{A}\mathbf{v}_1 = \mathbf{0} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\xrightarrow[\text{row operations}]{\text{Elementary}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The second row gives $b = 0$. The first row yields $a + c = 0$. The last row of zeros suggests a free variable exists, which can be either a or c. Choosing $a = 1$ results in $c = -1$ and

$$\mathbf{v}_1 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

For $\lambda_2 = 1$, we solve

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{v}_2 = \mathbf{0} \qquad (\mathbf{A} - \mathbf{I})\mathbf{v}_2 = \mathbf{0} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The first and third rows suggest $c = 0$ and $a = 0$, respectively. Therefore, b must be the free variable. Noting that an eigenvector cannot be the zero vector, we let $b = 1$ so that

$$\mathbf{v}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

For $\lambda_3 = 2$, solve

$$(\mathbf{A} - 2\mathbf{I})\mathbf{v}_3 = \mathbf{0} \qquad \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \overset{\text{EROs}}{} \quad \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The second row gives $b = 0$ whereas the first row yields $a = c$. The last row suggests one free variable, which can be either a or c. Choosing $a = 1$ results in $c = 1$, and

$$\mathbf{v}_3 = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

b.

```
>> A = [1 0 1;0 1 0;1 0 1];
```

To find the eigenvalues as well as the eigenvectors, we use the command `"eig"` in the form of `[V,D] = eig(A)` so that matrix `V` contains the eigenvectors (normalized as unit vectors) in its columns and diagonal matrix `D` has the eigenvalues along its main diagonal. The eigenvalue in the (1,1) entry of `D` corresponds to the eigenvector in the first column of `V`, and so on.

```
>> [V,D] = eig(A)
V =
    0.7071         0    0.7071
         0   -1.0000         0
   -0.7071         0    0.7071
D =
     0     0     0
     0     1     0
     0     0     2
```

As mentioned above, the eigenvectors are normalized to unit vectors by dividing the vector by its length (norm). For instance, eigenvector $\mathbf{v}_1$ in Part (a) has length $\sqrt{2}$ and

$$\frac{1}{\sqrt{2}}\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{Bmatrix}$$

This is returned by the `"eig"` command in MATLAB. Also note that $\mathbf{v}_2$ returned by `"eig"` matches the one in (a)—except for a negative multiple—because it is a unit vector to begin with. This, of course, is not a concern because they both have the same basis.

3.3.2 Algebraic Multiplicity and Geometric Multiplicity

The algebraic multiplicity (AM) of an eigenvalue is the number of times it occurs. Its geometric multiplicity (GM) is the number of linearly independent eigenvectors associated with it. For instance, in Example 3.12, each of the three eigenvalues has an AM of 1 because each occurs only once, and a GM of 1 because there is only one independent eigenvector for each. In general, we have GM ≤ AM. Therefore, any eigenvalue with an AM of 1 automatically has a GM of 1.

Example 3.13: AM, GM

Find all eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Solution

Because **A** is upper-triangular, $\lambda(\mathbf{A}) = 1,1,-2$ so that $\lambda = 1$ has AM of 2 and $\lambda = -2$ has AM of 1. For $\lambda = 1$, we solve

$$(\mathbf{A}-\mathbf{I})\mathbf{v} = \mathbf{0} \qquad \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \xrightarrow[\text{row operations}]{\text{Elementary}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The first row gives $c = 0$. Two zero rows indicate two free variables (a and b here) so that two linearly independent eigenvectors can be obtained. Letting $a = 1$, $b = 0$, and $a = 0$, $b = 1$ yields two independent eigenvectors associated with $\lambda = 1$:

$$\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Therefore, $\lambda = 1$ has a GM of 2. For $\lambda = -2$ the only independent eigenvector is obtained as

$$\begin{Bmatrix} 3 \\ -2 \\ 3 \end{Bmatrix}$$

Thus, $\lambda = -2$ has GM of 1.

3.3.2.1 Generalized Eigenvectors

Suppose AM of a certain eigenvalue λ is m so that m corresponding eigenvectors are expected for λ. However, if the GM of λ is $k<m$, then only k linearly independent eigenvectors will be generated for λ. This implies that there are $m-k$ missing eigenvectors. These missing eigenvectors are called generalized eigenvectors and can be found using a systematic approach (see Reference [6]). Any matrix with a generalized eigenvector is called defective. The following example illustrates how defective matrices can be handled in MATLAB.

Example 3.14: Generalized Eigenvectors

Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & -3 \\ 0 & 4 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

Solution

```
>> A = [1 -1 -3;0 4 0;-3 1 1];
>> [V,D] = eig(A)
V =
    0.7071    0.7071    0.7071
         0         0    0.0000
   -0.7071    0.7071   -0.7071
D =
    4.0000         0         0
         0   -2.0000         0
         0         0    4.0000
```

Because the first and third columns of `V` are the same eigenvector, we conclude that there is a generalized eigenvector. To find this missing eigenvector, we switch from `"eig"` to `"jordan,"` which is specifically designed for this purpose.

```
>> [V,J] = jordan(A)
V =
    0.1667   -1.0000    0.1667
         0         0    1.0000
    0.1667    1.0000   -0.1667
J =
    -2     0     0
     0     4     1
     0     0     4
```

Note that matrix `J` is no longer diagonal and is known as a Jordan matrix. The eigenvalues of **A** are –2,4,4. The first column of `V` contains the eigenvector for $\lambda = -2$. The next two columns correspond to $\lambda = 4$, with the last one being a generalized eigenvector.

3.3.2.2 Similarity Transformations

Two matrices $\mathbf{A}_{n\times n}$ and $\mathbf{B}_{n\times n}$ are similar if there exists a nonsingular matrix $\mathbf{S}_{n\times n}$ such that

$$\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} \tag{3.9}$$

We say that **B** is obtained from **A** through a similarity transformation. Eigenvalues of a matrix are preserved under similarity transformation. Similarity transformations are customarily used to transform a matrix into a diagonal matrix, with eigenvectors playing a key role in that process.

3.3.2.3 Matrix Diagonalization

Suppose $\mathbf{A}_{n\times n}$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_n$ with no generalized eigenvectors. Form the modal matrix $\mathbf{V}_{n\times n} = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \ldots \ \ \mathbf{v}_n]$, which is guaranteed to be nonsingular because its columns are linearly independent (see properties of the determinant). Then,

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots \\ \ldots & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix} = \mathbf{D} \tag{3.10}$$

Matrix **A** has clearly been transformed into a diagonal matrix **D** by a similarity transformation.

3.3.2.4 Defective Matrices

Suppose $\mathbf{A}_{n\times n}$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ including at least one generalized eigenvector. Again, the modal matrix $\mathbf{V}_{n\times n} = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \ldots \ \ \mathbf{v}_n]$ is guaranteed to be nonsingular because its columns are linearly independent, and

$$\mathbf{V}^{-1}\,\mathbf{A}\mathbf{V} = \mathbf{J} \tag{3.11}$$

where **J** is not diagonal and is called a Jordan matrix (see Example 3.14).

Example 3.15: Diagonalization

In Example 3.12, we found $\lambda(\mathbf{A}) = 0,1,2$ and no generalized eigenvectors. The modal matrix then transforms **A** into the diagonal matrix **D** as in Equation 3.10.

```
>> D = V\A*V    % Perform D = V^-1AV

D =

     0    0   0

     0    1   0

     0    0   2    % Diagonal matrix D comprises the eigenvalues of A
```

Example 3.16: Jordan Matrix

In Example 3.14, we found $\lambda(\mathbf{A}) = -2,4,4$ and one generalized eigenvector. The modal matrix transforms **A** into a Jordan matrix **J** as in Equation 3.11.

```
>> J = V\A*V    % Perform J = V^-1AV

J =

     -2    0    0

     0     4    1

     0     0    4    % Jordan matrix
```

PROBLEM SET 3.3

In Problems 1 through 10,

a. Find the eigenvalues and eigenvectors of the matrix.
b. Repeat in MATLAB.

1. $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 5 & 3 \end{bmatrix}$

2. $\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$

4. $\mathbf{A} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$, $\quad a =$ parameter

5. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. $\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & -1 \end{bmatrix}$

7. $\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ -4 & 4 & -6 \\ -2 & 1 & -1 \end{bmatrix}$

8. $\mathbf{A} = \begin{bmatrix} 2 & -2 & 4 \\ -1 & 3 & 2 \\ -5 & 10 & -1 \end{bmatrix}$

9. $\mathbf{A} = \begin{bmatrix} 2 & 2 & -2 & 0 \\ -1 & -3 & 2 & -2 \\ 0 & -3 & 2 & -3 \\ 1 & 2 & -2 & 1 \end{bmatrix}$

10. $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

In Problems 11 through 18, find the eigenvalues, eigenvectors, AM, and GM of each eigenvalue, and decide whether the matrix is defective or not. Then transform the matrix into either a diagonal or a Jordan matrix, whichever is applicable.

11. $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 6 & 6 \end{bmatrix}$

12. $\mathbf{A} = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}$

13. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 5 & 0 & -2 \end{bmatrix}$

14. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

15. $\mathbf{A} = \begin{bmatrix} -2 & -1 & -1 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

16. $\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

17. $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

18. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

19. Prove that if $\mathbf{A}_{2\times 2}$ has a repeated eigenvalue λ, then $\mathbf{A}$ must be defective.
20. Prove that a singular matrix must have at least one zero eigenvalue.

3.4 SUMMARY

A matrix is a collection of elements arranged in a rectangular array and enclosed by square brackets. Matrix $\mathbf{A}_{m\times n}$ has m rows and n columns, and is said to be of size $m \times n$. The abbreviated form of an $m \times n$ matrix is

$$\mathbf{A} = [a_{ij}]_{m\times n}, \quad i = 1,2, \ldots , m, \quad j = 1,2, \ldots , n$$

Matrix addition is performed entry-wise. If k is scalar, then $k\mathbf{A} = [ka_{ij}]_{m\times n}$. If $\mathbf{A} = [a_{ij}]_{m\times n}$ and $\mathbf{B} = [b_{ij}]_{n\times p}$, then $\mathbf{C} = \mathbf{AB}$ is $m \times p$ whose entries are obtained as

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad i = 1,2, \ldots , m, \quad j = 1,2, \ldots , p$$

The transpose of $\mathbf{A}_{m\times n}$, denoted by $\mathbf{A}^T$, is an $n \times m$ matrix whose rows are the columns of $\mathbf{A}$.

$\mathbf{A}_{n\times n} = [a_{ij}]$ is symmetric if $\mathbf{A}^T = \mathbf{A}$, upper-triangular if $a_{ij} = 0$ for all $i > j$, lower-triangular if $a_{ij} = 0$ for all $i < j$, and diagonal if $a_{ij} = 0$ for all $i \neq j$. Matrix transformations may be achieved by using EROs:

- ERO_1: Multiply a row by a nonzero constant.
- ERO_2: Interchange two rows.
- ERO_3: Multiply row i by $\alpha = \text{const} \neq 0$, add the result to row k, then replace row k with the outcome. In this process, row i is called the pivot row.

The rank of $\mathbf{A}$ is the number of nonzero rows in the row–echelon form of $\mathbf{A}$. The determinant of $\mathbf{A}_{n\times n}$ is a real scalar, calculated as

$$|\mathbf{A}| = \sum_{k=1}^{n} a_{ik}(-1)^{i+k} M_{ik}, \quad i = 1, 2, \ldots, n$$

where M_{ik} is the minor of a_{ik} and $(-1)^{i+k}M_{ik}$ is the cofactor of a_{ik}. The adjoint matrix of $\mathbf{A}$ is defined as

$$\text{adj}(\mathbf{A}) = \begin{bmatrix} (-1)^{1+1}M_{11} & (-1)^{2+1}M_{21} & \ldots & (-1)^{n+1}M_{n1} \\ (-1)^{1+2}M_{12} & (-1)^{2+2}M_{22} & \ldots & (-1)^{n+2}M_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ (-1)^{1+n}M_{1n} & (-1)^{2+n}M_{2n} & \ldots & (-1)^{n+n}M_{nn} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{bmatrix}$$

Then, the inverse of matrix $\mathbf{A}$ is obtained as

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\text{adj}(\mathbf{A})$$

The eigenvalue problem for matrix $\mathbf{A}$ is formulated as

$$\mathbf{A}_{n\times n}\,\mathbf{v}_{n\times 1} = \lambda_{1\times 1}\mathbf{v}_{n\times 1}, \quad \mathbf{v}_{n\times 1} \neq \mathbf{0}_{n\times 1}$$

where λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ is the corresponding eigenvector. The eigenvalues are the roots of the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$. The AM of an eigenvalue is the number of times it occurs. Its GM is the number of linearly independent eigenvectors associated with it. In general, we have GM ≤ AM. If GM < AM for an eigenvalue, then the matrix has at least one generalized eigenvector. A matrix with at least one generalized eigenvector is called defective. Any nondefective matrix $\mathbf{A}$ can be diagonalized via a similarity transformation $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{D}$ where $\mathbf{V}_{n\times n} = [\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \ldots \;\; \mathbf{v}_n]$ is the modal matrix of $\mathbf{A}$, whose columns are the (linearly independent) eigenvectors of $\mathbf{A}$, and $\mathbf{D}$ is a diagonal matrix comprised of the eigenvalues of $\mathbf{A}$. For defective matrices, matrix $\mathbf{D}$ is replaced with the Jordan matrix $\mathbf{J}$.

REVIEW PROBLEMS

1. Prove that if $\mathbf{A}$ is $n\times n$ and rank($\mathbf{A}$) < n, then $\mathbf{A}^{-1}$ does not exist.
2. Show that if $\mathbf{A}$ is lower triangular and one of its diagonal entries is zero, then $\mathbf{A}^{-1}$ does not exist.

3. Prove that the product of two symmetric matrices is not necessarily symmetric.
4. If **A** is $m\times m$ and symmetric, and **B** is a general $m\times n$ matrix, show that $\mathbf{B}^T\mathbf{AB}$ is $n\times n$ and symmetric.
5. Find the rank of

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & -3 \\ 4 & 9 & -2 & 7 \\ 3 & 4 & -1 & 5 \end{bmatrix}$$

6. Determine a such that rank(**A**) = 3, where

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & -1 & 0 \\ -2 & 3 & 4 & 1 \\ 3 & 5 & 0 & -2 \\ 2 & 6 & a & -1 \end{bmatrix}, \quad a = \text{parameter}$$

7. Find a such that the following homogeneous system has a nontrivial ($\mathbf{x} \neq \mathbf{0}$) solution:

$$\begin{bmatrix} 2 & 1 & -1 \\ -2 & 3 & a \\ 2 & 5 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

8. Find the value(s) of a for which the following system only has a trivial solution:

$$\begin{bmatrix} -1 & 4 & 3 \\ 2 & -2 & 1 \\ a & 2 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

9. Using Gauss elimination, solve

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} = \begin{bmatrix} 2 & 0 & 2 & 1 \\ 0 & 1 & 3 & -2 \\ 2 & 1 & 4 & 3 \\ 2 & -1 & -1 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 1 \\ -5 \\ 1 \\ 7 \end{Bmatrix}$$

10. Solve the system in Problem 9 using the inverse of the coefficient matrix.
11. Find the inverse of the rotation matrix

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

12. Solve the following system using
    a. Cramer's rule.
    b. The "\" operator.

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 0 & -2 \\ 0 & 3 & 1 & -1 \\ -1 & 0 & 2 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} -2 \\ 4 \\ 2 \\ -2 \end{Bmatrix}$$

13. Solve using Cramer's rule:

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} = \begin{bmatrix} -1 & 1 & 0 & 3 \\ 0 & 2 & 1 & -1 \\ 1 & 0 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 2 \\ -2 \\ 4 \\ 1 \end{Bmatrix}$$

14. Show that any matrix with distinct eigenvalues is nondefective.
15. a. Find all eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

    b. Repeat in MATLAB.
16. Find the eigenvalues and the AM and GM of each, and decide whether the matrix is defective:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 2 & 0 & -1 \end{bmatrix}$$

17. Prove that the eigenvalues of a matrix are preserved under a similarity transformation, that is, if $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{B}$, then the eigenvalues of **A** and **B** are the same. *Hint*: show that if λ is an eigenvalue of **B**, it is also an eigenvalue of **A**.
18. Find the modal matrix and use it to transform **A** into a diagonal or a Jordan matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -1 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -3 \end{bmatrix}$$

19. Prove that if $\mathbf{A}_{n\times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\mathbf{A}^{-1}$ exists if $\lambda_i \neq 0$ for $i = 1, \ldots, n$.
20. Prove that if k = const and $\mathbf{A}_{n\times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then the eigenvalues of $k\mathbf{A}$ are $k\lambda_1, \ldots, k\lambda_n$ with eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

4 System Model Representation

This chapter presents various forms involved in the representation of mathematical models of dynamic systems. The techniques to derive these models will be introduced in Chapters 5 through 7. The main topics covered in this chapter include the configuration form, state-space form, input–output (I/O) equation, transfer function, and block diagram representation. Also included is how any of these forms may be obtained from another. All of these will be discussed on the premise that the dynamic system is linear. Linearization of nonlinear systems is covered in the final section of this chapter.

4.1 CONFIGURATION FORM

A set of coordinates that completely describes the motion of a system is known as a set of generalized coordinates. This set is not unique so that more than one set of coordinates can be chosen for this purpose. The number of coordinates, however, remains the same regardless of the set selected for a specific system. If there are n generalized coordinates, they are usually denoted by $q_1, q_2, \ldots, q_n$. Suppose a dynamic system model is described by

$$\begin{cases} \ddot{q}_1 = f_1(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) \\ \ddot{q}_2 = f_2(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) \\ \qquad \cdots \\ \ddot{q}_n = f_n(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) \end{cases} \tag{4.1}$$

where $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$ are the generalized velocities and $f_1, f_2, \ldots, f_n$, known as the generalized forces, are algebraic functions of q_i, $\dot{q}_i$ ($i = 1,2, \ldots, n$) and time t. Assuming initial time is $t = 0$, Equation 4.1 is subjected to initial generalized coordinates $q_1(0), q_2(0), \ldots, q_n(0)$ and initial generalized velocities $\dot{q}_1(0), \dot{q}_2(0), \ldots, \dot{q}_n(0)$ and is called the configuration form.

Example 4.1: Configuration Form

The mechanical system shown in Figure 4.1 consists of blocks m_1 and m_2, linear springs with stiffness coefficients k_1 and k_2, a linear damper with coefficient of viscous damping c, and force $f(t)$ applied to block m_1. The equations of motion are derived as (Chapter 5)

$$\begin{cases} m_1\ddot{x}_1 + c\dot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = f(t) \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \end{cases}$$

where x_1, x_2 are the displacements of the blocks and $\dot{x}_1$, $\dot{x}_2$ are their respective velocities. The system is subjected to initial conditions $x_1(0) = x_{10}$, $x_2(0) = x_{20}$, $\dot{x}_1(0) = \dot{x}_{10}$, and $\dot{x}_2(0) = \dot{x}_{20}$. Obtain the configuration form.

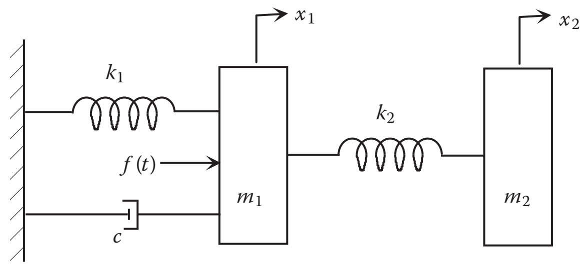


FIGURE 4.1 Mechanical system.

Solution

Comparison with Equation 4.1 reveals $n = 2$ so that there are two generalized coordinates for this system: $q_1 = x_1$ and $q_2 = x_2$. Simplifying and rewriting the equations of motion to resemble the form in Equation 4.1 yields

$$\begin{cases} \ddot{x}_1 = \dfrac{1}{m_1}[-c\dot{x}_1 - (k_1 + k_2)x_1 + k_2x_2 + f(t)] = f_1(x_1, x_2, \dot{x}_1, \dot{x}_2, t) \\ \ddot{x}_2 = \dfrac{1}{m_2}[-k_2x_2 + k_2x_1] = f_2(x_1, x_2, \dot{x}_1, \dot{x}_2, t) \end{cases}$$

where f_1 and f_2 are the generalized forces. These, together with the four initial conditions, constitute the configuration form.

4.1.1 Second-Order Matrix Form

Mathematical models of dynamic systems that are governed by n-dimensional systems of second-order differential equations can conveniently be expressed as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{4.2}$$

which is commonly known as the standard, second-order matrix form, where

- $\mathbf{x}_{n\times 1}$ = configuration vector, $\mathbf{f}_{n\times 1}$ = vector of external forces
- $\mathbf{M}_{n\times n}$ = mass matrix, $\mathbf{C}_{n\times n}$ = damping matrix, $\mathbf{K}_{n\times n}$ = stiffness matrix

Example 4.2: Second-Order Matrix Form

Express the equations of motion of the system in Example 4.1 in second-order matrix form.

Solution

Using matrix and vector notation, the equations of motion can be written as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f(t) \\ 0 \end{Bmatrix}$$

Consequently, the pertinent vectors and matrices can be properly identified as

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} f(t) \\ 0 \end{Bmatrix}$$

PROBLEM SET 4.1

In Problems 1 through 10, express the system model, assuming general initial conditions, in

a. Configuration form.
b. Standard, second-order matrix form.

1. $\begin{cases} \ddot{x}_1 + \dot{x}_1 + 2(x_1 - x_2) = 10e^{-t} \\ \ddot{x}_2 + \frac{1}{2}\dot{x}_2 - 2(x_1 - x_2) = 0 \end{cases}$

2. $\begin{cases} m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) - c(\dot{x}_2 - \dot{x}_1) = 0 \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_1) = f(t) \end{cases}$; Mechanical system in Figure 4.2

3. $\begin{cases} m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) = F_1(t) \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) = F_2(t) \end{cases}$; Mechanical system in Figure 4.3

4. $\begin{cases} \ddot{\theta} + \dot{\theta} + 2\theta + b(\theta - x) = \sin(\frac{1}{2}t) \\ 3\ddot{x} + \dot{x} - b(\theta - x) = 0 \end{cases}$ $(b = \text{const})$

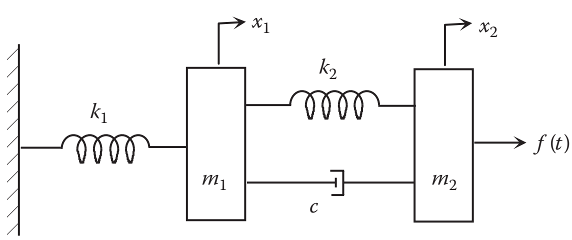


FIGURE 4.2 Mechanical system in Problem 2.

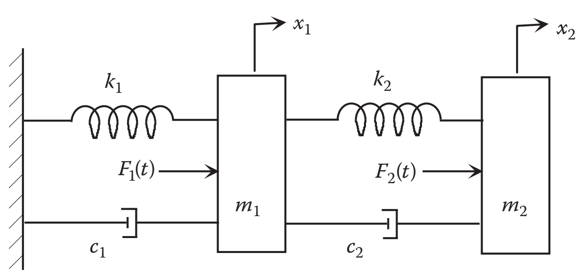


FIGURE 4.3 Mechanical system in Problem 3.

5. $\begin{cases} 3\ddot{I}_1 + \dot{I}_1 + 2I_1 + 2(\dot{I}_1 - \dot{I}_2) + \frac{1}{2}(I_1 - I_2) = \sin t \\ \ddot{I}_2 + 2(\dot{I}_2 - \dot{I}_1) + \frac{1}{2}(I_2 - I_1) = \sin 2t \end{cases}$

6. $\begin{cases} m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = 0 \\ m_2\ddot{x}_2 + k_3x_2 + k_2(x_2 - x_1) = f(t) \end{cases}$

7. $\begin{cases} m\ddot{x}_1 + kx_1 - k(x_2 - x_1) - c(\dot{x}_2 - \dot{x}_1) = 0 \\ m\ddot{x}_2 - k(x_3 - x_2) + k(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_1) = 0 \\ m\ddot{x}_3 + kx_3 + k(x_3 - x_2) + c\dot{x}_3 = f(t) \end{cases}$

8. $\begin{cases} m\ddot{x}_1 + kx_1 - k(x_2 - x_1) - c(\dot{x}_2 - \dot{x}_1) = F_1(t) \\ m\ddot{x}_2 - k(x_3 - x_2) + k(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_1) - c(\dot{x}_3 - \dot{x}_2) = 0 \\ m\ddot{x}_3 + kx_3 + k(x_3 - x_2) + c(\dot{x}_3 - \dot{x}_2) = F_2(t) \end{cases}$

9. $\begin{cases} \ddot{x}_1 + \dot{x}_1 + 3(x_1 - x_2) = e^{-t}\sin t \\ \frac{1}{3}x_2 - 3(x_1 - x_2) = 0 \end{cases}$

10. $\begin{cases} \ddot{x} = f(t) - \dot{x} \\ \ddot{\ } = \frac{1}{2}\dot{\ } + \quad - \ddot{x} \end{cases}$

4.2 STATE-SPACE FORM

Mathematical models of physical systems (Chapters 5 through 7) appear as systems of differential equations of various orders. However, this rather general system model form is not very convenient for analysis and simulation. One of the most convenient forms is achieved by transforming a system of ODEs of various orders into a larger system of first-order ODEs. And the first step toward that goal is the selection of a suitable set of state variables. State variables are denoted by x_i ($i = 1, 2, \ldots, n$).

4.2.1 State Variables, State-Variable Equations, State Equation

State variables form the smallest set of independent variables that completely describe the state of a system. More exactly, knowledge of the state variables at some fixed (reference) time t_0 and system inputs at all $t \geq t_0$ translates to knowing the state variables and system outputs at all $t \geq t_0$. It is important to note that state variables are independent, thus they cannot be expressible as algebraic functions of one another and the system inputs. Moreover, a set of state variables is not unique so that more than one set can be identified for a dynamic system. Given a system model, the state variables are determined as follows:

- The number of state variables = the number of initial conditions needed to completely solve the system model,
- The state variables are exactly those variables for which initial conditions are required.

Example 4.3: State Variables

The mathematical model of the mechanical system in Figure 4.4 is provided by its equation of motion, $\ddot{x} + \dot{x} + 2x = f(t)$. Identify the state variables.

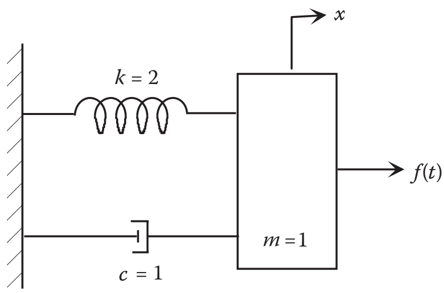


FIGURE 4.4 Mechanical system in Example 4.3.

Solution

Two initial conditions, $x(0)$ and $\dot{x}(0)$, are needed to completely solve the differential equation. Hence, there are two state variables, denoted x_1 and x_2. However, the state variables are those variables for which initial conditions are required. Because these variables are x and $\dot{x}$, the state variables are selected as $x_1 = x$ and $x_2 = \dot{x}$.

4.2.1.1 State-Variable Equations

There are as many state-variable equations as there are state variables. Each state-variable equation is a first-order differential equation whose left side is the first derivative of a state variable and whose right side is an algebraic function of the state variables, system inputs, and possibly time t. Suppose a dynamic system has n state variables $x_1, x_2, \ldots, x_n$ and m inputs $u_1, u_2, \ldots, u_m$. Then, the state-variable equations take the general form

$$\begin{cases} \dot{x}_1 = f_1(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \\ \dot{x}_2 = f_2(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \\ \qquad\qquad \cdots \\ \dot{x}_n = f_n(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \end{cases} \tag{4.3}$$

where $f_1, f_2, \ldots, f_n$ are algebraic functions of the state variables and inputs, and are generally nonlinear.

Example 4.4: State-Variable Equations (Example 4.3 Continued)

Referring to Example 4.3, because there are two state variables, there must be two state-variable equations in the form

$$\begin{cases} \dot{x}_1 = \cdots \\ \dot{x}_2 = \cdots \end{cases}$$

Recall that only state variables and inputs may appear on the right side of each differential equation. We know $x_1 = x$ so that $\dot{x}_1 = \dot{x}$. But $\dot{x} = x_2$, hence $\dot{x}_1 = \dot{x} = x_2$. This means the first equation is simply $\dot{x}_1 = x_2$, which is valid because the right side contains only a state variable. Next, we know that $x_2 = \dot{x}$ and thus $\dot{x}_2 = \ddot{x}$. However, $\ddot{x}$ is obtained from the equation of motion as

$$\dot{x}_2 = \ddot{x} \overset{\text{from the equation of motion}}{=} -\dot{x} - 2x + f(t) \underset{x_1 = x,\ x_2 = \dot{x}}{\overset{\text{use state variables}}{=}} -x_2 - 2x_1 + f(t)$$

With this, the state-variable equations can be written as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 - 2x_1 + f(t) \end{cases}$$

Noting that *f*(*t*) is the only system input here, this agrees with Equation 4.3.

4.2.1.2 State Equation

In general, when at least one of the functions $f_1, f_2, \dots, f_n$ is nonlinear, Equation 4.3 is expressed in vector form as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \tag{4.4}$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{Bmatrix}_{n\times 1}, \quad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ \cdots \\ u_m \end{Bmatrix}_{m\times 1}, \quad \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \end{Bmatrix}_{n\times 1}$$

However, if all elements of a dynamic system are linear, then $f_1, f_2, \dots, f_n$ in Equation 4.3 will be linear combinations of $x_1, x_2, \dots, x_n$ and $u_1, u_2, \dots, u_m$:

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1m}u_m \\ \dot{x}_2 = a_{21}x_1 + \cdots + a_{2n}x_n + b_{21}u_1 + \cdots + b_{2m}u_m \\ \cdots \\ \dot{x}_n = a_{n1}x_1 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nm}u_m \end{cases} \tag{4.5}$$

Rewriting Equation 4.5 in matrix form yields

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdots \\ \dot{x}_n \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n} \begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{Bmatrix}_{n\times 1} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}_{n\times m} \begin{Bmatrix} u_1 \\ u_2 \\ \cdots \\ u_m \end{Bmatrix}_{m\times 1}$$

Finally, this can be conveniently expressed as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{4.6}$$

Equation 4.6 is known as the state equation, where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{Bmatrix}_{n\times 1} = \text{state vector}, \quad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ \cdots \\ u_m \end{Bmatrix}_{m\times 1} = \text{input vector}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n} = \text{state matrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}_{n\times m} = \text{input matrix}$$

Example 4.5: State Equation (Example 4.4 Continued)

The state-variable equations at the conclusion of Example 4.4 are expressed in matrix form as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$

Therefore, the state equation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u = f(t)$$

Because there is only one input $f(t)$, input vector $\mathbf{u}$ is scalar and denoted by u.

4.2.2 Output Equation, State-Space Form

Consider a dynamic system with state variables $x_1, x_2, \ldots, x_n$ and inputs $u_1, u_2, \ldots, u_m$ as before. Suppose the system has p outputs $y_1, y_2, \ldots, y_p$. Outputs are sometimes called measured outputs, referring to physical quantities that are being measured. Then, the output equations generally appear in the form

$$\begin{cases} y_1 = g_1(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \\ y_2 = g_2(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \\ \qquad \cdots \\ y_p = g_p(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \end{cases} \tag{4.7}$$

where $g_1, g_2, \ldots, g_p$ are algebraic functions of the state variables and inputs, and are generally nonlinear.

4.2.2.1 Output Equation

If at least one of the functions $g_1, g_2, \ldots, g_p$ is nonlinear, Equation 4.7 is expressed in vector form as

$$\mathbf{y} = \mathbf{g}(\mathbf{x},\mathbf{u},t) \tag{4.8}$$

where

$$\mathbf{y} = \begin{Bmatrix} y_1 \\ y_2 \\ \cdots \\ y_p \end{Bmatrix}_{p\times 1}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{Bmatrix}_{n\times 1}, \quad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ \cdots \\ u_m \end{Bmatrix}_{m\times 1}, \quad \mathbf{g} = \begin{Bmatrix} g_1 \\ g_2 \\ \cdots \\ g_p \end{Bmatrix}_{p\times 1}$$

But if all elements are linear, then $g_1, g_2, \ldots, g_p$ in Equation 4.7 are linear combinations of $x_1, x_2, \ldots, x_n$ and $u_1, u_2, \ldots, u_m$:

$$\begin{cases} y_1 = c_{11}x_1 + \cdots + c_{1n}x_n + d_{11}u_1 + \cdots + d_{1m}u_m \\ y_2 = c_{21}x_1 + \cdots + c_{2n}x_n + d_{21}u_1 + \cdots + d_{2m}u_m \\ \qquad \cdots \\ y_p = c_{p1}x_1 + \cdots + c_{pn}x_n + d_{p1}u_1 + \cdots + d_{pm}u_m \end{cases}$$

This is conveniently expressed as

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{4.9}$$

Equation 4.9 is known as the output equation, where **y**, **x**, and **u** are defined as before, and

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}_{p\times n} = \text{output matrix}$$

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}_{p\times m} = \text{direct transmission matrix}$$

Example 4.6: Output Equation

For the mechanical system studied in Examples 4.3 through 4.5, suppose the output is the velocity $\dot{x}$ of the block. Find the output equation.

Solution

Because there is only one output, the output vector $\mathbf{y}$ is 1×1, hence denoted by y. The output is $\dot{x}$, therefore $y=\dot{x}$. But for this system we know $\dot{x}=x_2$, thus $y=x_2$. Finally, Equation 4.9 may be written as

$$y=[0 \quad 1]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}+0\cdot u$$

so that $\mathbf{C}=[0 \ \ 1]$ and $D=0$. Note that the direct transmission matrix $\mathbf{D}$ is 1×1, hence denoted by D.

4.2.2.2 State-Space Form

The combination of the state equation and output equation is called the state-space form. For a linear system with state variables $x_1, x_2, \ldots, x_n$, inputs $u_1, u_2, \ldots, u_m$, and outputs $y_1, y_2, \ldots, y_p$, the state-space form is

$$\begin{cases} \dot{\mathbf{x}}=\mathbf{A}_{n\times n}\mathbf{x}_{n\times1}+\mathbf{B}_{n\times m}\mathbf{u}_{m\times1} \\ \mathbf{y}_{p\times1}=\mathbf{C}_{p\times n}\mathbf{x}_{n\times1}+\mathbf{D}_{p\times m}\mathbf{u}_{m\times1} \end{cases} \tag{4.10}$$

Example 4.7: State-Space Form

The equations of motion for the mechanical system in Figure 4.5 are

$$\begin{cases} \ddot{x}_1+x_1-(x_2-x_1)-(\dot{x}_2-\dot{x}_1)=f_1 \\ 2\ddot{x}_2+(x_2-x_1)+(\dot{x}_2-\dot{x}_1)=f_2 \end{cases}$$

Assuming the (measured) outputs are x_2 and $\dot{x}_2$, derive the state-space form.

Solution

The system model comprises two second-order differential equations, hence a total of four initial conditions are needed for complete solution. There are therefore four state variables:

$$\begin{aligned} x_1&=x_1 \\ x_2&=x_2 \\ x_3&=\dot{x}_1 \\ x_4&=\dot{x}_2 \end{aligned}$$

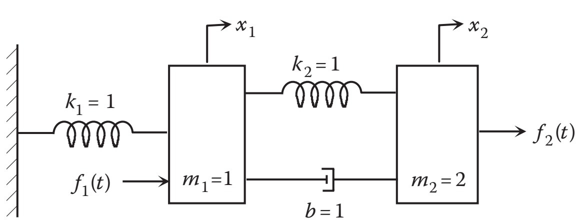


FIGURE 4.5 Mechanical system in Example 4.7.

When designating state variables, the above arrangement is the most commonly used.

The state-variable equations are then formed as

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = -2x_1 + x_2 + x_4 - x_3 + f_1 \\ \dot{x}_4 = \frac{1}{2}[-x_2 + x_1 - x_4 + x_3 + f_2] \end{cases}$$

The state equation is subsequently obtained as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_{4\times 1}, \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -1 & 1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}_{4\times 4}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}_{4\times 2}, \quad \mathbf{u} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}_{2\times 1}$$

Because the outputs are x_2 and $\dot{x}_2$, we have

$$\mathbf{y} = \begin{Bmatrix} x_2 \\ \dot{x}_2 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ x_4 \end{Bmatrix}$$

As a result, the output equation is

$$\mathbf{y}_{2\times 1} = \mathbf{C}_{2\times 4}\mathbf{x}_{4\times 1} + \mathbf{D}_{2\times 2}\mathbf{u}_{2\times 1}$$

where

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{2\times 4}, \quad \mathbf{D} = \mathbf{0}_{2\times 2}$$

Finally, combining the state equation and the output equation yields the state-space form.

Example 4.8: State-Space Form

The state-space form in Example 4.7 can be stored in MATLAB® under an assigned name, for example, `sys` as shown below. This stored form can then be used for analysis and simulation.

```
% Input matrices A,B,C and D
>> A = [0 0 1 0;0 0 0 1;-2 1 -1 1;1/2 -1/2 1/2 -1/2];
>> B = [0 0;0 0;1 0;0 1/2]; C = [0 1 0 0;0 0 0 1]; D = [0 0;0 0];
>> sys = ss(A,B,C,D)       % state-space form
```

```
a =
          x1     x2     x3      x4
    x1     0      0      1       0
    x2     0      0      0       1
    x3    -2      1     -1       1
    x4   0.5   -0.5    0.5    -0.5

b =
          u1     u2
    x1     0      0
    x2     0      0
    x3     1      0
    x4     0      0.5

c =
          x1   x2   x3   x4
    y1     0    1    0    0
    y2     0    0    0    1

d =
          u1   u2
    y1     0    0
    y2     0    0

Continuous-time state-space model.
```

4.2.3 Decoupling the State Equation

As mentioned at the outset, more than one set of state variables can be selected for a system model. In other words, a state vector $\tilde{\mathbf{x}}$, different from $\mathbf{x}$, still leads to a state-space form that is in the standard form of Equation 4.10. But any such set of state variables are still coupled through the entries of the resulting state matrix. The decoupling of the state equation is possible through the use of the modal matrix (Section 3.3.1) corresponding to the state matrix as follows. Given the state-space form (Equation 4.10) assume $\mathbf{A}_{n\times n}$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ so that the modal matrix $\mathbf{V} = [\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \ldots \;\; \mathbf{v}_n]_{n\times n}$ diagonalizes matrix $\mathbf{A}$, that is, $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \tilde{\mathbf{D}}$ (see Equation 3.10). Note that we have changed the notation of $\mathbf{D}$ to $\tilde{\mathbf{D}}$ to avoid confusion with the direct transmission matrix. Consider the transformation $\mathbf{x} = \mathbf{V}\tilde{\mathbf{x}}$ and substitute into the state-space form to obtain

$$\begin{cases}\mathbf{V}\dot{\tilde{\mathbf{x}}} = \mathbf{A}\mathbf{V}\tilde{\mathbf{x}} + \mathbf{B}\mathbf{u}\\ \mathbf{y} = \mathbf{C}\mathbf{V}\tilde{\mathbf{x}} + \mathbf{D}\mathbf{u}\end{cases} \quad \underset{\text{by } \mathbf{V}^{-1}}{\text{Pre-multiply the state equation}} \quad \begin{cases}\mathbf{V}^{-1}\mathbf{V}\dot{\tilde{\mathbf{x}}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}\tilde{\mathbf{x}} + \mathbf{V}^{-1}\mathbf{B}\mathbf{u}\\ \mathbf{y} = \mathbf{C}\mathbf{V}\tilde{\mathbf{x}} + \mathbf{D}\mathbf{u}\end{cases}$$

Noting $\mathbf{V}^{-1}\mathbf{V} = \mathbf{I}$ and $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \tilde{\mathbf{D}}$, and denoting $\tilde{\mathbf{B}} = \mathbf{V}^{-1}\mathbf{B}$ and $\tilde{\mathbf{C}} = \mathbf{C}\mathbf{V}$, the above becomes

$$\begin{cases}\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{D}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u}\\ \mathbf{y} = \tilde{\mathbf{C}}\tilde{\mathbf{x}} + \mathbf{D}\mathbf{u}\end{cases} \tag{4.11}$$

Because $\tilde{\mathbf{D}}$ is diagonal, with diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n$, the new state equation in Equation 4.11 is clearly decoupled, each row a first-order differential equation in one state variable, independent of the others. And the new state vector is $\tilde{\mathbf{x}}$.

Example 4.9: Decoupled State Equation

The state-space form for a system model is

$$\begin{cases}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u\\ y = \mathbf{C}\mathbf{x} + Du\end{cases}$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u = u, \quad \mathbf{C} = [1 \quad 0], \; D = 0$$

a. Find the decoupled state equation together with the corresponding output equation.
b. Confirm in MATLAB.

Solution

a. Proceeding as in Section 3.3, we find $\lambda(\mathbf{A}) = -1, -2$ and modal matrix $\mathbf{V} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ so that

$$\mathbf{V}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}, \quad \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \tilde{\mathbf{D}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \mathbf{V}^{-1}\mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \tilde{\mathbf{C}} = \mathbf{C}\mathbf{V} = [1 \quad 1]$$

It is then readily seen that the decoupled state-space form is

$$\begin{cases} \dot{\tilde{\mathbf{x}}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \tilde{\mathbf{x}} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \\ y = [1 \quad 1]\,\tilde{\mathbf{x}} + 0 \cdot u \end{cases}$$

b.

```
>> A = [0 1;-2 -3]; B = [0;1]; C = [1 0]; D = 0;
>> [V,Dtilda] = eig(A)
V =
     0.7071   -0.4472      % Eigenvectors are normalized
    -0.7071    0.8944

Dtilda =
       -1        0
        0       -2
>> Btilda = V\B; Ctilda = C*V;
>> dec_sys = ss(Dtilda,Btilda,Ctilda,D)       % Define decoupled system
a =
           x1    x2
    x1     -1     0
    x2      0    -2

b =
             u1
    x1     1.414
    x2     2.236

c =
              x1         x2
    y1     0.7071    -0.4472

d =
           u1
    y1     0

Continuous-time model.
```

Because the "eig" command returns normalized eigenvectors, the ensuing matrix V is different from **V** obtained in the solution of (a), and the eventual numerical results do not agree with those in (a), but the new system is decoupled nonetheless.

PROBLEM SET 4.2

In Problems 1 to 8, find a suitable set of state variables, derive the state-variable equations, and form the state equation.

1. $\ddot{x} + kx = e^{-t/3}, \quad k = \text{const} > 0$
2. $2\ddot{x} + \dot{x} + 3x = \sin t$
3. $2\dddot{x} + \ddot{x} + \dot{x} + 2x = f(t)$
4. $\begin{cases} \ddot{z}_1 + 2\dot{z}_1 + \frac{1}{2}(z_1 - z_2) = f_1(t) \\ \dot{z}_2 + z_2 + \frac{1}{2}(z_2 - z_1) = f_2(t) \end{cases}$
5. $\begin{cases} 2\ddot{x} + \dot{x} + x = z \\ \dot{z} + 3z + x = f(t) \end{cases}$
6. $\begin{cases} 3\ddot{x}_1 + \dot{x}_1 + \frac{2}{3}(x_1 - x_2) = f(t) \\ x_2 - \frac{2}{3}(x_1 - x_2) = 0 \end{cases}$
7. $\begin{cases} \ddot{z}_1 + \dot{z}_1 + k(z_1 - z_2) = e^{-t} \\ z_2 = k(z_1 - z_2) \end{cases}, \quad k = \text{const}$
8. $\begin{cases} \ddot{x}_1 + \dot{x}_1 + 2x_1 - \dot{x}_2 - 3x_2 = f_1(t) \\ 2\ddot{x}_2 - \dot{x}_1 - 2x_1 + \dot{x}_2 + 3x_2 = f_2(t) \end{cases}$
9. A nonlinear dynamic system is mathematically modeled as $\ddot{x} + \frac{1}{3}\dot{x} + 2x^3 = e^{-t/2}\sin t$. Derive the state-variable equations and express them in vector form.
10. The mathematical model of a nonlinear system is given below. Derive the state-variable equations and express them in vector form.

$$\begin{cases} \ddot{x}_1 + \dot{x}_1 = 2x_2^3 \\ \dot{x}_2 = x_1 + \frac{1}{2}\sin t \end{cases}$$

Problems 11 through 14 are concerned with the stability of systems. A linear dynamic system is called stable if the homogeneous solution of its mathematical model—subjected to the prescribed initial conditions—decays. More practically, a linear system is stable if the eigenvalues of its state matrix all have negative real parts, that is, they all lie in the left half-plane.

11. Determine the range of values of k for which the system in Problem 1 is stable.
12. Decide whether the system in Problem 2 is stable.
13. Find the range of values of a for which a system described by $\ddot{z} - (a-1)\dot{z} - z = f$ is stable.
14. Determine whether the system in Problem 6 is stable.

In Problems 15 through 18, find the state-space form of the mathematical model.

15. $\begin{cases} 2\ddot{x}_1 + \dot{x}_1 + 2(x_1 - x_2) = f(t) \\ \dot{x}_2 + x_2 - 2(x_1 - x_2) = 0 \end{cases}$, outputs are x_1 and x_2.

16. $\begin{cases} \ddot{q}_1 + \frac{1}{3}(q_1 - q_2) + \dot{q}_1 + 2q_1 = v(t) \\ \dot{q}_2 + \frac{1}{3}(q_2 - q_1) = 0 \end{cases}$, outputs are q_1 and $\dot{q}_1$.

17. $\begin{cases} \ddot{x}_1 + 2(x_1 - x_3) - 2(\dot{x}_2 - \dot{x}_1) - \frac{1}{2}(x_2 - x_1) = f(t) \\ \ddot{x}_2 + 2(\dot{x}_2 - \dot{x}_1) + \frac{1}{2}(x_2 - x_1) = 0 \\ x_3 - 2(x_1 - x_3) = 0 \end{cases}$, outputs are x_2 and $\dot{x}_2$.

18. $\begin{cases} \ddot{x}_1 + 2(x_1 - x_3) - 2(\dot{x}_2 - \dot{x}_1) - \frac{1}{2}(x_2 - x_1) = 0 \\ \ddot{x}_2 + 2(\dot{x}_2 - \dot{x}_1) + \frac{1}{2}(x_2 - x_1) = f(t) \\ \dot{x}_3 = 2(x_1 - x_3) \end{cases}$, outputs are x_1 and x_3.

19. A dynamic system model is described by $\ddot{x} + 4\dot{x} + 3x = f(t)$, where x is the output.
    a. Find the state-space form.
    b. Decouple the state equation and obtain the transformed state-space form.
20. The governing equations for a system are given as

$$\begin{cases} \ddot{x}_1 - \dot{x}_1 - x_1 + 3x_2 = f(t) \\ \dot{x}_2 = -x_1 - 3x_2 \end{cases}$$

    where the outputs are x_1 and $\dot{x}_1$.
    a. Find the state-space form.
    b. Decouple the state equation obtained in (a) and present the transformed state-space form.

4.3 INPUT–OUTPUT EQUATION, TRANSFER FUNCTION

An input–output (I/O) equation is a differential equation that relates a system input, a system output, and their time derivatives. If $u(t)$ is an input and $y(t)$ is an output, then the I/O equation is in the general form

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(m)} + b_1 u^{(m-1)} + \cdots + b_{m-1}\dot{u} + b_m u, \quad m \le n \tag{4.12}$$

where $a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ are constants, and $y^{(n)} = d^n y/dt^n$. A single-input/single-output (SISO) system, therefore, has only one I/O equation. A multi-input/multi-output (MIMO) system, on the other hand, has several I/O equations, one for each pair of input/output. In particular, a system with q inputs and r outputs has a total of qr I/O equations.

4.3.1 Input–Output Equations from the System Model

Because the generalized coordinates in a system model are normally coupled through the governing equations, finding one or more I/O equations is usually a difficult task. However, there is a systematic approach that can be used for this purpose. The idea is to take the Laplace transform

of the governing equations—assuming zero initial conditions—and eliminate the unwanted variables in the ensuing algebraic system. The new data is subsequently transformed back to the time domain and interpreted as one or more differential equations, which in turn are the desired I/O equations.

Example 4.10: SISO System

The mathematical model for the simple mechanical system in Example 4.3 is

$$\ddot{x} + \dot{x} + 2x = f(t)$$

where $f(t)$ is regarded as the input and $x(t)$ is the output. Find the I/O equation.

Solution

Because the governing equation is already in the form of Equation 4.12, there is no need for Laplace transformation.

Example 4.11: MIMO System

A system model is described by

$$\begin{cases} \ddot{x}_1 + \dot{x}_1 + \frac{1}{3}(x_1 - x_2) = f(t) \\ \dot{x}_2 - \frac{1}{3}(x_1 - x_2) = 0 \end{cases}$$

where f is the input and x_1 and x_2 are the outputs. Derive all possible I/O equations.

Solution

Because there are two outputs and one input, we expect two I/O equations. Assuming zero initial conditions, Laplace transformation of the governing equations yields

$$\begin{cases} (s^2 + s + \frac{1}{3})X_1(s) \; - \; \frac{1}{3}X_2(s) = F(s) \\ \qquad\qquad -\frac{1}{3}X_1(s) + (s + \frac{1}{3})X_2(s) = 0 \end{cases}$$

Because both x_1 and x_2 are the outputs, we solve the above system once for $X_1(s)$ and a second time for $X_2(s)$ using Cramer's rule (Section 3.2.3):

$$X_1(s) = \frac{\begin{vmatrix} F(s) & -\frac{1}{3} \\ 0 & s + \frac{1}{3} \end{vmatrix}}{\begin{vmatrix} s^2 + s + \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & s + \frac{1}{3} \end{vmatrix}} = \frac{(s + \frac{1}{3})F(s)}{s^3 + \frac{4}{3}s^2 + \frac{2}{3}s} \quad \xrightarrow{\text{cross multiply}} \quad (3s^3 + 4s^2 + 2s)X_1(s) = (3s + 1)F(s)$$

$$X_2(s) = \frac{\begin{vmatrix} s^2+s+\frac{1}{3} & F(s) \\ -\frac{1}{3} & 0 \end{vmatrix}}{\begin{vmatrix} s^2+s+\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & s+\frac{1}{3} \end{vmatrix}} = \frac{\frac{1}{3}F(s)}{s^3+\frac{4}{3}s^2+\frac{2}{3}s} \xrightarrow{\text{cross multiply}} (3s^3+4s^2+2s)X_2(s) = F(s)$$

Interpretation of these two equations in time domain results in the two desired I/O equations:

$$3\dddot{x}_1 + 4\ddot{x}_1 + 2\dot{x}_1 = 3\dot{f} + f, \quad 3\dddot{x}_2 + 4\ddot{x}_2 + 2\dot{x}_2 = f$$

4.3.2 Transfer Functions from the System Model

A transfer function is defined as the ratio of the Laplace transforms of an output and an input, with the assumption that initial conditions are zero. Therefore, if $y(t)$ is an output and $u(t)$ is an input, the corresponding transfer function is formed as

$$G(s) = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{u(t)\}} = \frac{Y(s)}{U(s)}$$

Consider the general form of I/O equation given by Equation 4.12. Taking the Laplace transform assuming zero initial conditions yields

$$(s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n)Y(s) = (b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m)U(s)$$

The transfer function is then formed as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}, \quad m \le n \tag{4.13}$$

As mentioned earlier, there is a transfer function for each input/output pair. A SISO system has only one transfer function, whereas a MIMO system has several, one for each possible input/output pair. If a system has q inputs and r outputs, then there are a total of qr transfer functions, assembled in an $r \times q$ transfer function matrix (also known as a transfer matrix), denoted by $\mathbf{G}(s) = [G_{ij}(s)]$, where $i = 1,2, \ldots, r$ and $j = 1,2, \ldots, q$.

Example 4.12: Transfer Function

A system model is described by

$$2\ddot{x} + \tfrac{1}{2}\dot{x} + 3x = f(t)$$

Assuming f is the input and x is the output, find the transfer function.

Solution

Taking the Laplace transform of the equation yields

$$(2s^2 + \tfrac{1}{2}s + 3)X(s) = F(s)$$

The transfer function is then formed as

$$\frac{X(s)}{F(s)} = \frac{1}{2s^2 + \frac{1}{2}s + 3} = \frac{2}{4s^2 + s + 6}$$

Example 4.13: Transfer Matrix

The equations of motion for the mechanical system in Figure 4.5, Example 4.7, are

$$\begin{cases} \ddot{x}_1 + x_1 - (x_2 - x_1) - (\dot{x}_2 - \dot{x}_1) = f_1 \\ 2\ddot{x}_2 + (x_2 - x_1) + (\dot{x}_2 - \dot{x}_1) = f_2 \end{cases}$$

If f_1 and f_2 are inputs and x_1 and x_2 are outputs, find the transfer matrix.

Solution

We expect a 2×2 transfer matrix with the following structure:

$$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}_{2\times 2} = \begin{bmatrix} \left.\dfrac{X_1(s)}{F_1(s)}\right|_{F_2=0} & \left.\dfrac{X_1(s)}{F_2(s)}\right|_{F_1=0} \\ \left.\dfrac{X_2(s)}{F_1(s)}\right|_{F_2=0} & \left.\dfrac{X_2(s)}{F_2(s)}\right|_{F_1=0} \end{bmatrix}$$

To find the four transfer functions listed above, we take the Laplace transform of the governing equations, with zero initial conditions,

$$\begin{cases} (s^2 + s + 2)X_1(s) \qquad -(s+1)X_2(s) = F_1(s) \\ \quad -(s+1)X_1(s) + (2s^2 + s + 1)X_2(s) = F_2(s) \end{cases}$$

Solving for $X_1(s)$, we find

$$X_1(s) = \frac{\begin{vmatrix} F_1(s) & -(s+1) \\ F_2(s) & 2s^2 + s + 1 \end{vmatrix}}{(s)} = \frac{2s^2 + s + 1}{(s)}F_1(s) + \frac{s+1}{(s)}F_2(s) \qquad \text{(a)}$$

where

$$(s) = \begin{vmatrix} s^2 + s + 2 & -(s+1) \\ -(s+1) & 2s^2 + s + 1 \end{vmatrix} = 2s^4 + 3s^3 + 5s^2 + s + 1$$

Solving for $X_2(s)$ yields

$$X_2(s) = \frac{\begin{vmatrix} s^2+s+2 & F_1(s) \\ -(s+1) & F_2(s) \end{vmatrix}}{(s)} = \frac{s^2+s+2}{(s)}F_2(s) + \frac{s+1}{(s)}F_1(s) \tag{b}$$

where $\Delta(s)$ is as before. The four transfer functions are then determined as follows.

Equation a: $\left.\dfrac{X_1(s)}{F_1(s)}\right|_{F_2=0} = \dfrac{2s^2+s+1}{(s)}, \quad \left.\dfrac{X_1(s)}{F_2(s)}\right|_{F_1=0} = \dfrac{s+1}{(s)}$

Equation b: $\left.\dfrac{X_2(s)}{F_1(s)}\right|_{F_2=0} = \dfrac{s+1}{(s)}, \quad \left.\dfrac{X_2(s)}{F_2(s)}\right|_{F_1=0} = \dfrac{s^2+s+2}{(s)}$

Ultimately, the transfer matrix is formed as

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{2s^2+s+1}{(s)} & \dfrac{s+1}{(s)} \\ \dfrac{s+1}{(s)} & \dfrac{s^2+s+2}{(s)} \end{bmatrix}$$

PROBLEM SET 4.3

In Problems 1 through 8, find all possible I/O equations.

1. $\begin{cases} \ddot{x}_1 + \dot{x}_1 + x_1 - x_2 = f(t) \\ \ddot{x}_2 + 2\dot{x}_2 - x_1 + x_2 = 0 \end{cases}, \quad f(t) = \text{input}, \ x_2 = \text{output}$

2. $\begin{cases} \ddot{x}_1 + \dot{x}_1 + 2(x_1 - x_2) = 0 \\ \ddot{x}_2 + \dot{x}_2 - 2(x_1 - x_2) = f(t) \end{cases}, \quad f(t) = \text{input}, \ x_1 = \text{output}$

3. $\begin{cases} \ddot{x}_1 + \dot{x}_1 + 2(x_1 - x_2) = f_1(t) \\ \ddot{x}_2 - 2(x_1 - x_2) = f_2(t) \end{cases}, \quad f_1(t), f_2(t) = \text{inputs}, \ x_2 = \text{output}$

4. $\begin{cases} \ddot{x}_1 + \dot{x}_1 + 2(x_1 - x_2) = f_1(t) \\ \ddot{x}_2 + \dot{x}_2 - 2(x_1 - x_2) = f_2(t) \end{cases}, \quad f_1(t), f_2(t) = \text{inputs}, \ x_1 = \text{output}$

5. $\begin{cases} \ddot{x}_1 + \dot{x}_1 + 3(x_1 - x_2) = f_1(t) \\ \ddot{x}_2 - 3(x_1 - x_2) = f_2(t) \end{cases}, \quad f_1(t), f_2(t) = \text{inputs}, \ x_1, x_2 = \text{outputs}$

6. $\begin{cases} \ddot{\theta}_1 + \dot{\theta}_1 + \frac{1}{2}(\theta_1 - \theta_2) = h(t) \\ \theta_2 = \frac{1}{2}(\theta_1 - \theta_2) \end{cases}, \quad h(t) = \text{input}, \ \theta_1 = \text{output}$

7. $\begin{cases} \ddot{q}_1 + \dot{q}_1 + 2(q_1 - q_2) + \frac{1}{2}q_1 = v(t) \\ \dot{q}_2 = 2(q_1 - q_2) \end{cases}, \quad v(t) = \text{input}, \ q_1, q_2 = \text{outputs}$

8. $\begin{cases} \ddot{x}_1 + x_1 - x_3 - (\dot{x}_2 - \dot{x}_1) - \frac{1}{2}(x_2 - x_1) = u(t) \\ \ddot{x}_2 + \dot{x}_2 - \dot{x}_1 + \frac{1}{2}(x_2 - x_1) = 0 \\ \dot{x}_3 = x_1 - x_3 \end{cases}$, $u(t)$ = input, x_1, x_3 = outputs

9. Find the transfer matrix for the system described in Example 4.11.
10. Derive all possible I/O equations in Example 4.13.

In Problems 11 through 14, the mathematical model of a system, as well as its inputs and outputs, are provided. Find the appropriate transfer matrix. Do *not* cancel any terms involving s in the numerator and denominator.

11. $\begin{cases} \ddot{x}_1 + \dot{x}_1 + \frac{1}{3}(x_1 - x_2) = 0 \\ \ddot{x}_2 + \frac{1}{2}\dot{x}_2 + \frac{1}{3}(x_2 - x_1) = f(t) \end{cases}$, $f(t)$ = input, x_1 = output

12. $\begin{cases} \ddot{q}_1 + \dot{q}_1 + \frac{1}{2}(q_1 - q_2) + q_1 = v(t) \\ \dot{q}_2 - \frac{1}{2}(q_1 - q_2) = 0 \end{cases}$, $v(t)$ = input, q_1 = output

13. $\begin{cases} \ddot{x}_1 + x_1 - x_3 - 2(\dot{x}_2 - \dot{x}_1) = f_1(t) \\ \ddot{x}_2 + 2(\dot{x}_2 - \dot{x}_1) = f_2(t) \\ \dot{x}_3 = x_1 - x_3 \end{cases}$, $f_1(t), f_2(t)$ = inputs, x_2, x_3 = outputs

14. $\begin{cases} \ddot{x}_1 + \dot{x}_1 + 2(x_1 - x_2) = f_1(t) \\ \ddot{x}_2 - 2(x_1 - x_2) = f_2(t) \end{cases}$, $f_1(t), f_2(t)$ = inputs, x_1, x_2 = outputs

15. A mechanical system model is derived as $m\ddot{x} + b\dot{x} + kx = f(t)$ where $m = \frac{1}{3}$, $b = 2$, $k = 3$, and applied force $f(t) = \frac{1}{2}\sin 2t$, all in consistent physical units. Assuming x is the output, find the transfer function.
16. In Problem 15, find the transfer function if $\dot{x}$ is the output.
17. The governing equation for an electric circuit is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(t)\,dt = v(t)$$

where L, R, and C are the inductance, resistance, and capacitance, all constants, $i(t)$ is the current, and $v(t)$ is the applied voltage. Assuming $i(t)$ and $v(t)$ are the system output and input, respectively, find the transfer function.
18. Electric charge q and electric current i are related via $i = dq/dt$. In Problem 17, find the transfer function if $q(t)$ and $v(t)$ are the system output and input, respectively.
19. The I/O equation for a dynamic system is given as $2\ddot{x} + 5\dot{x} + 3x = f(t)$ where f and x denote the input and output, respectively.
    a. Find the system's transfer function.
    b. Assuming $f(t)$ is the unit-step, find the expression for $X(s)$ using Part (a).
    c. Find the steady-state value x_{ss} of $x(t)$ using the final-value theorem.

20. The state-space representation of a system model is described as

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} + Du \end{cases}$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [0 \quad 1], \quad D = 0, \quad u = f$$

Find

a. The I/O equation.
b. The transfer function.

4.4 RELATIONS BETWEEN STATE-SPACE FORM, INPUT–OUTPUT EQUATION AND TRANSFER MATRIX

Thus far in this chapter, we have learned to derive the state-space form, I/O equation(s) and transfer function(s), directly from the system's mathematical model. In this section, we will develop and implement two systematic techniques to obtain (1) the state-space form from the I/O equation, and (2) the transfer matrix from the state-space form.

4.4.1 Input–Output Equation to State-Space Form

Consider the I/O equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u \tag{4.14}$$

where u is the input, y is the output, $a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ are all constants, and $y^{(n)} = d^n y/dt^n$. Note that this agrees with Equation 4.12, except that the same highest order of differentiation for y and u is now allowed, that is, $m = n$. The goal is to derive the state-space form directly from the I/O equation (Equation 4.14). With the assumption of zero initial conditions, the transfer function is readily obtained as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Rewrite this expression as

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)}\frac{V(s)}{U(s)} = (b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n)\left(\frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}\right)$$

so that

$$\frac{Y(s)}{V(s)} = b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n, \quad \frac{V(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Time–domain interpretation of the first gives

$$y = b_0 v^{(n)} + b_1 v^{(n-1)} + \cdots + b_{n-1}\dot{v} + b_n v \tag{4.15}$$

and the second one yields

$$v^{(n)} + a_1 v^{(n-1)} + \cdots + a_{n-1}\dot{v} + a_n v = u \tag{4.16}$$

Equation 4.16 is an nth-order differential equation in v; hence, n initial conditions are required for complete solution, and by Section 4.2, there are n state variables, which are selected as

$$\begin{aligned} x_1 &= v \\ x_2 &= \dot{v} \\ &\ldots \\ x_{n-1} &= v^{(n-2)} \\ x_n &= v^{(n-1)} \end{aligned}$$

The resulting state-variable equations are then formed as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\ldots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_n x_1 - a_{n-1}x_2 - \cdots - a_1 x_n + u \end{aligned} \tag{4.17}$$

Subsequently, the state equation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \ldots \\ x_{n-1} \\ x_n \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{bmatrix}_{n\times n}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}, \quad u = u \tag{4.18}$$

The state matrix in Equation 4.18 is called the lower companion matrix. The output is given by Equation 4.15. Using the state variables in Equation 4.15, we find

$$\begin{aligned} y &= b_0 v^{(n)} + b_1 v^{(n-1)} + \cdots + b_{n-1}\dot{v} + b_n v \\ &= b_0 \dot{x}_n + b_1 x_n + \cdots + b_{n-1} x_2 + b_n x_1 \end{aligned}$$

Because the output equation cannot contain $\dot{x}_n$, we substitute for $\dot{x}_n$ using the last relation in Equation 4.17. The result is

$$
\begin{aligned}
y &= b_0(-a_n x_1 - a_{n-1}x_2 - \cdots - a_1 x_n + u) + b_1 x_n + \cdots + b_{n-1}x_2 + b_n x_1 \\
&\overset{\text{collect like terms}}{=} (-b_0 a_n + b_n)x_1 + (-b_0 a_{n-1} + b_{n-1})x_2 + \cdots + (-b_0 a_1 + b_1)x_n + b_0 u
\end{aligned}
\tag{4.19}
$$

Finally, the output equation is obtained as

$$y = \mathbf{Cx} + Du$$

where

$$\mathbf{C} = [-b_0 a_n + b_n \quad -b_0 a_{n-1} + b_{n-1} \quad \ldots \quad -b_0 a_1 + b_1]_{1\times n}, \quad D = b_0 \tag{4.20}$$

Equations 4.18 and 4.20 describe all four matrices involved in the state-space form.

4.4.1.1 Controller Canonical Form

The process of obtaining state-space form from I/O equations can be handled in MATLAB using the "`tf2ss`" command. It calls for the transfer function, which is available from the I/O equation, and returns the state-space form in controller canonical form. This form is different from that in Equations 4.18 and 4.20 because it is based on the state variables selected in the reverse order of what we have become accustomed to. As a result, the state and input matrices appear as

$$
\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} & -a_n \\ 1 & 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 \end{bmatrix}_{n\times n}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ \ldots \\ 0 \\ 0 \end{bmatrix}_{n\times 1}
$$

When the state matrix is in the above form, it is known as the upper companion matrix. Note that the resulting input matrix in the controller canonical form also differs from that in Equation 4.18.

Example 4.14: I/O Equation to State-Space Form

Find the state-space form from the I/O equation

$$\dddot{y} + \ddot{y} + \dot{y} + 2y = 2\dot{u} + u$$

Solution

Comparing with Equation 4.14, we have $n = 3$, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $b_0 = 0$, $b_1 = 0$, $b_2 = 2$, $b_3 = 1$. By Equation 4.18,

$$
\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad u = u
$$

By Equation 4.20,

$$\mathbf{C} = [1 \;\; 2 \;\; 0], \quad D = 0$$

The state-space form is therefore obtained as

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y = [1 \;\; 2 \;\; 0]\mathbf{x} \end{cases}$$

Example 4.15: I/O Equation to State-Space Form

A system's I/O equation is provided as

$$\ddot{y} + 2\dot{y} + y = \ddot{u} + 3\dot{u} + 2u$$

a. Find the state-space representation.
b. Repeat in MATLAB.

Solution

a. By comparison with Equation 4.14, we find $n = 2$, $a_1 = 2$, $a_2 = 1$, $b_0 = 1$, $b_1 = 3$, $b_2 = 2$. Therefore, by Equation 4.18,

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u = u$$

Equation 4.20 yields

$$\mathbf{C} = [1 \;\; 1], \quad D = 1$$

Therefore, the state-space form is derived as

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = [1 \;\; 1]\mathbf{x} + u \end{cases}$$

b. The transfer function is directly obtained from the I/O equation,

$$\frac{s^2 + 3s + 2}{s^2 + 2s + 1}$$

```
>> Num = [1 3 2];    % Define numerator
>> Den = [1 2 1];    % Define denominator
>> [A,B,C,D] = tf2ss(Num,Den)
A =
    -2   -1
     1    0      % Controller canonical form
B =
     1
     0
C =
     1    1
D =
     1
```

4.4.2 State-Space Form to Transfer Matrix

The transfer function (for SISO systems) or transfer matrix (for MIMO systems) can be systematically derived from the state-space form. Consider the state-space form as in Equation 4.10,

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_{n\times n}\mathbf{x}_{n\times 1} + \mathbf{B}_{n\times m}\mathbf{u}_{m\times 1} \\ \mathbf{y}_{p\times 1} = \mathbf{C}_{p\times n}\mathbf{x}_{n\times 1} + \mathbf{D}_{p\times m}\mathbf{u}_{m\times 1} \end{cases}$$

Because the system has m inputs and p outputs, there are a total of mp transfer functions and the transfer matrix $\mathbf{G}(s)$ is $p \times m$, which will be derived shortly. First, we note that the Laplace transform of a vector such as $\mathbf{x}$ is handled as follows:

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{Bmatrix} \xrightarrow{\text{Laplace transform}} \mathcal{L}\{\mathbf{x}\} = \mathbf{X}(s) = \begin{Bmatrix} \mathcal{L}\{x_1\} \\ \mathcal{L}\{x_2\} \\ \cdots \\ \mathcal{L}\{x_n\} \end{Bmatrix}$$

Assuming zero initial state vector, $\mathbf{x}(0) = \mathbf{0}_{n\times 1}$, Laplace transformation of the state-space form yields

$$\begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

The first equation is manipulated as

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) \xrightarrow[(s\mathbf{I}-\mathbf{A})^{-1}]{\text{Premultiply by}} \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

Inserting this into the second equation results in

$$\begin{aligned} \mathbf{Y}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) \\ &= [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) \end{aligned}$$

For a SISO system with input u and output y, the transfer function is $G(s) = Y(s)/U(s)$ so that $Y(s) = G(s)U(s)$. This idea can be extended to MIMO systems with input vector $\mathbf{u}$ and output vector $\mathbf{y}$. The above relation, $\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s)$, suggests that the transfer matrix is defined as

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{4.21}$$

The fact that the size of $\mathbf{G}(s)$ is $p \times m$ can also be easily verified.

Example 4.16: State-Space Form to Transfer Function

A system's state-space representation is

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} + Du \end{cases}$$

with

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \quad \mathbf{C} = [0 \quad 1], \quad D = 0, \quad u = u$$

a. Find the transfer function.
b. Repeat in MATLAB.

Solution

a. Because u and y are both 1×1, the system is SISO; hence, there is only one transfer function. Given the matrix sizes, $\mathbf{G}(s)$ in Equation 4.21 is easily confirmed to be 1×1 and simply denoted by $G(s)$. Noting $D = 0$, Equation 4.21 reduces to

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

Using the adjoint matrix (Section 3.1), we find

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + s + 3}\begin{bmatrix} s+1 & 1 \\ -3 & s \end{bmatrix}$$

With this, the transfer function is obtained as

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = [0 \quad 1]\frac{1}{s^2 + s + 3}\begin{bmatrix} s+1 & 1 \\ -3 & s \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} = \frac{s}{2(s^2 + s + 3)}$$

b. The above procedure can be followed step-by-step in MATLAB:

```
>> A = [0 1;-3 -1]; B = [0;1/2]; C = [0 1]; syms s;
>> TF = C*inv(s*eye(2)-A)*B     % Transfer function
TF =
s/(2*(s^2 + s + 3))
```

The other option is to use the "ss2tf" command—state-space to transfer function—which directly finds the transfer function or transfer matrix from the state-space description.

```
>> A = [0 1;-3 -1]; B = [0;1/2]; C = [0 1]; D = 0;
>> [Num, Den] = ss2tf(A,B,C,D)       % Num = numerator, Den = denominator
Num =
          0   0.5000   -0.0000     % Num = s/2
Den =
     1.0000   1.0000    3.0000     % Den = s^2 + s + 3
```

Example 4.17: State-Space Form to Transfer Matrix

The state-space form for a system model is

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} = \mathbf{Cx} + \mathbf{Du} \end{cases}$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -2 & 0 \\ -1 & 2 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \mathbf{0}_{2\times 2}$$

Find the transfer matrix using

a. Equation 4.21.
b. The "ss2tf" command.

Solution

It is observed that $\mathbf{y}$ and $\mathbf{u}$ are both 2×1; hence there are two outputs and two inputs and the transfer matrix $\mathbf{G}(s)$ is 2×2.

a.

```
>> A = [0 0 1;1 -2 0;-1 2 -1]; B = [0 0;2 0;0 1];
>> C = [1 0 0;0 0 1]; D = zeros(2,2); syms s;
>> TM = C*inv(s*eye(3)-A)*B

TM =
[4/(s^3 + 3*s^2 + 3*s), (s + 2)/(s^3 + 3*s^2 + 3*s)]
[      4/(s^2 + 3*s + 3),        (s + 2)/(s^2 + 3*s + 3)]
```

The denominators in the second row entries, in comparison with the first row, suggest that there has been a cancellation of s terms, and the transfer matrix is actually

$$\mathbf{G}(s) = \frac{1}{s^3 + 3s^2 + 3s} \begin{bmatrix} 4 & s+2 \\ 4s & s^2 + 2s \end{bmatrix}$$

b.

```
>> [Num1, Den1] = ss2tf(A,B,C,D,1)      % Contribution by input 1
Num1 =
       0         0         0    4.0000     % 4
       0         0    4.0000   -0.0000     % 4s
Den1 =
  1.0000    3.0000    3.0000   -0.0000
>> [Num2, Den2] = ss2tf(A,B,C,D,2)       % Contribution by input 2
Num2 =
       0         0    1.0000    2.0000     % s + 2
       0    1.0000    2.0000   -0.0000     % s2 + 2s
Den2 =
  1.0000    3.0000    3.0000   -0.0000
```

The two rows returned in `Num1` translate to 4 and $4s$. These represent the numerators in the two transfer functions relating the first input to the two outputs. Recall from previous work that these two transfer functions occupy the first column of the transfer matrix. The two rows returned in `Num2` translate to $s + 2$ and $s^2 + 2s$, which are the numerators in the two transfer functions relating the second input to the two outputs. These two transfer functions occupy the second column of the transfer matrix. Therefore, the transfer matrix is formed as

$$\mathbf{G}(s) = \frac{1}{s^3 + 3s^2 + 3s}\begin{bmatrix} 4 & s+2 \\ 4s & s^2 + 2s \end{bmatrix}$$

PROBLEM SET 4.4

In Problem 1 through 6, find the state-space form directly from the I/O equation.

1. $\ddot{y} + \frac{1}{3}\dot{y} + y = 3u$
2. $2\ddot{y} + \dot{y} + y = \dot{u}$
3. $2\dddot{y} + \ddot{y} + 3\dot{y} + y = \ddot{u} + 2u$
4. $4\ddot{y} + \dot{y} + y = 4\ddot{u} + u$
5. $\dddot{y} + 2\ddot{y} + \dot{y} + 3y = 2\dddot{u} + 3\dot{u}$
6. $3\dddot{y} + 5\ddot{y} + 2\dot{y} + y = 2u$

In Problems 7 through 12, given the transfer function $Y(s)/U(s)$, find

a. The I/O equation.
b. The state-space form directly from the I/O equation in Part (a).

7. $\dfrac{2s+1}{s^2+3s+1}$
8. $\dfrac{1}{2s^2+s+2}$
9. $\dfrac{s^2+2}{2s^3+s^2+s+3}$
10. $\dfrac{s^2+1}{3s^2+s+1}$

11. $\dfrac{s^3+s}{s^3+s+2}$

12. $\dfrac{s^2+2s}{s^2+1}$

In Problems 13 through 18, given matrices **A**, **B**, **C**, and **D** in the state-space description of a system model, find the transfer function or transfer matrix using

a. Equation 4.21.
b. The "`ss2tf`" command in MATLAB.

13. $\mathbf{A}=\begin{bmatrix}0 & 1\\ -\frac{3}{4} & 0\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0\\ \frac{1}{3}\end{bmatrix},\ \mathbf{C}=[1\ \ 0],\ D=0$

14. $\mathbf{A}=\begin{bmatrix}0 & 1\\ -\frac{1}{2} & -1\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0\\ 2\end{bmatrix},\ \mathbf{C}=[0\ \ 1],\ D=0$

15. $\mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2}\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0\\ 0\\ \frac{3}{2}\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\ \mathbf{D}=\mathbf{0}_{2\times 1}$

16. $\mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0 & 0\\ \frac{1}{2} & 0\\ 0 & 1\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix},\ \mathbf{D}=\mathbf{0}_{2\times 2}$

17. $\mathbf{A}=\begin{bmatrix}0 & 1\\ -\frac{1}{3} & -\frac{2}{3}\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0\\ \frac{2}{3}\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}-\frac{1}{9} & -\frac{2}{9}\end{bmatrix},\ D=\frac{1}{3}$

18. $\mathbf{A}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0\\ 2\end{bmatrix},\ \mathbf{C}=[-1\ \ 2],\ D=2$

In Problems 19 through 22, given the I/O equation, find

a. The state-space form.
b. The transfer function from the state-space form in Part (a).
c. The transfer function from the given I/O equation and compare with Part (b).

19. $\ddot{y}+2\dot{y}+3y=\dot{u}+3u$
20. $\ddot{y}+2y=\ddot{u}+\dot{u}+2u$
21. $2\ddot{y}+\dot{y}+y=\ddot{u}+2\mathrm{u}$
22. $\dddot{y}+\ddot{y}+y=\dot{u}+3u$
23. The state-variable equations and the output equation for a dynamic system are given as

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=-x_2-x_1+u\end{cases},\quad y=x_1+2x_2$$

Find the transfer function (or transfer matrix) by determining the Laplace transforms of x_1 and x_2 in the state-variable equations and using them in the Laplace transform of the output equation.

24. Repeat Problem 23 for

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{1}{2}x_2 - x_1 + \frac{1}{2}u \end{cases}, \quad \mathbf{y} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

4.5 BLOCK DIAGRAM REPRESENTATION

A block diagram that represents a dynamic system is an interconnection of blocks, each block corresponding to an operation carried out by a component in such a way that the block diagram as a whole agrees with the mathematical model of the system. Each block is identified with a transfer function $G(s) = O(s)/I(s)$, also called the gain of the block, as shown in Figure 4.6. The output of the block is therefore

$$O(s) = G(s)I(s)$$

4.5.1 Block Diagram Operations

The principal operations in block diagrams include signal amplification, algebraic summation of signals, integration of signals, replacing series and parallel block combinations with equivalent single blocks, and treatment of loops.

4.5.1.1 Summing Junction

The output of a summing junction (or summer) is the algebraic sum of signals entering the summing junction. Each signal is accompanied by a positive or negative sign (see Figure 4.7). A summing junction may have as many inputs (with the same units) as desired, but only one single output.

4.5.1.2 Series Combinations of Blocks

Consider two blocks, with transfer functions $G_1(s)$ and $G_2(s)$, in a series combination as in Figure 4.8. The block $G_1(s)$ has input $U(s)$ and output $X(s)$, which is the input to the block $G_2(s)$. The output of block $G_2(s)$ is $Y(s)$. Therefore, the overall input is $U(s)$, whereas the overall output is $Y(s)$. The goal is to replace this arrangement with a single block that has $U(s)$ as input and $Y(s)$ as output, that is, a single block with transfer function $Y(s)/U(s)$.

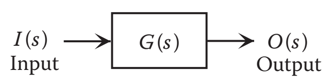


FIGURE 4.6 Schematic of a transfer function block.

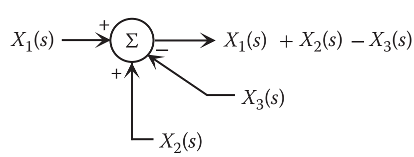


FIGURE 4.7 Summing junction.

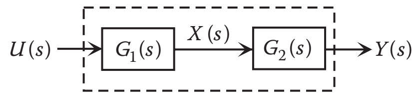


FIGURE 4.8 Blocks in series.

The output of the second block is $Y(s) = G_2(s)X(s)$. But because $X(s)$ is the output of the first block, and given by $X(s) = G_1(s)U(s)$, we have

$$Y(s) = G_2(s)X(s) = G_2(s)\,[G_1(s)U(s)] = [G_1(s)G_2(s)]U(s)$$

Therefore,

$$\frac{Y(s)}{U(s)} = G_1(s)G_2(s)$$

The series configuration in Figure 4.8 can thus be replaced with a single block $G_1(s)G_2(s)$, overall input $U(s)$ and overall output $Y(s)$ (Figure 4.9).

Example 4.18: Blocks in Series

Consider two blocks in a series connection with transfer functions

$$G_1(s) = \frac{1}{2s+3}, \quad G_2(s) = \frac{s+1}{s^2+s+2}$$

The transfer function of the equivalent single block is determined in MATLAB as follows:

```
>> Num1 = 1; Den1 = [2 3]; Num2 = [1 1]; Den2 = [1 1 2];
>> sysG1 = tf(Num1, Den1);         % Define system w/transfer function G1(s)
>> sysG2 = tf(Num2, Den2);         % Define system w/transfer function G2(s)
>> sysEq = series(sysG1,sysG2)     % Find the equivalent single TF

Transfer function:
         s + 1
_ _ _ _ _ _ _ _ _ _ _ _  -
2 s^3 + 5 s^2 + 7 s + 6
```

Of course, the result agrees with $G_1(s)G_2(s) = \dfrac{s+1}{(2s+3)(s^2+s+2)}$.

4.5.1.3 Parallel Combinations of Blocks

Consider two blocks with transfer functions $G_1(s)$ and $G_2(s)$ in a parallel combination as shown in Figure 4.10. Once again, the objective is to replace the arrangement with a single block that has $U(s)$ as input and $Y(s)$ as output, that is, a single block with transfer function $Y(s)/U(s)$. Note that the point

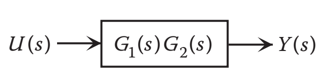


FIGURE 4.9 A single block replacing two blocks in series.

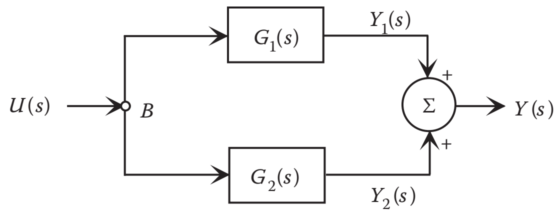


FIGURE 4.10 Blocks in parallel.

B in Figure 4.10 is called a branch point. The outputs of the two blocks, labeled $Y_1(s)$ and $Y_2(s)$, are simply $Y_1(s) = G_1(s)U(s)$ and $Y_2(s) = G_2(s)U(s)$. These two are the inputs to the summing junction, whose output is then calculated as

$$Y(s) = Y_1(s) + Y_2(s) = G_1(s)U(s) + G_2(s)U(s) = [G_1(s) + G_2(s)]\ U(s)$$

Therefore,

$$\frac{Y(s)}{U(s)} = G_1(s) + G_2(s)$$

The parallel configuration in Figure 4.10 can thus be replaced with a single block $G_1(s) + G_2(s)$, overall input $U(s)$ and overall output $Y(s)$ (Figure 4.11).

Example 4.19: Blocks in Parallel

Consider two blocks in a parallel connection with transfer functions

$$G_1(s) = \frac{3}{s+1}, \quad G_2(s) = \frac{2s+1}{s^2+2}$$

The transfer function for the equivalent single block is found in MATLAB as follows:

```
>> Num1 = 3; Den1 = [1 1]; Num2 = [2 1]; Den2 = [1 0 2];
>> sysG1 = tf(Num1, Den1);
>> sysG2 = tf(Num2, Den2);
>> sysEq = parallel(sysG1,sysG2)

Transfer function:
  5 s^2 + 3 s + 7
— — — — — — — — — —
s^3 + s^2 + 2 s + 2
```

Of course, the result agrees with $G_1(s) + G_2(s)$.

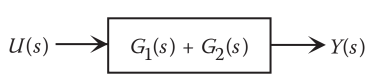


FIGURE 4.11 A single block replacing two blocks in parallel.

4.5.1.4 Integration

Consider a signal $u(t)$ integrated form initial time 0 to the current time t to produce

$$y(t) = \int_0^t u(t)\,dt$$

Then, as we learned in Section 2.3,

$$Y(s) = \frac{1}{s}U(s) \quad \xrightarrow[\text{transfer function}]{\text{Integrator}} \quad \frac{Y(s)}{U(s)} = \frac{1}{s}$$

An integrator is therefore represented by a single block with transfer function $\frac{1}{s}$ as in Figure 4.12.

4.5.1.5 Closed-Loop Systems

Figure 4.13 shows a closed-loop (or feedback) system. The input is $U(s)$. The output $Y(s)$ is fed back through the feedback element $H(s)$, whose output $C(s)$ is compared with the input $U(s)$ at the summing junction. The difference, $E(s) = U(s) - C(s)$, is known as the error signal. Because of the negative sign associated with $C(s)$ at the summing junction, the configuration in Figure 4.13 is known as a negative feedback system.

There are two important transfer functions in the closed-loop system in Figure 4.13:

$$\text{feed-forward transfer function} = \frac{Y(s)}{E(s)} = G(s)$$

and

$$\text{open-loop transfer function} = \frac{C(s)}{E(s)} = G(s)H(s)$$

This last relation can be verified by noting that $Y(s) = E(s)G(s)$ is the input to the feedback element $H(s)$, thus $C(s) = H(s)Y(s) = H(s)E(s)G(s)$.

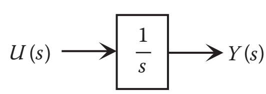


FIGURE 4.12 Integrator.

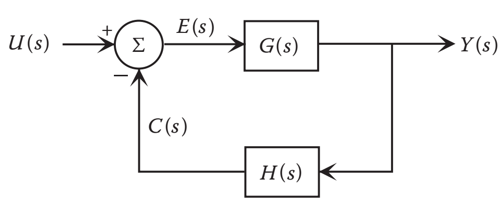


FIGURE 4.13 Schematic of a negative feedback system.

4.5.1.5.1 Closed-Loop Transfer Function

The closed-loop transfer function (CLTF) provides the direct relation between the input $U(s)$ and the output $Y(s)$, and is determined as follows (see Figure 4.13):

$$Y(s) = G(s)E(s) \overset{E(s)=U(s)-C(s)}{=} G(s)[U(s) - C(s)] \overset{C(s)=H(s)Y(s)}{=} G(s)[U(s) - H(s)Y(s)]$$

Manipulating the above equation, we find

$$[1 + G(s)H(s)]Y(s) = G(s)U(s)$$

Finally, the CLTF is formed as

$$\text{Negative feedback} \quad \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{4.22}$$

Similarly, in the case of a positive feedback, it can be easily shown that

$$\text{Positive feedback} \quad \frac{Y(s)}{U(s)} = \frac{G(s)}{1 - G(s)H(s)} \tag{4.23}$$

Example 4.20: Negative Feedback

Consider the negative feedback system shown in Figure 4.13, and assume that

$$G(s) = \frac{1}{s^2 + 2s + 2}, \quad H(s) = s + 1$$

a. Find the CLTF.
b. Repeat in MATLAB.

Solution

a. By Equation 4.22,

$$\frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{1}{s^2+2s+2}}{1 + \frac{s+1}{s^2+2s+2}} = \frac{1}{s^2 + 3s + 3}$$

b.

```
>> NumG = 1; DenG = [1 2 2]; sysG = tf(NumG,DenG);
>> NumH = [1 1]; DenH = 1; sysH = tf(NumH,DenH);
>> sysEq = feedback(sysG,sysH)

Transfer function:
     1
_ _ _ _ _ _ _
s^2 + 3 s + 3
```

4.5.2 Block Diagram Reduction Techniques

As dynamic systems become more complex in nature, so do their block-diagram representations. In these situations, the block diagram can potentially contain several summing junctions, blocks in series or parallel connections, and positive and negative feedback loops. There are a few basic rules that facilitate the process of simplifying a block diagram. These include, among others, moving a branch point and moving a summing junction, as explained below.

4.5.2.1 Moving a Branch Point

Consider the branch point B in Figure 4.14a, located to the left of the block $G(s)$. The branch point may be moved to the right side of the block $G(s)$ as demonstrated in Figure 4.14b. The key is for signals $Y_1(s)$ and $Y_2(s)$ to carry the same information before and after B is moved. It is readily seen that $Y_1(s) = G(s)U(s)$ in both arrangements. Also $Y_2(s) = U(s)$ before B was moved, and $Y_2(s) = [1/G(s)]G(s)U(s) = U(s)$ after the move.

4.5.2.2 Moving a Summing Junction

Consider the summing junction in the arrangement shown in Figure 4.15a. The summing junction may be moved to the left side of the block $G_1(s)$ as demonstrated in Figure 4.15b. Once again, the key is for the signal $Y(s)$ to carry the same information before and after the summing junction is moved. It is easily seen that $Y(s) = G_1(s)U_1(s) + G_2(s)U_2(s)$ in Figure 4.15a. In Figure 4.15b, we have

$$Y(s) = G_1(s)\left[U_1(s) + \frac{G_2(s)}{G_1(s)}U_2(s)\right] = G_1(s)U_1(s) + G_2(s)U_2(s)$$

This validates the equivalence of the two configurations.

Example 4.21: Block Diagram Reduction

Using reduction techniques, simplify the block diagram shown in Figure 4.16 to a single block with input $U(s)$ and output $Y(s)$, and determine the overall transfer function $Y(s)/U(s)$.

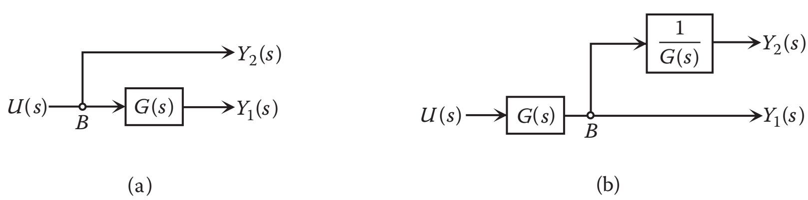


FIGURE 4.14 (a) Branch point and (b) branch point moved.

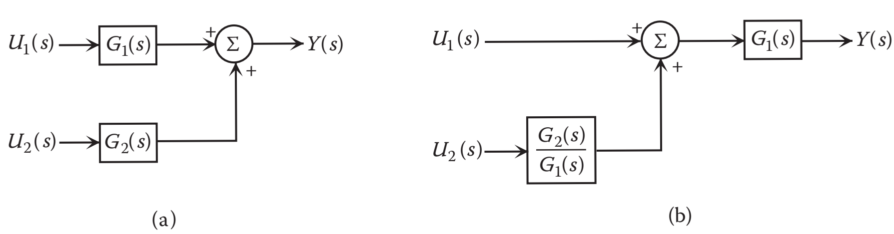


FIGURE 4.15 (a) Summing junction and (b) summing junction moved.

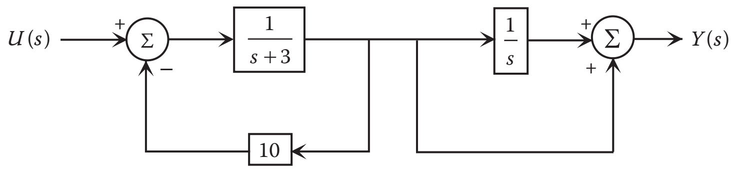


FIGURE 4.16 Block diagram in Example 4.21.

Solution

There are several ways this block diagram can be reduced. The end result, however, is independent of the choices of reduction techniques and the order in which they are used. We will simplify it as follows. The block diagram is composed of a negative feedback and a parallel connection, as shown in Figure 4.17. The negative feedback is replaced with a single block with a transfer function

$$\frac{\frac{1}{s+3}}{1+\frac{10}{s+3}} = \frac{1}{s+13}$$

The parallel connection consists of blocks $1/s$ and 1, hence replaced with a single block

$$\frac{1}{s}+1=\frac{s+1}{s}$$

These two single blocks are shown in Figure 4.18. The series connection in Figure 4.18 is next replaced with a single block whose transfer function is the product of the individual block transfer functions, that is,

$$\frac{s+1}{s(s+13)}$$

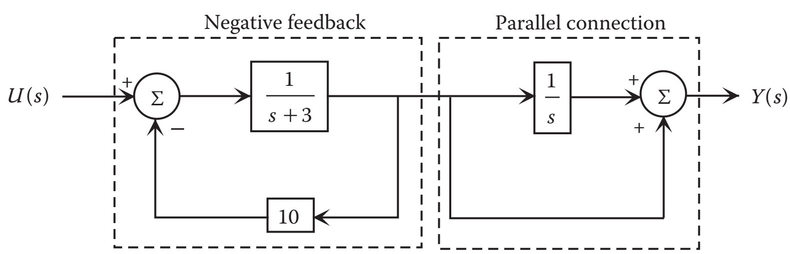


FIGURE 4.17 Feedback and parallel connection identified.

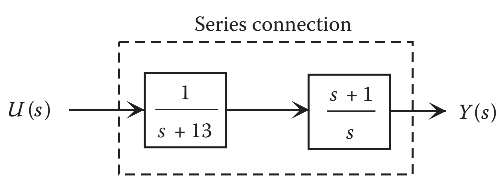


FIGURE 4.18 Feedback loop and parallel connection replaced.

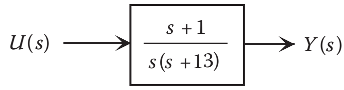


FIGURE 4.19 Simplified block diagram in Example 4.21.

This yields Figure 4.19, from which the overall transfer function is easily found as

$$\frac{Y(s)}{U(s)} = \frac{s+1}{s(s+13)}$$

4.5.2.3 Mason's Rule

Thus far, we have learned that when a block diagram contains several loops, each loop can be replaced with a single block with a transfer function given by either Equation 4.22 or Equation 4.23 for negative and positive feedbacks, respectively. This, in conjunction with other tactics mentioned earlier, can then help us find the overall transfer function for the block diagram. An alternative approach, however, is to employ Mason's rule, as outlined in the next subsection. We first define a forward path as one that originates from the overall input leading to the overall output, never moving in the opposite direction. A loop path (or a loop) is one that originates from a certain variable and returns to the same variable. The gain of a forward path or a loop path is the product of the gains of the individual blocks that constitute the path.

4.5.2.3.1 Mason's Rule: Special Case

Suppose that all forward paths and loops in a block diagram are coupled, that is, they all have a common segment. Then, the overall transfer function is determined as

$$\text{overall transfer function} = \frac{\sum \text{forward path gains}}{1 - \sum \text{loop gains}} \tag{4.24}$$

Example 4.22: Mason's Rule (Special Case)

Using Mason's rule, determine the overall transfer function $Y(s)/U(s)$ for the block diagram in Figure 4.20.

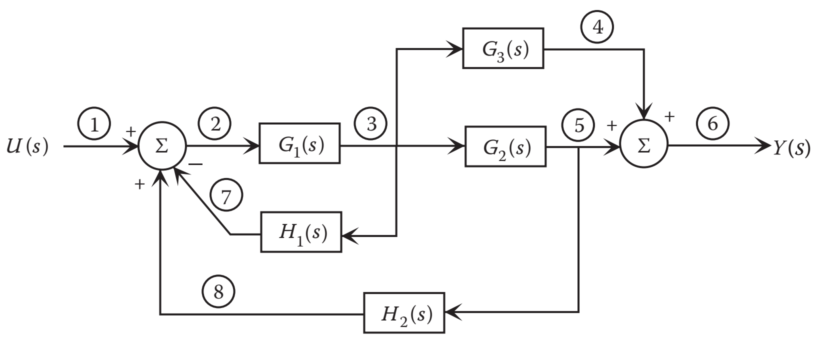


FIGURE 4.20 Block diagram in Example 4.22.

Solution

Path segments have been assigned numbers for easier identification. There are two forward paths and two loops:

Forward Path	Gain	Loop	Gain
12356	G_1G_2	2372	$-G_1H_1$
12346	G_1G_3	23582	$G_1G_2H_2$

Note that the loop labeled 2372 is a negative feedback, thus its gain is negative. Because all forward paths and loops have a common segment, labeled 23, the overall transfer function is found with Equation 4.24 as

$$\frac{Y(s)}{U(s)} = \frac{G_1G_2 + G_1G_3}{1 + G_1H_1 - G_1G_2H_2}$$

4.5.2.3.2 Mason's Rule: General Case

In general, when all forward paths and loops are not coupled, the overall transfer function is obtained as

$$\text{overall transfer function} = \frac{\sum_{k=1}^{m} F_k D_k}{D}, \quad m = \text{number of forward paths} \tag{4.25}$$

where

$F_k =$ gain of the kth forward path

$$D = 1 - \sum \text{single-loop gains} + \sum \text{gain products of all non-touching two-loops} - \sum \text{gain products of all non-touching three-loops} + \cdots$$

$D_k =$ same as D when the block diagram is restricted to the portion not touching the kth forward path

Example 4.23: Mason's Rule (General Case)

Using Mason's rule, find the overall transfer function $Y(s)/U(s)$ for the block diagram in Figure 4.21.

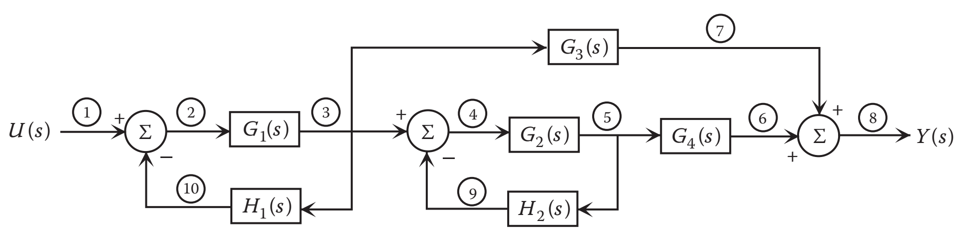


FIGURE 4.21 Block diagram in Example 4.23.

Solution

We note that not all forward paths are coupled; in particular, the two negative feedback loops do not share a common segment. Therefore, the overall transfer function must be found using Mason's general rule (Equation 4.25). Two forward paths and two loops are identified:

Forward Path	Gain	Loop	Gain
1234568	$G_1G_2G_4$	23(10)2	$-G_1H_1$
12378	G_1G_3	4594	$-G_2H_2$

The quantities in Equation 4.25 are calculated as follows:

$$F_1 = G_1G_2G_4, \quad F_2 = G_1G_3$$

$$D = 1 + \underbrace{G_1H_1 + G_2H_2}_{\text{Single-loop gains}} + \underbrace{G_1H_1G_2H_2}_{\substack{\text{Gain product of} \\ \text{nontouching two-loops}}}$$

To find D_1, consider the portion of the block diagram that does not touch the first forward path, 1234568. Because there are no forward paths or loops in the restricted segment, we conclude

$$D_1 = 1$$

To determine D_2, consider the portion of the block diagram that does not touch the second forward path, 12378. Because the restricted section contains only one single loop, 4594, we have

$$D_2 = 1 + G_2H_2$$

Finally, Equation 4.25 yields

$$\frac{Y(s)}{U(s)} = \frac{G_1G_2G_4 + G_1G_3[1 + G_2H_2]}{1 + G_1H_1 + G_2H_2 + G_1H_1G_2H_2}$$

4.5.3 Block Diagram Construction from System Model

Block diagrams disclose many characteristics of dynamic systems that may not be observable using their mathematical models, such as the interrelation between the different components and variables. In what follows, we will learn how to construct partial blocks corresponding to specific segments of a system model, and subsequently assemble them properly to generate the complete block diagram. Based on the block diagram, a model can then be constructed in Simulink® for analysis and simulation purposes. This process will be systematically used in the subsequent chapters.

Example 4.24: Block Diagram from System Model

A first-order system with input u and output y is governed by its I/O equation as

$$3\dot{y} + y = 2u$$

a. Construct a block diagram which includes a feedback loop.
b. Using the block diagram in Part (a), find the transfer function $Y(s)/U(s)$.

Solution

a. The overall input $U(s)$ must appear on the far left of the diagram and the overall output $Y(s)$ on the far right. Laplace transformation of the I/O equation yields

$$3sY(s) = 2U(s) - Y(s) \qquad sY(s) = \tfrac{1}{3}[2U(s) - Y(s)]$$

We first concentrate on the term $2U(s) - Y(s)$. The input $U(s)$ must go through a constant block of 2 to generate $2U(s)$. A summing junction with inputs $2U(s)$, with a positive sign, and the overall output $Y(s)$, with a negative sign, will produce the output $2U(s) - Y(s)$ (Figure 4.22a). This output subsequently goes through a constant block of $\frac{1}{3}$ to generate $\frac{1}{3}[2U(s) - Y(s)]$, which is $sY(s)$ (Figure 4.22b). Finally, $sY(s)$ goes through $1/s$, an integrator, to produce $Y(s)$. Ultimately, the block diagram can be completed by feeding back $Y(s)$ to the summing junction (Figure 4.22c).

b. Because there is only one forward path and one loop, and they are coupled, the special case of Mason's rule may be applied to find the transfer function as

$$\frac{Y(s)}{U(s)} = \frac{\frac{2}{3s}}{1 + \frac{1}{3s}} = \frac{2}{3s+1}$$

This clearly agrees with the transfer function directly obtained from the I/O equation.

Example 4.25: Block Diagram from System Model

A dynamic system with input f and output x is described by

$$\ddot{x} + 2\dot{x} + 3x = f(t)$$

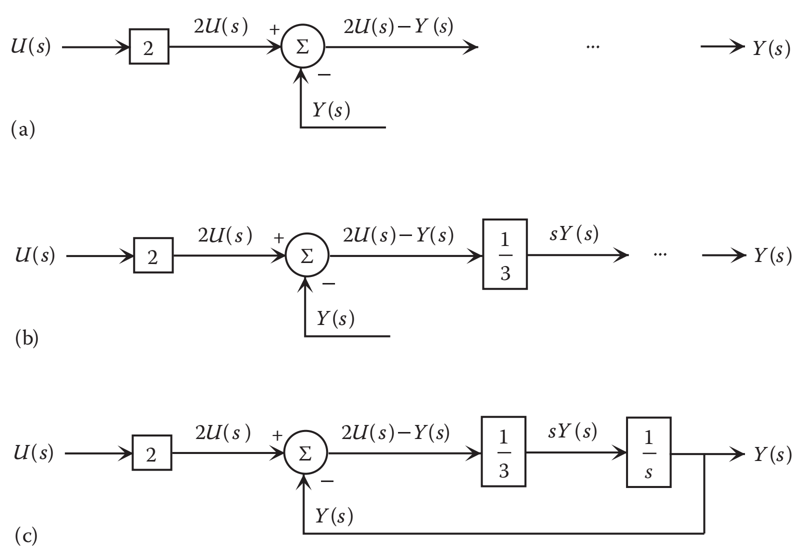


FIGURE 4.22 (a) Block diagram initiated, (b) partial block, and (c) diagram completed.

a. Derive the state-variable equations and the output equation.
b. Construct a block diagram using the information in Part (a).
c. Build a Simulink model based on the block diagram of Part (b).

Solution

a. The state variables are selected as $x_1 = x$ and $x_2 = \dot{x}$. Noting that the output is $y = x = x_1$, the state-variable equations and the output equation are derived as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -3x_1 - 2x_2 + f \end{cases}, \quad y = x_1$$

b. Taking the Laplace transform of the equations in Part (a), we have

$$\begin{cases} sX_1(s) = X_2(s) \\ sX_2(s) = -3X_1(s) - 2X_2(s) + F(s) \end{cases}, \quad Y(s) = X_1(s)$$

The overall input $F(s)$ must appear on the far left of the diagram, and the overall output $X_1(s)$ on the far right. The block diagram is built from the left to right; hence, we need to start with an equation that contains the overall input $F(s)$. Thus, we start with $sX_2(s) = -3X_1(s) - 2X_2(s) + F(s)$, which may be constructed by using a summing junction. The inputs to the summing junction are $F(s)$ with a positive sign, $3X_1(s)$ with a negative sign, and $2X_2(s)$ with a negative sign. Then, the output of the summing junction is $sX_2(s)$ (Figure 4.23a). This output

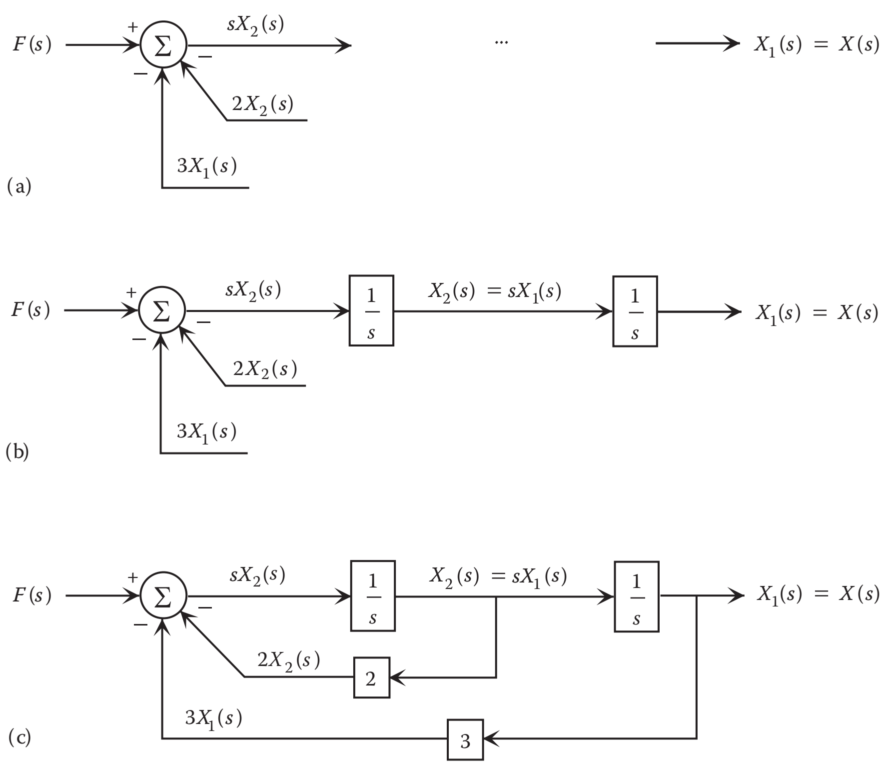


FIGURE 4.23 (a) Block diagram initiated, (b) partial block, and (c) diagram completed.

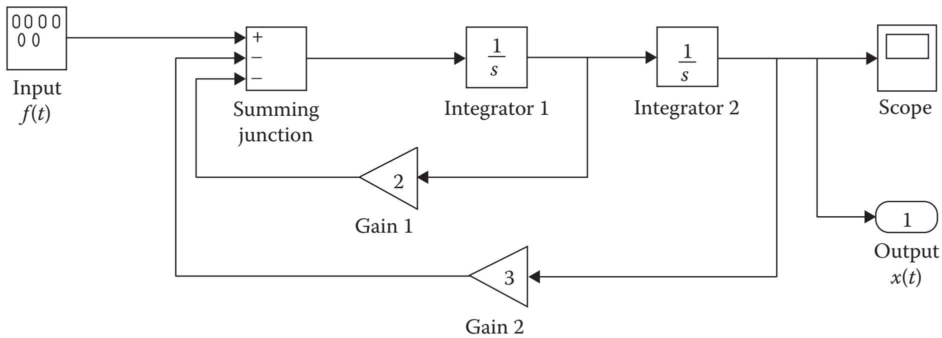


FIGURE 4.24 Simulink model in Example 4.25.

subsequently goes through a block of $1/s$, an integrator, to generate $X_2(s)$. But, $X_2(s) = sX_1(s)$. Therefore, $X_2(s)$ will go through yet another block of $1/s$ to produce $X_1(s)$ (Figure 4.23b). Because $X_1(s)$ is the overall output, we can complete the diagram as depicted in Figure 4.23c.

c. Figure 4.24 shows a Simulink model based on the block diagram in Figure 4.23c. The input is represented by a signal generator (Sources library). The summing junction is represented by the Add block (Math Operations library). The output is stored in port 1 (Commonly Used Blocks library), as well as in Scope for simulation purposes. Using this model, the output corresponding to a specified input is easily generated in MATLAB.

PROBLEM SET 4.5

1. Reduce the block diagram in Figure 4.25 by moving the constant block K to the right of the summing junction. Subsequently, find the transfer function $Y(s)/U(s)$.
2. For the block diagram in Figure 4.26, find the transfer function $Y(s)/U(s)$ using
   a. Block diagram reduction techniques.
   b. Mason's rule.
3. Find the overall transfer function of the block diagram in Figure 4.16, Example 4.21, using Mason's rule.

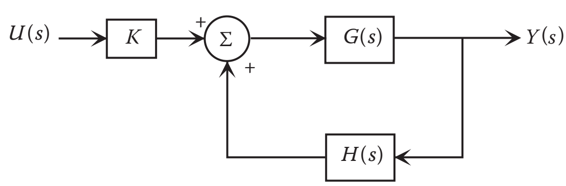


FIGURE 4.25 Problem 1.

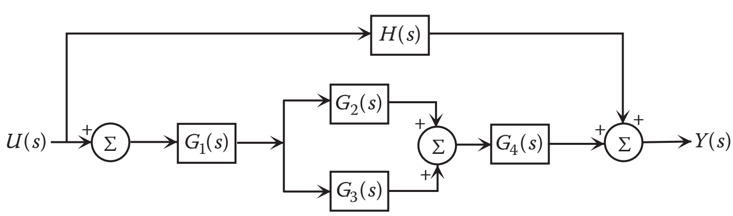


FIGURE 4.26 Problem 2.

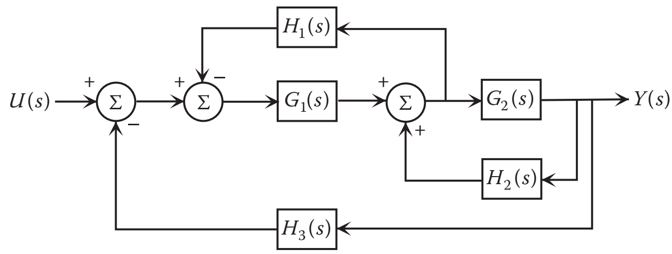


FIGURE 4.27 Problems 4 and 5.

4. Using Mason's rule, find $Y(s)/U(s)$ in Figure 4.27.
5. Consider the block diagram in Figure 4.27. Use block diagram reduction techniques listed below to find the overall transfer function.
   a. Move the summing junction of the positive feedback loop containing $H_2(s)$ outside of the negative loop containing $H_1(s)$.
   b. In the ensuing diagram, replace the loop containing $H_1(s)$ with a single block.
   c. Similarly, replace the two remaining loops with single equivalent blocks to obtain one single block whose input is $U(s)$ and whose output is $Y(s)$.
6. Using block diagram reduction techniques, find $Y(s)/U(s)$ in Figure 4.28.
7. Find $Y(s)/U(s)$ in Figure 4.28 using Mason's rule.
8. Find the overall transfer function in Figure 4.29 using Mason's rule.
9. For the block diagram in Figure 4.30, find $Y(s)/U(s)$ using
   a. Block diagram reduction techniques.
   b. Mason's rule.

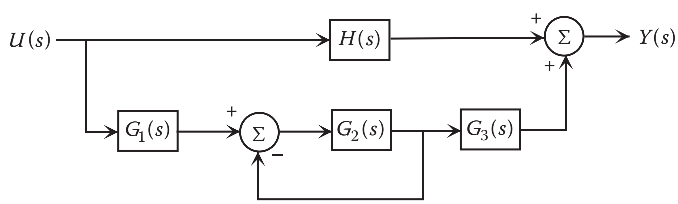


FIGURE 4.28 Problems 6 and 7.

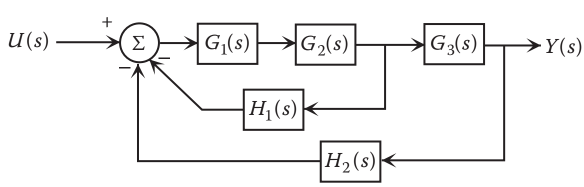


FIGURE 4.29 Problem 8.

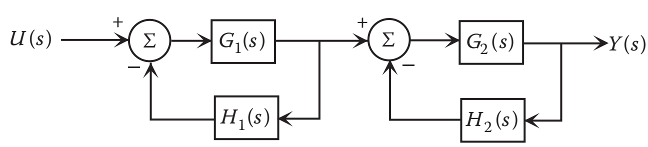


FIGURE 4.30 Problem 9.

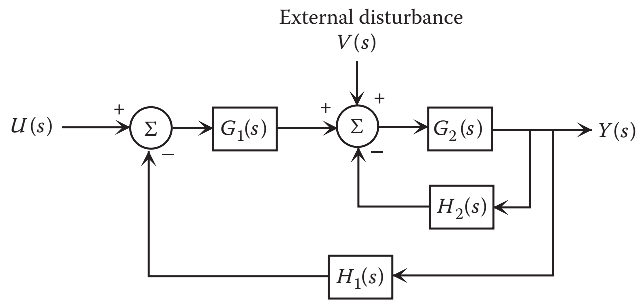


FIGURE 4.31 Problem 10.

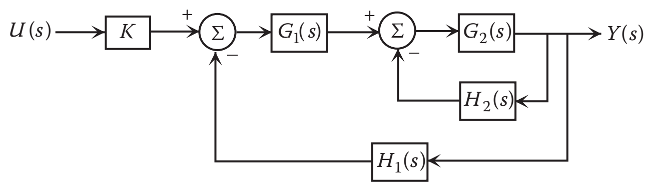


FIGURE 4.32 Problem 11.

10. Consider the block diagram in Figure 4.31 in which $V(s)$ represents an external disturbance.
    a. Find the transfer function $Y(s)/U(s)$ by setting $V(s) = 0$.
    b. Find $Y(s)/V(s)$ by setting $U(s) = 0$.
11. For the block diagram shown in Figure 4.32, where K is a constant, obtain the transfer function $Y(s)/U(s)$.
12. The I/O equation for a SISO system is described by

$$\dot{y} + \tfrac{1}{2}y = \tfrac{1}{3}u$$

    a. Construct a block diagram containing a feedback loop.
    b. Find the transfer function $Y(s)/U(s)$ directly from the block diagram.
13. The state-variable equations and output equation of a system are given as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - 2x_2 + u \end{cases}, \quad y = x_1 + 2x_2$$

    where u and y are the input and output, respectively.
    a. Build the block diagram.
    b. Find the transfer function $Y(s)/U(s)$ directly from the block diagram.
14. The state-space representation of a system model is given as

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} + Du \end{cases}$$

    with

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{1}{3} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}, \quad \mathbf{C} = [1 \quad 1], \qquad D = 0, \quad u = u$$

    a. Build the block diagram.
    b. Find the transfer function directly from the block diagram.

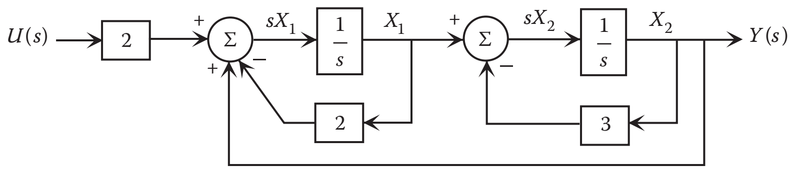


FIGURE 4.33 Problem 20.

In Problems 15 through 18, the system's I/O equation is provided.

a. Find the state-variable equations and the output equation.
b. Construct the block diagram based on the information in Part (a).
c. Determine the transfer function directly from the block diagram in Part (b).

15. $2\ddot{y}+3\dot{y}+y=\dot{u}$
16. $\ddot{y}+\dot{y}+2y=2\dot{u}+u$
17. $\dddot{y}+2\ddot{y}+y=\ddot{u}+\dot{u}$
18. $\dddot{y}+\dot{y}+2y=\ddot{u}+u$
19. A system is described by its transfer function

$$\frac{Y(s)}{U(s)}=\frac{s+1}{s(s^2+s+1)}$$

 a. Find the I/O equation.
 b. Find the state-space form from the I/O equation.
 c. Build a block diagram based on the information in Part (b). Find the transfer function directly from the block diagram and compare with the given transfer function.
20. The block diagram representation of a system model is presented in Figure 4.33, where U, X_1, X_2, and Y denote the Laplace transforms of the input, the two state variables, and the output.
 a. Derive the state-space form directly from the block diagram.
 b. Find the transfer function directly from the block diagram.

4.6 LINEARIZATION

Up to this point, we have mainly studied linear systems, whose analysis was somewhat straightforward. Many dynamic systems, however, contain elements that are inherently nonlinear, which cannot be treated as linear except for a constrained range of operating conditions. In this section, we present a systematic approach to derive a linear approximation of a nonlinear model. We will also learn how to linearize nonlinear elements in Simulink. In Section 8.5, we will see how the linearized model compares with the nonlinear model.

4.6.1 Linearization of a Nonlinear Element

Consider a nonlinear function f of a single variable x as shown in Figure 4.34a. The linearization of $f(x)$ will be done with respect to a specific point $P:(\bar{x},\bar{f})$ known as an operating point. For now, we assume that the (constant) values of $\bar{x}$ and $\bar{f}$ are available, but we will learn shortly how to determine an operating point for a given system.

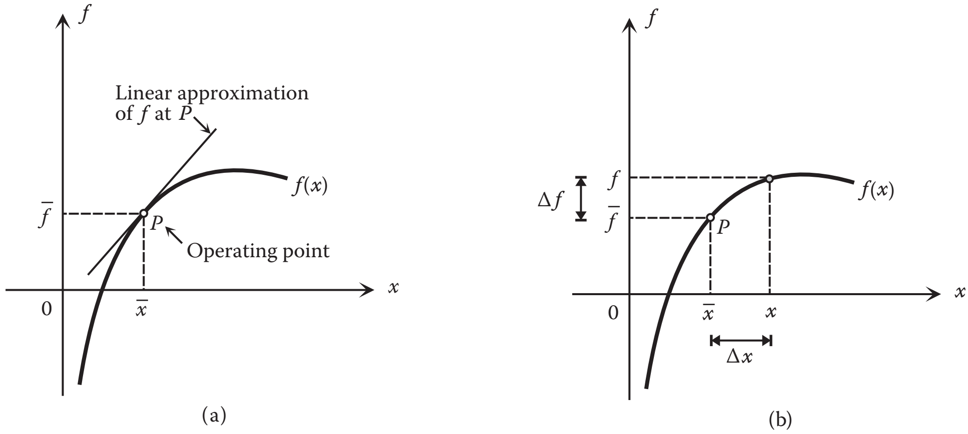


FIGURE 4.34 (a) Linearization about an operating point and (b) incremental variables.

The values $\bar{x}$ and $\bar{f} = f(\bar{x})$ are called the nominal values of x and f, respectively. As shown in Figure 4.34b, any point (x, f) on the graph of $f(x)$ can be expressed as

$$x(t) = \bar{x} + \Delta x(t), \quad f(t) = \bar{f} + \Delta f(t) \tag{4.26}$$

where the time-varying quantities $\Delta x(t)$ and $\Delta f(t)$ are the incremental variables for x and f. Graphically, the linear approximation of $f(x)$ is provided by the tangent line to the curve at the operating point P, with a reasonably good accuracy in a small neighborhood of P, that is, small values for $\Delta x(t)$ and $\Delta f(t)$. Analytically, this is justified by writing the Taylor series expansion of $f(x)$ about the operating point as

$$f(x) = \underbrace{f(\bar{x}) + \left.\frac{df}{dx}\right|_P (x - \bar{x})}_{\text{Linear terms}} + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_P (x - \bar{x})^2 + \ldots$$

Assuming $\Delta x = x - \bar{x}$ is small, the linear approximation of $f(x)$ is achieved by retaining the first two terms, while neglecting the remaining higher powers of $x - \bar{x}$. Therefore,

$$f(x) \cong f(\bar{x}) + \left.\frac{df}{dx}\right|_{\bar{x}} \Delta x \tag{4.27}$$

Example 4.26: Nonlinear Function of a Single Variable

Linearize $f(x) = x|x|$ about the operating point P:(2,4). Examine the accuracy of the linear approximation for $x = 2.1$ and $x = 1.9$.

Solution

We first note that

$$f(x) = \big|x\big| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases} \quad \xrightarrow{\text{Differentiate}} \quad \frac{df}{dx} = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

Because $\bar{x} = 2$, Equation 4.27 yields

$$f(x) \cong f(2) + [2x]_{x=2}\Delta x = 4 + 4\Delta x$$

For $x = 2.1$, we have $x = x - \bar{x} = 2.1 - 2 = 0.1$ and thus

$$f(2.1) \cong 4 + 4(0.1) = 4.40 \quad [\text{Exact value} = (2.1)^2 = 4.41]$$

For $x = 1.9$, we have $\Delta x = 1.9 - 2 = -0.1$, hence

$$f(1.9) \cong 4 + 4(-0.1) = 3.60 \quad [\text{Exact value} = (1.9)^2 = 3.61]$$

Therefore, as expected, if x is sufficiently close to $\bar{x}$, the approximation has a reasonably good accuracy.

4.6.1.1 Functions of Two Variables

Consider a nonlinear function $f(x,y)$ of two independent variables. In this case, the operating point is represented by $(\bar{x}, \bar{y}, \bar{f})$ and the incremental variables are defined as

$$\Delta x(t) = x(t) - \bar{x}, \quad \Delta y(t) = y(t) - \bar{y}, \quad \Delta f(t) = f(t) - \bar{f}$$

Taylor series expansion about the operating point yields

$$f(x,y) = \underbrace{f(\bar{x},\bar{y}) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{y}}(x-\bar{x}) + \left.\frac{\partial f}{\partial y}\right|_{\bar{x},\bar{y}}(y-\bar{y})}_{\text{Linear terms}}$$
$$+\frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{\bar{x},\bar{y}}(x-\bar{x})^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial y^2}\right|_{\bar{x},\bar{y}}(y-\bar{y})^2 + \left.\frac{\partial^2 f}{\partial x\,\partial y}\right|_{\bar{x},\bar{y}}(x-\bar{x})(y-\bar{y}) + \ldots$$

Assuming $x - \bar{x}$ and $y - \bar{y}$ are small, the linear approximation of $f(x,y)$ in a small neighborhood of the operating point is given by

$$f(x,y) \cong f(\bar{x},\bar{y}) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{y}}\Delta x + \left.\frac{\partial f}{\partial y}\right|_{\bar{x},\bar{y}}\Delta y \tag{4.28}$$

4.6.2 Linearization of a Nonlinear Model

Linearization of the model of a nonlinear system can be performed systematically by following a standard procedure outlined shortly. We first need to elaborate on the determination of the operating point(s), which is a significant part of the said procedure.

4.6.2.1 Operating Point

To find the operating point, we first replace the dependent variables such as $x(t)$ with $\bar{x} = \text{const}$ to be determined. It is also desired that a system operates not far from its equilibrium state. For that, we set the time-varying portion of the input $u(t)$ to zero. The result is an algebraic equation that can be solved for variables such as $\bar{x}$. Note that a system may have more than one operating point.

Example 4.27: Operating Point

Find the operating point(s) for a nonlinear system whose model is described by

$$\ddot{x} + \dot{x} + x|x| = 1 + \sin t$$

Solution

Note that the input here is 1 + sin$t$. Because $x(t)$ is the only dependent variable, the operating point is identified by $\bar{x}$. For the purpose of finding the operating point, we set the time-varying portion of the input, which is sin$t$, equal to zero. To find the operating point, we replace x with $\bar{x}$ and the input with 1 in the given system model. Because $\bar{x} = \text{const}$, we have $\ddot{\bar{x}} = 0$ and $\dot{\bar{x}} = 0$. Therefore,

$$\bar{x}|\bar{x}| = 1 \tag{a}$$

We solve this equation as follows:

Case (1)—If $\bar{x} > 0$, then $|\bar{x}| = \bar{x}$ and Equation a reduces to $\bar{x}^2 = 1$. This has two solutions $\bar{x} = \pm 1$. But the assumption in this case is $\bar{x} > 0$ so that only the positive solution is acceptable, and

$$\bar{x} = 1$$

Case (2)—If $\bar{x} < 0$, then $|\bar{x}| = -\bar{x}$ and Equation a becomes $-\bar{x}^2 = 1$, which has no real solution. Therefore, the only valid operating point corresponds to $\bar{x} = 1$.

4.6.2.2 Linearization Procedure

A nonlinear system model is linearized as follows:

1. Find the operating point as previously explained.
2. Linearize the nonlinear term(s) about the operating point with Taylor series expansions; Equation 4.27 for functions of a single variable and Equation 4.28 for two variables.
3. In the original nonlinear model, replace variables such as x with $\bar{x} + \Delta x$, nonlinear terms with their linear approximations of Step 2, and include the time-varying portions of the input that were previously set to zero to calculate the operating point in Step 1. The resulting system is linear in the incremental variables such as Δx.
4. Finally, use the initial conditions of the original model to calculate those for the linearized model. For instance, knowing $x(0)$ in the original system, find $\Delta x(0)$ by noting that $\Delta x(t) = x(t) - \bar{x}$ so that $\Delta x(0) = x(0) - \bar{x}$.

Example 4.28: Linearization

A nonlinear system model is given as

$$\ddot{x} + \dot{x} + x|x| = 1 + \sin t, \quad x(0) = 0, \quad \dot{x}(0) = 1$$

Derive a linearized model.

Solution

We will follow the procedure outlined above.

1. The operating point was determined in Example 4.27 as $\bar{x} = 1$.
2. The only nonlinear element in the model is $f(x) = x|x|$, which is linearized about the operating point via Equation 4.27. Recalling Example 4.26, we find

$$f(x) \cong f(1) + [2x]_{x=1}\Delta x = 1 + 2\Delta x$$

3. In the original model, replace x with $\bar{x} + \ \ x = 1 + \ \ x$, the nonlinear term with $1 + 2\Delta x$, and reconsider the time-varying portion $\sin t$, which was previously set to zero, to obtain

$$\frac{d^2(1+ \ \ x)}{dt^2} + \frac{d(1+ \ \ x)}{dt} + (1 + 2 \ \ x) = 1 + \sin t$$

This simplifies to

$$\ddot{x} + \ \ \dot{x} + 2 \ \ x = \sin t$$

4. The initial conditions are adjusted as follows:

$$x(t) = x(t) - 1 \quad \overset{\text{Differentiate}}{\qquad} \quad \dot{x}(t) = \dot{x}(t) \qquad \begin{aligned} x(0) &= x(0) - 1 = -1 \\ \dot{x}(0) &= \dot{x}(0) = 1 \end{aligned}$$

In summary, the linearized model is derived as

$$\ddot{x} + \ \ \dot{x} + 2 \ \ x = \sin t, \qquad x(0) = -1, \qquad \dot{x}(0) = 1$$

This is a second-order differential equation with constant coefficients, which is then easily solved to generate $\Delta x(t)$. It is important to note, however, that $\Delta x(t)$ is not compatible with the solution $x(t)$ of the original nonlinear system. To make them compatible, we must recall that $x(t) = 1 + \Delta x(t)$ in this problem. Therefore, the solution $\Delta x(t)$ of the linear model needs to be increased by one unit to be compatible with $x(t)$. We will elaborate on this and many other related issues in Section 8.5.

Example 4.29: Linearization

A system is governed by its nonlinear state-variable equations as

$$\begin{cases} \dot{x}_1 = 2x_2 - 2, & x_1(0) = 1 \\ \dot{x}_2 = (x_1 - 1)^3 x_2 + 1, & x_2(0) = -1 \end{cases}$$

Derive a linearized model.

Solution

We will follow the standard procedure outlined earlier. Note that the inputs are –2 in the first equation, and 1 in the second, both constants.

1. Replace x_1 and x_2 with $\bar{x}_1$ and $\bar{x}_2$, respectively. Because there are no time-varying portions in the input(s), no further modification is needed.

$$\begin{cases} 0 = 2\bar{x}_2 - 2 \\ 0 = (\bar{x}_1 - 1)^3\bar{x}_2 + 1 \end{cases} \qquad \begin{cases} \bar{x}_2 = 1 \\ (\bar{x}_1 - 1)^3 = -1 \end{cases} \qquad \begin{matrix} \bar{x}_2 = 1 \\ \bar{x}_1 = 0 \end{matrix}$$

Therefore, the operating point is $(\bar{x}_1, \bar{x}_2) = (0,1)$.

2. The only nonlinear element is $f(x_1,x_2) = (x_1 - 1)^3 x_2$, which will be linearized about the operating point (0,1). Following Equation 4.28, we have

$$\begin{aligned} f(x_1,x_2) &= f(0,1) + \left.\frac{\partial f}{\partial x_1}\right|_{(0,1)} \quad x_1 + \left.\frac{\partial f}{\partial x_2}\right|_{(0,1)} \quad x_2 \\ &= -1 + [3(x_1 - 1)^2 x_2]_{(0,1)} \quad x_1 + [(x_1 - 1)^3]_{(0,1)} \quad x_2 \\ &= -1 + 3 \quad x_1 - \quad x_2 \end{aligned}$$

3. In the original model, replace x_1 with $\bar{x}_1 + \quad x_1 = \quad x_1$, x_2 with $\bar{x}_2 + \quad x_2 = 1 + \quad x_2$, and the nonlinear term with its linear approximation, $-1 + 3\Delta x_1 - \Delta x_2$. No other adjustments need be made because no time-varying portions of input were set to zero. This yields

$$\begin{cases} \dot{x}_1 = 2(1 + \quad x_2) - 2 \\ \dot{x}_2 = -1 + 3 \quad x_1 - \quad x_2 + 1 \end{cases} \qquad \begin{cases} \dot{x}_1 = 2 \quad x_2 \\ \dot{x}_2 = 3 \quad x_1 - \quad x_2 \end{cases}$$

4. The initial conditions are modified as

$$\begin{matrix} x_1(0) = x_1(0) - \bar{x}_1 \\ x_2(0) = x_2(0) - \bar{x}_2 \end{matrix} \qquad \begin{matrix} x_1(0) = 1 \\ x_2(0) = -2 \end{matrix}$$

The linearized model is therefore

$$\begin{cases} \dot{x}_1 = 2 \quad x_2, & x_1(0) = 1 \\ \dot{x}_2 = 3 \quad x_1 - \quad x_2, & x_2(0) = -2 \end{cases}$$

This linear system is then solved for $\Delta x_1(t)$ and $\Delta x_2(t)$. Once again, it must be noted that $\Delta x_1(t)$ and $\Delta x_2(t)$ are not compatible with the coordinates $x_1(t)$ and $x_2(t)$ of the original nonlinear system. Compatibility is achieved by taking into account that $x_1 = \Delta x_1$ and $x_2 = 1 + \Delta x_2$.

4.6.2.3 Small-Angle Linearization

In some cases, mathematical models of dynamic systems contain nonlinear elements that involve trigonometric functions of an angle. One particular area in which such cases occur is in the rotational motion of a mechanical system (see Section 5.4). These types of models can be linearized as long as the angle remains small. In particular, if $\theta \ll 1$ radian, then

$$\begin{aligned} \sin\theta &\cong \theta \\ \cos\theta &\cong 1 \\ \theta\dot{\theta}^2 &\cong 0 \end{aligned} \tag{4.29}$$

Example 4.30: Small-Angle Linearization

The governing equations for a dynamic system have been derived as

$$\begin{cases} 2\ddot{x} + \ddot{\theta} - \dot{\theta}^2 \sin\theta + \dot{x} + x = f(t) \\ \ddot{\theta} + \ddot{x}\cos\theta + 10\sin\theta = 0 \end{cases}$$

a. Derive the linearized model for $\theta \ll 1$ radian.
b. Find the state equation for the linear model.

Solution

a. Using the first two approximations in Equation 4.29, the governing equations reduce to

$$\begin{cases} 2\ddot{x} + \ddot{\theta} - \dot{\theta}^2\theta + \dot{x} + x = f(t) \\ \ddot{\theta} + \ddot{x} + 10\theta = 0 \end{cases} \quad \overset{\dot{\theta}^2\theta \cong 0}{} \quad \begin{matrix} (a) \\ (b) \end{matrix} \begin{cases} 2\ddot{x} + \ddot{\theta} + \dot{x} + x = f(t) \\ \ddot{\theta} + \ddot{x} + 10\theta = 0 \end{cases}$$

b. In its present form, the linearized model cannot be transformed into state-variable equations because both $\ddot{x}$ and $\ddot{\theta}$ appear in the same equation. The remedy, however, is to manipulate the two equations, labeled Equations a and b, to eliminate the unwanted variables as follows. From Equation b, we find $\ddot{\theta} = -10\theta - \ddot{x}$ and insert into Equation a and simplify to obtain

$$\ddot{x} + \dot{x} + x - 10\theta = f(t) \tag{c}$$

This equation is now in the correct form. Next, from Equation b, we find $\ddot{x} = -10\theta - \ddot{\theta}$ and insert into Equation a and simplify to arrive at

$$\ddot{\theta} + 20\theta - \dot{x} - x = -f(t) \tag{d}$$

In summary,

$$\begin{matrix} (c) \\ (d) \end{matrix} \begin{cases} \ddot{x} + \dot{x} + x - 10\theta = f(t) \\ \ddot{\theta} + 20\theta - \dot{x} - x = -f(t) \end{cases}$$

We may now proceed as in Section 4.2 by selecting the state variables as $x_1 = x$, $x_2 = \theta$, $x_3 = \dot{x}$, and $x_4 = \dot{\theta}$. The state-variable equations are subsequently formed as

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = -x_3 - x_1 + 10x_2 + f(t) \\ \dot{x}_4 = -20x_2 + x_3 + x_1 - f(t) \end{cases}$$

Noting that there is only one input, $f(t)$, the state equation is written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 10 & -1 & 0 \\ 1 & -20 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad u = f(t)$$

These types of systems will also be discussed in greater detail in Section 8.5.

4.6.3 Linearization with MATLAB Simulink

The state-space linear model of a nonlinear system can be extracted in Simulink using the `'linearize'` built-in function:

```
LIN = linearize('sys',OP,IO) takes a Simulink model name, 'sys', an
  operating point object, OP, and an I/O object, IO, as inputs and returns
  a linear time-invariant state-space model, LIN. The operating point
  object is created with the function OPERPOINT or FINDOP. The
  linearization I/O object is created with the function GETLINIO or LINIO.
  Both OP and IO must be associated with the same Simulink model, sys.
```

The function is used in conjunction with `'findop'` and `'getlinio'`. The function call for `'findop'` is as follows:

```
[op,opreoprt] = findop(sys,opspec) finds an operating point,
  op, of the model, 'sys', from specifications given in opspec.
```

`opspec` is an operating point specification object, and is created with the function `'operspec'`. Specifications on the operating points, such as minimum and maximum values, initial guesses, and known values are specified by editing `opspec`. To find equilibrium (or steady-state) operating points, set the `SteadyState` property of the states and inputs in `opspec` to 1. The `findop` function uses optimization to find operating points that closely meet the specifications listed in `opspec`. The function call for `'getlinio'` is as follows:

```
IO = getlinio('sys') finds all linearization annotations in a Simulink
  model, 'sys'. The vector of objects returned from this function call
  has an entry for each linearization annotation in a model.
```

To initiate the process, we first build a Simulink model, `'sys'`, and identify a linearization input point and a linearization output point. This is done by right-clicking on the desired location, selecting "Linearization Points" from the pull-down menu, and choosing the appropriate category from the list.

Example 4.31: Linearization in MATLAB

Consider the nonlinear system in Example 4.28:

$$\ddot{x} + \dot{x} + x\left|x\right| = 1 + \sin t, \quad x(0) = 0, \quad \dot{x}(0) = 1$$

To build a Simulink model, the nonlinear state-variable equations must first be formed. Choosing state variables $x_1 = x$ and $x_2 = \dot{x}$, we find

$$\begin{cases} \dot{x}_1 = x_2 & x_1(0) = 0 \\ \dot{x}_2 = -x_2 - x_1\left|x_1\right| + 1 + \sin t' & x_2(0) = 1 \end{cases} \tag{a}$$

Following the procedure outlined in Section 4.5, we build the model shown in Figure 4.35 for the nonlinear system in Equation a and save it as `'Example431'`. Double-clicking on an integrator block allows us to input the appropriate initial condition for the output signal of that block. Because the first integrator has x_2 as its output, we use $x_2(0) = 1$. For the second integrator, $x_1(0) = 0$. The nonlinear element $x_1\left|x_1\right|$ is handled by the function block from the Simulink User-Defined Functions

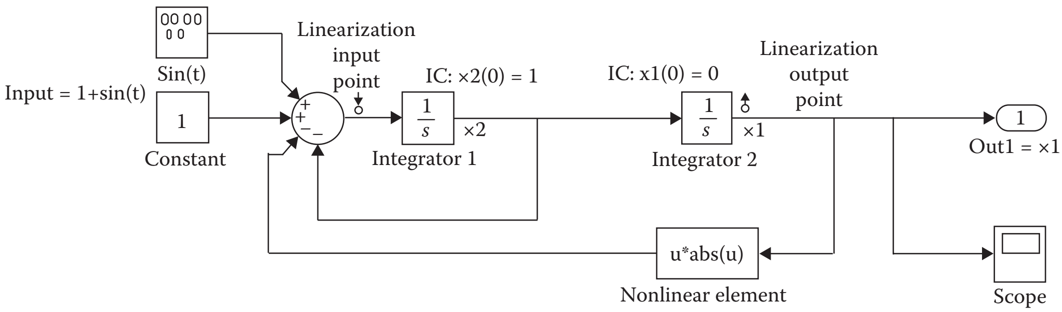


FIGURE 4.35 Simulink model in Example 4.31.

library, in which `u` is used as the input variable name. The linearization input point is chosen at the output signal of the summing junction, the linearization output point at the output signal `x1`.

The linear model is extracted from the nonlinear system in Equation a as follows. Note that the model, `'Example431'`, must be open before we can proceed.

```
>> sys = 'Example431';
>> load_system(sys);
>> opspec = operspec(sys);

% Specify the properties of the first state variable.
>> opspec.States(1).SteadyState = 1;
>> opspec.States(1).x = 0;        % Initial value
>> opspec.States(1).Min = 0;      % Minimum value

% Do the same for the second state variable
>> opspec.States(2).SteadyState = 1;
>> opspec.States(2).x = 1;
>> opspec.States(2).Min = 0;

% Find the operating point based on the specifications listed above.
>> [op,opreport] = findop(sys,opspec);
Operating Point Search Report:


_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _
Operating Report for the Model Example431.
(Time-Varying Components Evaluated at time t=0)
Operating point specifications were successfully met.
States:
_ _ _ _ _
(1.) Example431/Integrator 1
     x:      0 dx:   0 (0)
(2.) Example431/Integrator 2
     x:      1 dx:   0 (0)
Inputs: None


_ _ _ _ _
Outputs:
_ _ _ _ _
(1.) Example431/Out1 = x1
     y:      1 [-Inf Inf]

% Get the linearization annotations
>> IO=getlinio(sys);

% Extract the linear state-space model
>> LIN = linearize('Example431',op,IO)
```

```
a =
           Integrator 1 Integrator 2
   Integrator 1        -1           -2
   Integrator 2         1            0     % Controller canonical form

b =
                Sum 1
   Integrator 1     1
   Integrator 2     0

c =
    Integrator 1 Integrator 2
     x1         0            1

d =
       Sum 1
     x1    0

Continuous-time state-space model.
```

Note that

- the state matrix is in the controller canonical form (Section 4.4), and
- the input in the linear model is composed of the time-varying portion of the input in the original nonlinear system. In the current example, the input for the nonlinear system is $1+\sin t$. This implies that the input for the linear model is simply $\sin t$.

We will validate the above MATLAB results as follows. Recall from Example 4.28 that the operating point is (1,0) and the linearized model is

$$\ddot{x} + \dot{x} + 2 x = \sin t$$

To derive the controller canonical form, choose the state variables as $x_1 = \dot{x}$ and $\Delta x_2 = \Delta x$, and the state-variable equations for the linearized model are obtained as

$$\begin{cases} \dot{x}_1 = - x_1 - 2 x_2 + \sin t \\ \dot{x}_2 = x_1 \end{cases}$$

In vector form,

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{B}u \\ y = \mathbf{C} \mathbf{x} + Du \end{cases} \tag{b}$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [0 \ \ 1], \quad D = 0, \quad u = \sin t$$

This confirms the set of results realized in MATLAB Simulink. Linear analysis of nonlinear systems will be further elaborated in Section 8.5.

PROBLEM SET 4.6

In Problems 1 through 10, the mathematical model of a nonlinear dynamic system is given. Follow the procedure outlined in this section to derive the linearized model.

1. $2\ddot{x} + \dot{x} + x^3 = 1 + \sin 2t, \quad x(0) = 0, \quad \dot{x}(0) = 1$
2. $\ddot{x} + \dot{x} + 2x|x| = 2 + \cos t, \quad x(0) = 0, \quad \dot{x}(0) = 1$
3. $\ddot{x} + \dot{x} + 2x|x| = -2 + \sin t, \quad x(0) = 0, \quad \dot{x}(0) = 1$
4. $\ddot{x} + \dot{x} + x\sqrt{|x|} = 1 + \sin t, \quad x(0) = 1, \quad \dot{x}(0) = 0$
5. $\ddot{x} + \dot{x} + f(x) = 1 + \cos t, \quad x(0) = \frac{1}{4}, \quad \dot{x}(0) = 1, \; f(x) = \begin{cases} 2\sqrt{x} & \text{if} \quad x \geq 0 \\ -2\sqrt{|x|} & \text{if} \quad x \leq 0 \end{cases}$
6. $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_1|x_1| - x_2^3 - 2 + \sin t \end{cases}, \quad \begin{matrix} x_1(0) = -1 \\ x_2(0) = 1 \end{matrix}$
7. $\ddot{x} + 3\dot{x} + g(x) = 3 + \cos t, \quad x(0) = 1, \quad \dot{x}(0) = 0, \quad g(x) = \begin{cases} 3(1 - e^{-x}) & \text{if} \quad x \geq 0 \\ -3(1 - e^{-x}) & \text{if} \quad x < 0 \end{cases}$
8. $\begin{cases} \dot{x}_1 = x_2 - x_1 \\ \dot{x}_2 = 2x_2^{-1} + 1 + t \end{cases}, \quad \begin{matrix} x_1(0) = 0 \\ x_2(0) = -1 \end{matrix}$
9. $\begin{cases} \dot{x}_1 = -x_1 + 3x_2^3 \\ \dot{x}_2 = x_1 + 24 + \sin t \end{cases}, \quad \begin{matrix} x_1(0) = 1 \\ x_2(0) = 0 \end{matrix}$
10. $\begin{cases} \dot{x}_1 = -x_2 - x_1|x_1| - 2 + \cos 3t \\ \dot{x}_2 = x_1 - x_2 - 2 \end{cases}, \quad \begin{matrix} x_1(0) = 1 \\ x_2(0) = -2 \end{matrix}$

In Problems 11 through 14, a nonlinear model is provided.

a. Obtain the linear state-space form using MATLAB Simulink.
b. Derive the linearized model analytically to confirm the findings in Part (a).

11. $2\ddot{x} + \dot{x} + x^3 = 1 + \sin 2t, \; x(0) = 0, \; \dot{x}(0) = 1$
12. $\ddot{x} + \dot{x}^3 + x = 2 + \sin t, \; x(0) = 0, \; \dot{x}(0) = 1$
13. $\ddot{x} + \dot{x}|\dot{x}| + x = 1 + \cos 2t, \; x(0) = 0, \; \dot{x}(0) = 1$
14. $\ddot{x} + x^3 = 1 + \sin 2t, \; x(0) = 0, \; \dot{x}(0) = 1$

4.7 SUMMARY

A set of coordinates that completely describes the motion of a system is known as a set of generalized coordinates. If there are n generalized coordinates $q_1, q_2, \ldots, q_n$, the dynamic system model is described by

$$\begin{cases} \ddot{q}_1 = f_1(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) \\ \ddot{q}_2 = f_2(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) \\ \qquad \cdots \\ \ddot{q}_n = f_n(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) \end{cases}$$

where $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$ are the generalized velocities and $f_1, f_2, \ldots, f_n$, known as the generalized forces, are algebraic functions of q_i and $\dot{q}_i$ ($i = 1,2, \ldots, n$) and time t. This model, subjected to initial generalized coordinates and initial generalized velocities, is called the configuration form.

Mathematical models of dynamic systems that are governed by n-dimensional systems of second-order differential equations can conveniently be expressed as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}$$

which is commonly known as the standard, second-order matrix form, where

- $\mathbf{x}_{n\times 1}$ = configuration vector, $\mathbf{f}_{n\times 1}$ = vector of external forces
- $\mathbf{M}_{n\times n}$ = mass matrix, $\mathbf{C}_{n\times n}$ = damping matrix, $\mathbf{K}_{n\times n}$ = stiffness matrix

State variables, denoted by x_i ($i = 1, 2, \ldots, n$), form the smallest set of independent variables that completely describes the state of a system. There are as many state variables for a system as there are initial conditions required to completely solve the system's model. The state variables are those variables for which initial conditions are needed. For a linear system with state variables $x_1, x_2, \ldots, x_n$, inputs $u_1, u_2, \ldots, u_m$, and outputs $y_1, y_2, \ldots, y_p$, the state-space form is

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_{n\times n}\mathbf{x}_{n\times 1} + \mathbf{B}_{n\times m}\mathbf{u}_{m\times 1} \\ \mathbf{y}_{p\times 1} = \mathbf{C}_{p\times n}\mathbf{x}_{n\times 1} + \mathbf{D}_{p\times m}\mathbf{u}_{m\times 1} \end{cases}$$

where

- $\mathbf{x}_{n\times 1}$ = state vector, $\mathbf{A}$ = state matrix, $\mathbf{B}$ = input matrix, $\mathbf{y}$ = output vector
- $\mathbf{C}$ = output matrix, $\mathbf{D}$ = direct transmission matrix, $\mathbf{u}$ = input vector

If $u(t)$ is an input and $y(t)$ is an output of a system, then the I/O equation is

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(m)} + b_1 u^{(m-1)} + \cdots + b_{m-1}\dot{u} + b_m u, \quad m \le n$$

where $a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ are constants, and $y^{(n)} = d^n y/dt^n$.

A transfer function is the ratio of the Laplace transforms of the output and input. If a system has q inputs and r outputs, the qr transfer functions are assembled in an $r\times q$ transfer matrix, denoted by $\mathbf{G}(s) = [G_{ij}(s)]$, where $i = 1,2, \ldots, r$ and $j = 1,2, \ldots, q$.

A block diagram is an interconnection of blocks, each block corresponding to an operation carried out by a component, such that the block diagram as a whole agrees with the system's mathematical model.

The linearization of a nonlinear system model is performed as follows:

- Find the operating point(s) by assuming the dependent variables are constants and setting the time-varying portions of inputs equal to zero.
- Linearize the nonlinear term(s) about the operating point(s).
- In the original nonlinear model, express variables in terms of incremental variables, and replace nonlinear terms with their linear approximations. The resulting system is linear in the incremental variables.
- Find the initial conditions for the incremental variables.

The state-space linear model of a nonlinear system can be extracted in Simulink using the `'linearize'` built-in function. The function is used in conjunction with `'findop'` and `'getlinio'`. The inputs to the `'findop'` function are the model, `'sys'`, and `'opspec'`, which is an operating point specification object, and is created with the function `'operspec'`.

Specifications on the operating points, such as minimum and maximum values, initial guesses, and known values, are specified by editing `'opspec'`. To find equilibrium (or steady-state) operating points, set the `'SteadyState'` property of the states and inputs in `'opspec'` to 1. The `'findop'` function uses optimization to find operating points that closely meet the specifications listed in `'opspec'`.

REVIEW PROBLEMS

1. The governing equations for a dynamic system are given as

$$\begin{cases} 2\ddot{x}_1 + \dot{x}_1 + x_1 - 3x_2 = f(t) \\ 3\dot{x}_2 - x_1 - 2x_2 = 0 \end{cases}$$

   a. Assuming $f(t)$ and $\dot{x}_1$ are the input and output, respectively, obtain the state-space form.
   b. Determine whether the system is stable.
   c. Find the transfer function directly from the governing equations.

2. A dynamic system with input f and output x is described by

$$\tfrac{1}{3}\ddot{x} + \dot{x} + 2x = f(t)$$

   a. Find the state-space form.
   b. Find the transfer function directly from the state-space form.
   c. Decide if the system is stable by examining the state matrix in Part (a).

3. A system's transfer function is defined as

$$\frac{Y(s)}{U(s)} = \frac{2s+1}{3s^2+s+1}$$

   a. Find the I/O equation.
   b. Find the state-space form directly from the I/O equation.
   c. Find the transfer function directly from the state-space form.

4. A system is described by its governing equations

$$\begin{cases} \ddot{x}_1 + x_1 - 3(x_2 - x_1) = f(t) \\ \ddot{x}_2 + 3(x_2 - x_1) = 0 \end{cases}$$

   where f is the input, whereas x_1 and $\dot{x}_1$ are the outputs.
   a. Obtain the state-space form.
   b. Find the transfer matrix directly from the state-space form.
   c. Find the transfer matrix using the governing equations, and compare with Part (b).

5. A system's I/O equation is given as

$$\ddot{y} + 3\dot{y} + \tfrac{1}{4}y = \dot{u} + 2u$$

   a. Find the state-space form.
   b. Find the transfer function directly from the state-space form.

6. Repeat Problem 5 for the I/O equation $\frac{2}{3}\dddot{y}+\dot{y}+y=\frac{1}{2}\ddot{u}+u$.
7. A system's transfer function is given as

$$\frac{Y(s)}{U(s)}=\frac{s^2(s+1)}{s^3+2s^2+3s+1}$$

   a. Find the I/O equation.
   b. Find the state-space form directly from Part (a).
   c. Find the transfer function from Part (b).
8. Find the I/O equation of a SISO system whose state-space form is given as

$$\begin{cases}\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{B}u\\ y=\mathbf{C}\mathbf{x}+Du\end{cases}$$

with

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -1 & -\frac{1}{2}\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0\\ \frac{2}{3}\end{bmatrix},\ u=u,\ \mathbf{C}=\ [1\ \ \tfrac{1}{3}],\ D=0$$

9. Repeat Problem 8 for

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0\\ 2\end{bmatrix},\ u=u,\ \mathbf{C}=\ [1\ \ -1],\ D=1$$

10. Find all possible I/O equations for a system with state-space form

$$\begin{cases}\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{B}u\\ \mathbf{y}=\mathbf{C}\mathbf{x}+\mathbf{D}u\end{cases}$$

where

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -1 & -4\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0\\ 1\end{bmatrix},\ u=u,\ \mathbf{C}=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix},\ \mathbf{D}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

11. Find the transfer function for the system in Figure 4.36 using
   a. Block diagram reduction.
   b. Mason's rule.

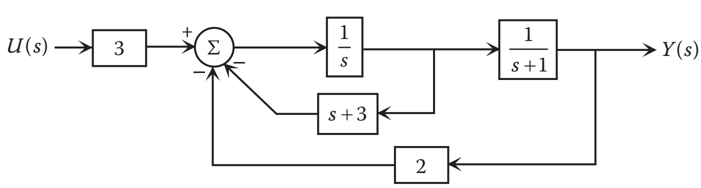


FIGURE 4.36 Problem 11.

In Problems 12 through 15, block diagram representation of a system is provided. Find the transfer function using Mason's rule.

12. Figure 4.37
13. Figure 4.38
14. Figure 4.39
15. Figure 4.40

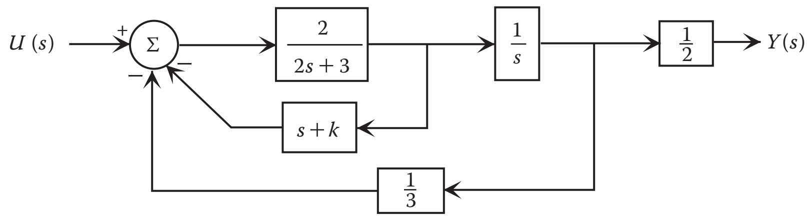


FIGURE 4.37 Problem 12.

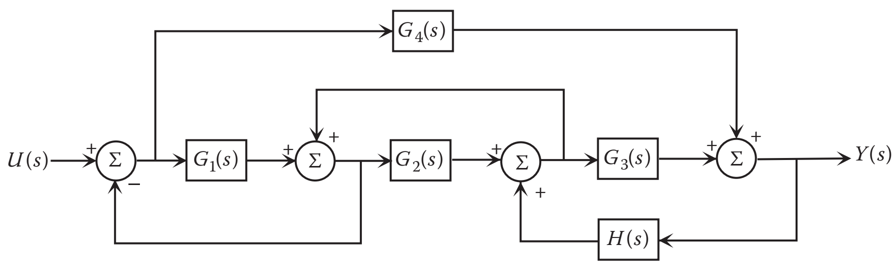


FIGURE 4.38 Problem 13.

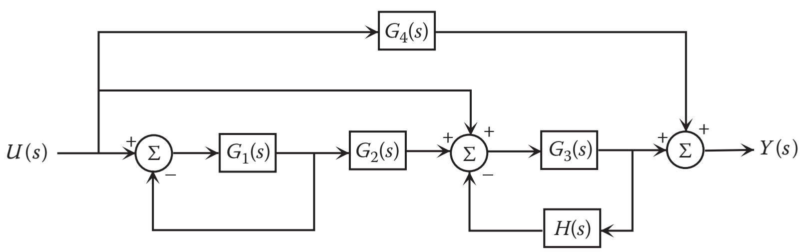


FIGURE 4.39 Problem 14.

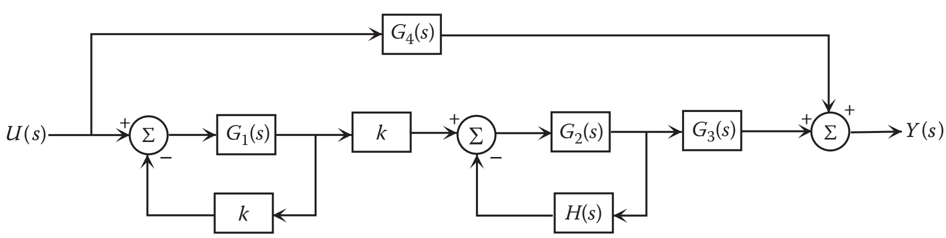


FIGURE 4.40 Problem 15.

In Problems 16 through 19, the I/O equation or the state-space form of a system model is provided. Construct the appropriate block diagram, and directly use it to find the transfer matrix.

16. State-space form is

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} + Du \end{cases}$$

with

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & -1 & -\frac{1}{2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \;\; 2 \;\; 0], \quad D = 0, \quad u = u$$

17. I/O equation is

$$2\ddot{x} + \dot{x} + 3x = \dot{u} + u$$

18. I/O equation is

$$2\ddot{x} + \dot{x} + 3x = \ddot{u} + u$$

19. State-space form is

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}u \end{cases}$$

with

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u = u$$

20. A system is stable if the poles of the overall transfer function lie in the left half-plane; these poles are the same as the eigenvalues of the state matrix. Consider the block diagram representation of a system shown in Figure 4.41, where $k > 0$ is a parameter. Determine the range of values of k for which the system is stable.

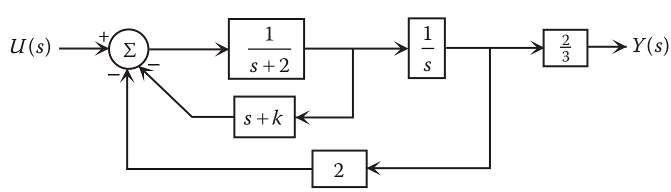


FIGURE 4.41 Problem 20.

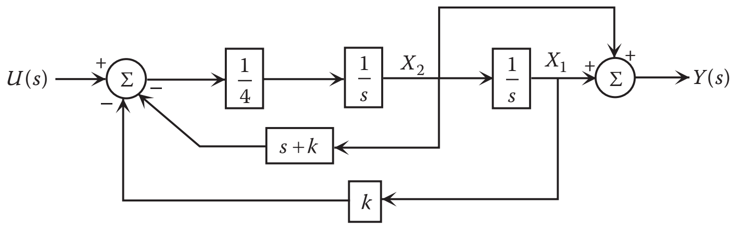


FIGURE 4.42 Problem 21.

21. Repeat Problem 20 for the block diagram in Figure 4.42, where $k > 0$ is a parameter.
22. Derive the linearized model for a nonlinear system described by

$$\begin{cases} \dot{x}_1 = x_1\left|x_1\right| + x_2 - 1 + \cos t \\ \dot{x}_2 = -x_1 - x_2 - 1 \end{cases}, \quad \begin{matrix} x_1(0) = 2 \\ x_2(0) = -2 \end{matrix}$$

23. Repeat Problem 22 for

$$\begin{cases} \dot{x}_1 = x_2 + 2\sin t \\ \dot{x}_2 = x_1^3 + 3x_2 - 1 \end{cases}, \quad \begin{matrix} x_1(0) = 1 \\ x_2(0) = 0 \end{matrix}$$

5 Mechanical Systems

The modeling techniques for mechanical systems are discussed in this chapter. Mechanical systems are in either translational or rotational motion, or both. We begin this chapter by introducing mechanical elements, which include mass elements, spring elements, and damper elements. The concept of equivalence is discussed, which simplifies the modeling of systems in many applications. We then review Newton's second law and apply it to translational systems. For rotational systems, moment equations are used to obtain dynamic models. For systems involving both translational and rotational motions, equations of motion can be derived using the force/moment approach based on Newtonian mechanics or the energy method based on analytical mechanics. Examples are given to illustrate both methods, followed by a brief coverage of gear–train systems. The chapter concludes with a simulation of mechanical systems using MATLAB®, Simulink®, and Simscape™ computer tools.

5.1 MECHANICAL ELEMENTS

The objective of this chapter is to show how one can obtain mathematical models of mechanical systems. Because a real mechanical system is usually complicated, simplifying assumptions must be made to reduce the system to an idealized model, which consists of interconnected elements. The behavior of the mathematical model can then approximate that of the real system.

A mathematical model of a mechanical system can be constructed based on physical laws (such as Newton's laws and the conservation of energy) that the elements and their interconnections must obey. Elements can be broadly divided into three classes according to whether element forces are proportional to accelerations, proportional to displacements, or proportional to velocities. Correspondingly, they can be divided into elements that store and release kinetic energy, store and release potential energy, and dissipate energy. In this section, element equations relating the external forces to the associated element variables are presented.

5.1.1 Mass Elements

Figure 5.1 shows a mass m traveling with a velocity v. The basic variables used to describe the dynamic behavior of a translational mechanical system are the acceleration vector $\mathbf{a}$, the velocity vector $\mathbf{v}$, and the position vector $\mathbf{r}$. They are related by the time derivatives

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}, \tag{5.1}$$

which can also be represented in the simple dot notation

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}. \tag{5.2}$$

Assume that the motion of the mass in Figure 5.1 is under the influence of an externally applied force, and is constrained in only one direction. According to Newtonian mechanics, the resulting force f acting on the mass is equal to the time rate of change of momentum. For a constant mass, Newton's second law is expressed as

$$f = \frac{\mathrm{d}}{\mathrm{d}t}(mv) = m\frac{\mathrm{d}v}{\mathrm{d}t} = ma. \tag{5.3}$$

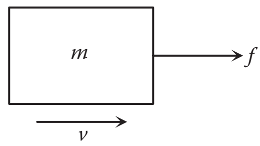


FIGURE 5.1 A mass traveling with a velocity v.

Note that the acceleration a is absolute and must be measured with respect to an inertial reference frame. For ordinary systems at or near the surface of the Earth, the ground can be approximated as a reference for motion.

Mass elements store mechanical energy. The energy stored in a mass is kinetic energy if the mass is in motion. The kinetic energy is expressed as

$$T = \frac{1}{2}mv^2, \tag{5.4}$$

which implies that the mass stores kinetic energy as its velocity increases, and releases kinetic energy as its velocity decreases. If a mass has a vertical displacement relative to a reference position, the energy stored in the mass is potential energy given by

$$V_g = mgh, \tag{5.5}$$

where g is the gravitational acceleration (9.81 m/s^2 or 32.2 ft/s^2) and h is the height measured from the reference position or datum to the center of mass. Subscript g is used to denote that the potential energy is associated with gravity.

For rotational mechanical systems, the basic variables used to describe system dynamics are the angular acceleration vector $\boldsymbol{\alpha}$, the angular velocity vector $\boldsymbol{\omega}$, and the angular position vector $\boldsymbol{\theta}$. The direction of an angular vector can be determined using the right-hand rule as shown in Figure 5.2. The sense of rotation follows the curve of the four fingers, and the rotational vector points in the direction of the thumb. In this chapter, we consider the rigid bodies that are constrained to rotate about only one axis. Then in scalar form, we have

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \tag{5.6}$$

or

$$\alpha = \dot{\omega} = \ddot{\theta}. \tag{5.7}$$

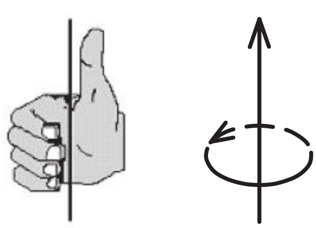


FIGURE 5.2 Right-hand rule.

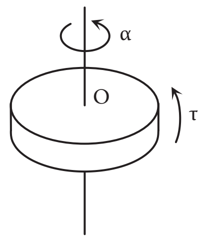


FIGURE 5.3 A disk rotating about an axis through a fixed point O.

Figure 5.3 shows a disk rotating about an axis through a fixed point O. The relation between the torque τ about the fixed point O and the angular acceleration α of the disk about O is

$$\tau = I_O \alpha, \tag{5.8}$$

where I_O is the mass moment of inertia of the body about the fixed point O, and common units used are kg·m^2 or slug·ft^2. Similar to a translational mass, a rotational mass can store kinetic energy and potential energy. The kinetic energy for a rotational mass about a fixed point O is expressed as

$$T = \frac{1}{2} I_O \omega^2. \tag{5.9}$$

The potential energy for a rotational mass has the same form as Equation 5.5.

5.1.2 Spring Elements

Figure 5.4a shows a translational spring element, which is fixed at one end and is subjected to a tensile (or compressive) force f at the other end. The spring has a free length x_0, and the deflection of the spring caused by the force f is denoted by x. Assume that the spring is massless, or of negligible mass. For a linear spring, Hooke's law states that

$$f = kx, \tag{5.10}$$

where k is the spring stiffness in units of N/m or lb/ft. If the spring is connected to a mass, due to Newton's third law, the force exerted on the mass by the spring has the same magnitude as f, but opposite in direction. When the two ends of a spring are displaced by x_1 and x_2, as shown in Figure 5.4b, the forces at the two ends are equal in magnitude but opposite in direction. If $x_2 > x_1 > 0$, the spring is under elongation, and the force applied to the spring is

$$f = k(x_2 - x_1), \tag{5.11}$$

where $x_{rel} = x_2 - x_1$ is the relative displacement between the two ends of the spring.

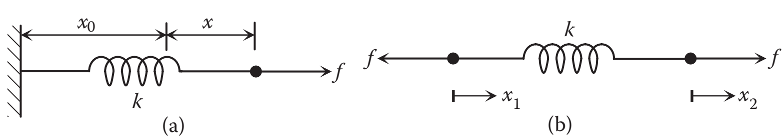


FIGURE 5.4 A translational spring element with (a) one fixed end and (b) two free ends.

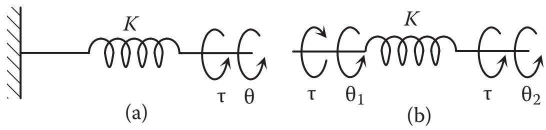


FIGURE 5.5 A torsional spring element with (a) one fixed end and (b) two free ends.

When a spring is stretched or compressed, potential energy is stored in the spring and is given by

$$V_e = \frac{1}{2}kx^2, \tag{5.12}$$

where subscript e denotes that the potential energy is associated with elastic elements.

For a torsional spring as shown in Figure 5.5a, we have

$$\tau = K\theta, \tag{5.13}$$

where τ is the applied torque, K is the torsional spring stiffness in units of N·m/rad or ft·lb/rad, and θ is the angular deformation of the spring. Figure 5.5b shows a torsional spring with both ends twisted. Assume that θ_1 and θ_2 are the angular displacements of respective ends corresponding to the applied torque. If $\theta_2 > \theta_1 > 0$, then

$$\tau = K(\theta_2 - \theta_1) \tag{5.14}$$

and the spring is twisted in the counterclockwise direction when viewed from the right-hand side. The potential energy stored in a torsional spring element is expressed as

$$V_e = \frac{1}{2}K\theta^2. \tag{5.15}$$

5.1.3 Damper Elements

A spring element exerts a reaction force that is dependent on the relative displacement between two ends of the spring. In contrast, a force that depends on the relative velocity between two bodies is modeled by a damper element. Figure 5.6 shows a mass sliding on a fixed horizontal surface, where the two surfaces are separated by a film of liquid. The mass is subjected to a friction force generated between the two surfaces, and the friction caused by the liquid is called viscous damping. The direction of the damping force is opposite to the direction of the motion and its magnitude depends

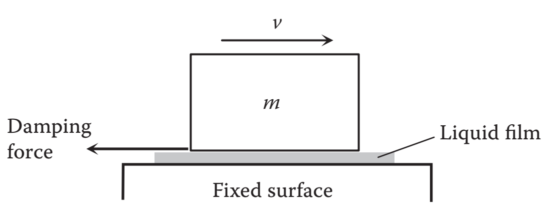


FIGURE 5.6 A mass sliding on a lubricated fixed surface.

on the nature of fluid flow between the two surfaces. The exact viscous damping force is complex; thus, for modeling in system dynamics, we use a linear relationship

$$f = bv, \tag{5.16}$$

where the symbol b is used to denote the viscous damping coefficient in units of N·s/m or lb·s/ft. The damping force exerted on the mass in Figure 5.6 is to the left. Note that the symbol c is also often used to denote the viscous damping coefficient. Therefore, both b and c will be used interchangeably in this book.

The viscous friction can be modeled using a viscous damper (or a dashpot). The symbol in Figure 5.7a is the representation of a viscous damper, which is like a piston moving through a liquid-filled cylinder as shown in Figure 5.7b. There are small holes in the piston through which the liquid flows as the parts move relative to each other. If $v_2 > v_1 > 0$, then the right end of the damper moves to the right with respect to the left end. The force applied to the right end is dependent on the relative velocity $v_{\text{rel}} = v_2 - v_1$. The force has a magnitude of

$$f = b(v_2 - v_1) \tag{5.17}$$

and points to the right. Assume that the damper is massless, or of negligible mass. Then the forces at the two ends of the damper are equal in magnitude but opposite in direction.

For a torsional damper as shown in Figure 5.8a, the linear relationship between the externally applied torque and the angular velocity is given by

$$\tau = B\omega, \tag{5.18}$$

where B is the rotational viscous damping coefficient in units of N·m·s/rad or ft·lb·s/rad. The symbol in Figure 5.8b represents a rotational viscous damper, which can be used to model the viscous friction between two rotating surfaces separated by a film of liquid. If $\omega_2 > \omega_1 > 0$, the magnitude of the applied torque is

$$\tau = B(\omega_2 - \omega_1) \tag{5.19}$$

and the direction is as shown.

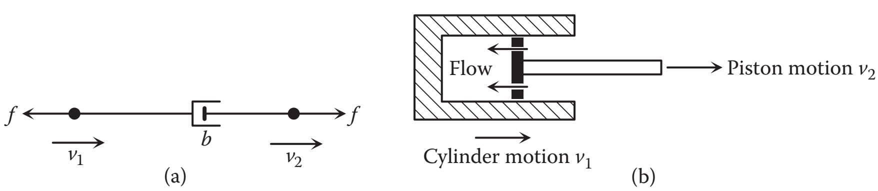


FIGURE 5.7 A viscous damper: (a) symbol and (b) physical system.

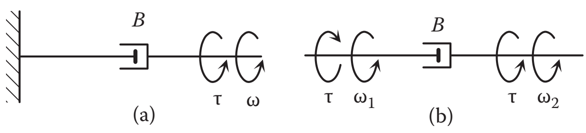


FIGURE 5.8 A rotational viscous damper with (a) one fixed end and (b) two free ends.

Note that the damping dissipates the energy of the system. Besides viscous damping, there are two other types of damping in engineering mechanics: Coulomb damping associated with dry friction and structural damping. The former will be discussed in Chapter 9, and the latter is beyond the scope of this text.

5.1.4 Equivalence

In many mechanical systems, multiple springs or dampers are used. In such cases, an equivalent spring stiffness constant or damping coefficient can be obtained to represent the combined elements.

Example 5.1: Springs in Parallel

Consider a system of two springs, k_1 and k_2, in parallel as shown in Figure 5.9. Prove that the system is equivalent to a single spring whose stiffness is

$$k_{eq} = k_1 + k_2.$$

Proof

Because of parallel interconnection, the bottom ends of the springs are attached to the same fixed body, and their top ends are also attached to a common body. This implies that both springs have the same deflection x. Assume that the forces applied to the two springs are f_1 and f_2, respectively. Because the system is in static equilibrium, the total force is given by

$$f = f_1 + f_2 = k_1x + k_2x = (k_1 + k_2)x.$$

Comparing it with the equivalent system,

$$f = k_{eq}x,$$

we obtain the equivalent spring stiffness

$$k_{eq} = k_1 + k_2.$$

The result can be extended to n springs. For a system of n springs in parallel, the equivalent spring stiffness k_{eq} is equal to the sum of all the individual spring stiffness coefficients k_i:

$$k_{eq} = k_1 + k_2 + \cdots + k_n.$$

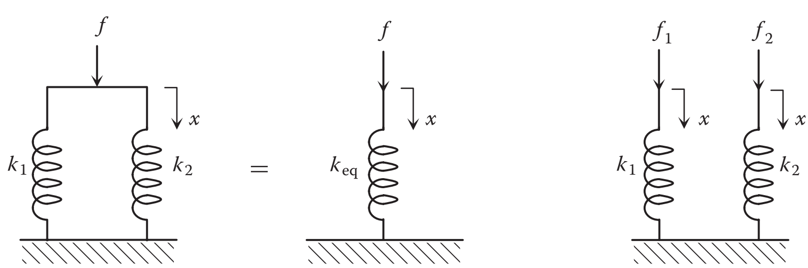


FIGURE 5.9 Equivalence for two springs in parallel.

Example 5.2: Springs in Series

Consider a system of two springs, k_1 and k_2, in series as shown in Figure 5.10. Prove that the equivalent spring stiffness of the system is

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2}.$$

Proof

Because both springs are in static equilibrium, they are subjected to the same force f. Assume that the two springs are deformed by x_1 and x_2, respectively. The total deformation of the system is given by

$$x = x_1 + x_2 = \frac{f}{k_1} + \frac{f}{k_2} = f\left(\frac{1}{k_1} + \frac{1}{k_2}\right).$$

For the equivalent system, the deformation is

$$x = \frac{f}{k_{eq}}.$$

Thus,

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

or

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2}.$$

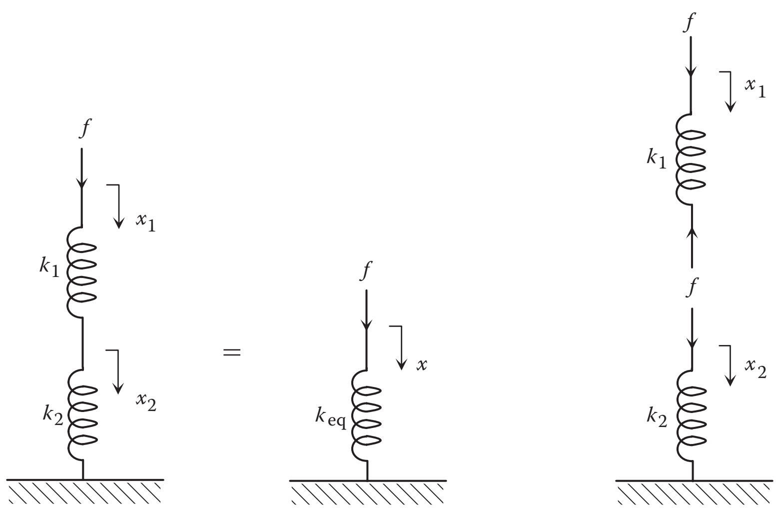


FIGURE 5.10 Equivalence for two springs in series.

The result can also be extended to n springs. For a system of n springs in series, the reciprocal of the equivalent spring stiffness k_{eq} is equal to the sum of all the reciprocals of the individual spring stiffness coefficients k_i:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}.$$

The above two examples show how one can derive the equivalent spring stiffness for springs connected in parallel or in series. For a system of dampers, the equivalent damping coefficient can be derived using the same logic and similar steps.

Springs are the most familiar elastic elements. However, many engineering applications involving elastic elements do not contain springs but other mechanical elements, such as beams and rods, which can be modeled as springs. The equivalent spring constants can be determined using the results from the study of mechanics of materials [2,11].

Example 5.3: Equivalent Spring Constant of a Cantilever Beam

Consider a uniform cantilever beam of length L, width b, and thickness h in Figure 5.11. Assume that a force f is applied to the free end of the beam, and the corresponding deflection is x. Derive the equivalent spring constant k_{eq}.

Solution

The force–deflection relation of a cantilever beam can be found in the mechanics of materials references. The relation is

$$x = \frac{L^3}{3EI_A} f,$$

where x is the deflection at the free end of the beam, f is the force applied at the free end of the beam, E is the modulus of elasticity of beam material, and I_A is the area moment of inertia about the beam's longitudinal axis. For a beam having a rectangular cross-section with width b and thickness h, the area moment of inertia is

$$I_A = \frac{bh^3}{12}.$$

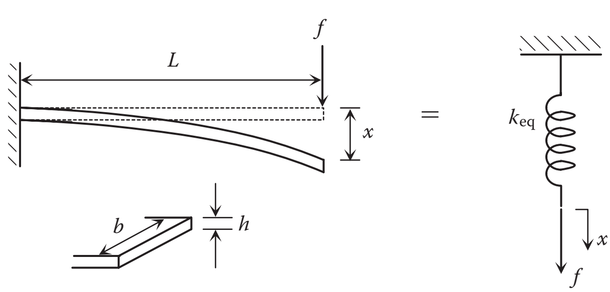


FIGURE 5.11 A beam in bending under a transverse force.

Thus, the force–deflection relation reduces to

$$x = \frac{4L^3}{Ebh^3} f.$$

For the equivalent system, the force–deflection relation is

$$x = \frac{f}{k_{eq}}.$$

Thus, the equivalent spring stiffness is

$$k_{eq} = \frac{Ebh^3}{4L^3}.$$

PROBLEM SET 5.1

1. If the 50-kg block in Figure 5.12 is released from rest at A, determine its kinetic energy and velocity after it slides 5 m down the plane. Assume that the plane is smooth.
2. Repeat Problem 1 if the coefficient of kinetic friction between the block and the plane is $\mu_k = 0.1$.
3. The ball in Figure 5.13 has a mass of 5 kg and is fixed to a rod having a negligible mass. Assume that the ball is released from rest when $\theta = 60°$.
   a. Determine the gravitational potential energy of the ball when $\theta = 60°$ and 30°. The datum is shown in Figure 5.13.
   b. Determine the kinetic energy and the velocity of the ball when $\theta = 30°$.

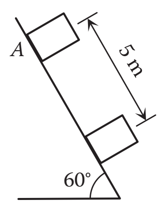


FIGURE 5.12 Problem 1.

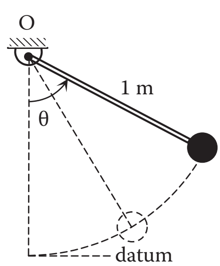


FIGURE 5.13 Problem 3.

4. The 5-kg slender rod in Figure 5.14 is released from rest when $\theta = 60°$.
   a. Determine the gravitational potential energy of the rod when $\theta = 60°$ and 30°. The datum is shown in Figure 5.14.
   b. Determine the kinetic energy and the angular velocity of the rod when $\theta = 30°$. The mass moment of inertia of the slender rod about the fixed point O is $I_O = \frac{1}{3}mL^2$, where L is the length of the rod.
5. Determine the elastic potential energy of the system shown in Figure 5.15 if the 10-kg block moves downward a distance of 0.1 m. Assume that the block is originally in static equilibrium.
6. If the disk in Figure 5.16 rotates in the clockwise direction by 5°, determine the elastic potential energy of the system. Assume that the springs are originally undeformed.

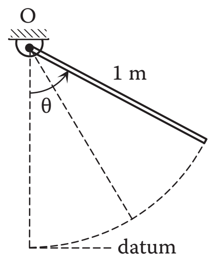


FIGURE 5.14 Problem 4.

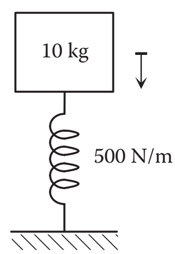


FIGURE 5.15 Problem 5.

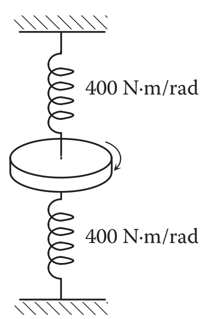


FIGURE 5.16 Problem 6.

7. Determine the equivalent spring constant for the system shown in Figure 5.17.
8. Determine the equivalent spring constant for the system shown in Figure 5.18.
9. Derive the spring constant expression for the axially loaded bar shown in Figure 5.19. Assume that the cross-sectional area is A and the modulus of elasticity of the material is E.
10. The uniform circular shaft in Figure 5.20 acts as a torsional spring. Assume that the elastic shear modulus of the material is G. Derive the equivalent spring constant corresponding to a pair of torques applied at the two free ends.
11. A rod is made of two uniform sections, as shown in Figure 5.21. The two sections are made of the same material, and the modulus of elasticity of the rod material is E. The areas for the two sections are A_1 and A_2, respectively. Derive the equivalent spring constant corresponding to a tensile force f applied at the free end.

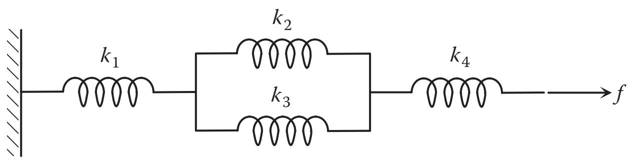


FIGURE 5.17 Problem 7.

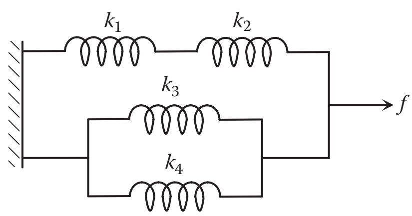


FIGURE 5.18 Problem 8.

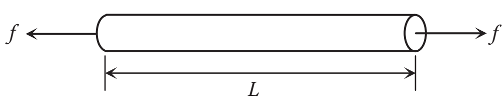


FIGURE 5.19 Problem 9.

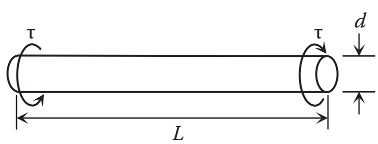


FIGURE 5.20 Problem 10.

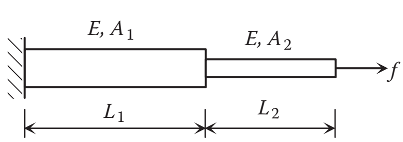


FIGURE 5.21 Problem 11.

12. Derive the spring constant expression for the fixed-fixed beam in Figure 5.22. The Young's modulus of the material is E and the moment of inertia of cross-sectional area is I. Assume that the force f and the deflection x are at the center of the beam.
13. Derive the spring constant expression for the simply supported beam in Figure 5.23. The Young's modulus of the material is E and the moment of inertia of cross-sectional area is I. Assume that the force f and the deflection x are at the center of the beam.
14. Derive the spring constant expression for the simply supported beam in Figure 5.24. The Young's modulus of the material is E and the moment of inertia of cross-sectional area is I. Assume that the applied load f is anywhere between the supports.
15. Determine the equivalent damping coefficient for the system shown in Figure 5.25.
16. Determine the equivalent damping coefficient for the system shown in Figure 5.26.

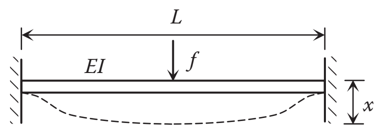


FIGURE 5.22 Problem 12.

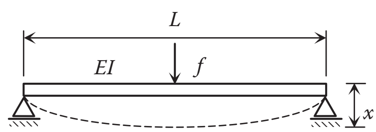


FIGURE 5.23 Problem 13.

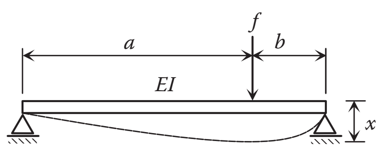


FIGURE 5.24 Problem 14.

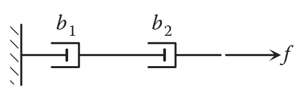


FIGURE 5.25 Problem 15.

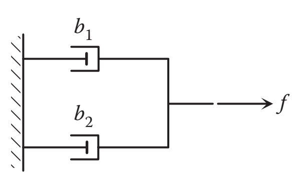


FIGURE 5.26 Problem 16.

5.2 TRANSLATIONAL SYSTEMS

With appropriate simplifying assumptions, a translational mechanical system can be modeled as a system of interconnected mechanical elements. The dynamic behavior of the system must obey the physical laws, and the dynamic equations of motion can be obtained by applying these physical laws, such as Newton's second law or D'Alembert's principle. The number of equations of motion is determined by the number of degrees of freedom of the system.

5.2.1 Degrees of Freedom

The number of degrees of freedom of a dynamic system is defined as the number of independent generalized coordinates that specify the configuration of the system. Generalized coordinates need not be restricted only to the actual position coordinates, which are physical coordinates. They could be anything, for example, position coordinates, translational displacement, rotational displacement, pressure, voltage, or current. The generalized coordinates of a system need not be of the same type.

Figure 5.27a shows a translational mechanical system, in which the mass m moves in the horizontal direction, and x is the displacement measured from the static equilibrium position of the mass. The displacement x is the generalized coordinate, and the number of degrees of freedom of the system is 1. When a pendulum consisting of a massless rod of length L and a point mass of M is attached to the block of mass m (see Figure 5.27b), one displacement coordinate x is not enough to describe the motion of the system. On the one hand, the pendulum moves together with the block in the horizontal direction. On the other hand, the pendulum rotates, and the rotational motion can be described using an angular displacement θ. Thus, for the system in Figure 5.27b, x and θ are the two independent generalized coordinates, and the number of degrees of freedom of the system is 2.

When generalized coordinates are independent, they are equal in number to the degrees of freedom. If generalized coordinates are dependent, then the number of degrees of freedom is the difference between the number of dependent coordinates and the number of constraints. For instance, to describe the motion of the pendulum in Figure 5.28, we can use the rectangular displacements x and y instead of the angular displacement θ. However, those two rectangular displacements are related by the constraint $x^2 + y^2 = L^2$. Thus, the number of degrees of freedom is 1 (i.e., number of dependent

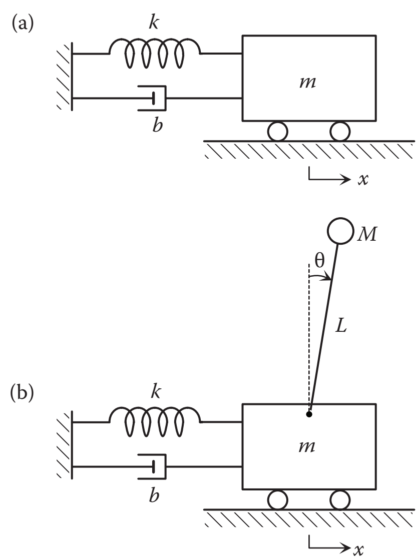


FIGURE 5.27 A mechanical system with (a) displacement as the generalized coordinate and (b) mixed types of generalized coordinates.

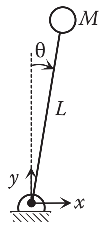


FIGURE 5.28 Independence of generalized coordinates.

generalized coordinates – number of constraints), which is the same as the result obtained by using the angular displacement as the generalized coordinate.

5.2.2 Newton's Second Law

Newton's second law states that the acceleration of a mass is proportional to the resultant force vector acting on it and is in the direction of this force. Assume that the translational motion of a particle or a rigid body is restricted in a plane. For a particle, which is a mass of negligible dimensions, Newton's second law can be expressed in vector form

$$\sum \mathbf{F} = m\mathbf{a} \tag{5.20}$$

or in scalar form

$$\sum F_x = ma_x, \quad \sum F_y = ma_y, \tag{5.21}$$

where $\sum F_x$ and $\sum F_y$ are summations of the applied forces decomposed along the x and y directions, respectively, and a_x and a_y are the x and y components of the acceleration of the particle, respectively.

For a rigid body, Newton's second law is given by

$$\sum \mathbf{F} = m\mathbf{a}_\mathrm{C} \tag{5.22}$$

or

$$\sum F_x = ma_{\mathrm{C}x}, \quad \sum F_y = ma_{\mathrm{C}y}, \tag{5.23}$$

where subscript C denotes the center of mass. In many engineering applications, the gravity field is considered to be uniform, and the center of mass coincides with the center of gravity.

5.2.3 Free-Body Diagrams

To apply Newton's second law to a mechanical system, it is useful to draw a free-body diagram for each mass in the system, showing all external forces. The noninput forces can be described in terms of displacements or velocities using the expressions associated with the basic spring or damper elements. Drawing correct free-body diagrams is the most important step in analyzing mechanical systems by the force/moment approach (as opposed to the energy approach).

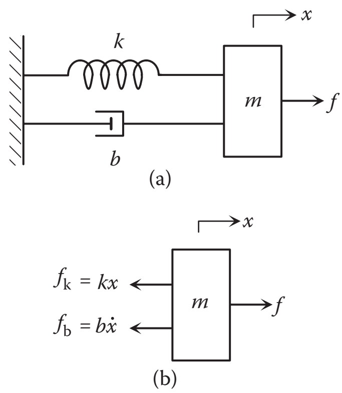


FIGURE 5.29 A mass–spring–damper system: (a) physical system and (b) free-body diagram.

Let us consider a simple system consisting of a block of mass m, a spring of stiffness k, and a viscous damper of viscous damping coefficient b. Figure 5.29 shows the physical mass–spring–damper system and the free-body diagram drawn for the mass. The motion of the system can be described using the displacement variable x, which is chosen as the generalized coordinate. The positive direction is the direction shown by the arrow next to the displacement x. Assume that the positive direction is to the right as shown. This sign convention implies that the displacement x, velocity $\dot{x}$, and acceleration $\ddot{x}$ are positive to the right.

Three forces included in the free-body diagram are the applied force f, the force exerted by the spring f_k, and the force exerted by the damper f_b. The magnitudes of the forces are shown in the free-body diagram, and their physical directions are indicated by the arrows. The force f is externally applied to the mass–spring–damper system, and the positive direction is given to the right. To determine the forces f_k and f_b, we can imagine the mass to be displaced along the positive direction, $x > 0$. Thus, the spring is in tension, and there is a spring force $f_k = kx$ applied to the mass. The force exerted by the spring is to the left because it tends to restore to the undeformed position. Similarly, the assumption of $\dot{x} > 0$ indicates that the mass moves to the right with a velocity $\dot{x}$. Remember that the damping force for a moving mass is always opposite to the direction of motion. Thus, the force exerted by the damper is to the left, and the magnitude is $f_b = b\dot{x}$.

The above analysis shows that an assumption about the motion of all masses in a mechanical system must be made to draw the free-body diagrams. It is customary to assume that all displacements are in the assumed positive directions when determining the proper magnitudes and directions for the forces. Applying Newton's second law to the correct free-body diagrams leads to differential equations of motion, which can be converted to other system representations, such as the transfer function form and the state-space form.

Example 5.4: A Single-Degree-of-Freedom Mass–Spring–Damper System

Consider the simple mass–spring–damper system subjected to an input force f, as shown in Figure 5.29a.

a. Apply Newton's second law to derive the differential equation of motion.
b. Determine the transfer function form. Assume that the system output is the displacement x and the initial conditions are $x(0) = 0$ and $\dot{x}(0) = 0$.
c. Determine the state-space representation. Assume that the system output is the displacement x and the state variables are $x_1 = x$ and $x_2 = \dot{x}$.
d. Use Simulink and Simscape to construct block diagrams to find the displacement output $x(t)$ of the system subjected to an applied force $f(t) = 10u(t)$, where $u(t)$ is the unit-step function. The parameter values are $m = 1$ kg, $b = 2$ N·s/m, and $k = 5$ N/m. Assume zero initial conditions.

Solution

a. Let us choose the displacement of the mass as the coordinate x. The free-body diagram of the mass is shown in Figure 5.29b. Applying Newton's second law in the x direction gives

$$+\rightarrow x: \sum F_x = ma_x,$$

$$f(t) - kx - b\dot{x} = m\ddot{x},$$

which can be rearranged into the standard input–output differential equation form

$$m\ddot{x} + b\dot{x} + kx = f(t).$$

b. Taking the Laplace transform of both sides of the preceding equation with zero initial conditions results in

$$(ms^2 + bs + k)\, X(s) = F(s).$$

Thus, the transfer function relating the input $f(t)$ to the output $x(t)$ is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}.$$

c. As specified, the state, the input, and the output are

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}, \quad u = f, \quad y = x.$$

The state-variable equations are thus formed as

$$\dot{x}_1 = \dot{x} = x_2,$$

$$\dot{x}_2 = \ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x} + \frac{1}{m}f = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u.$$

The output equation is

$$y = x = x_1.$$

Writing the state-variable equations and the output equation in matrix form yields

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u,$$

$$y = [1 \ \ 0] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + 0 \cdot u.$$

d. We start with building a Simulink block diagram using the mathematical model obtained in Part (a). Solving for the highest derivative of the output x gives

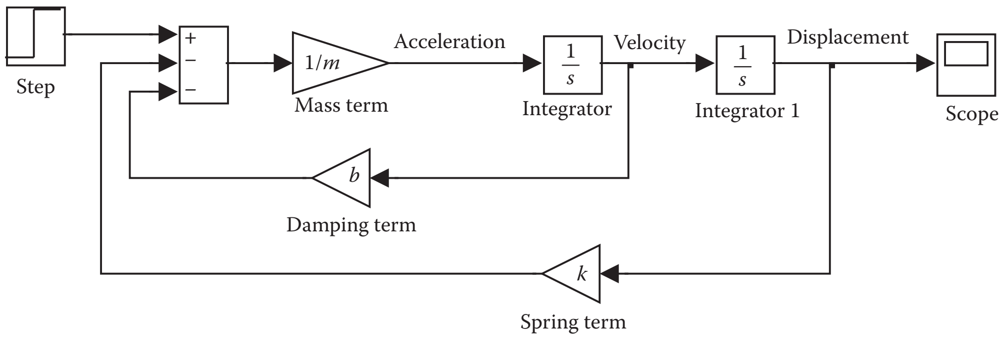


FIGURE 5.30 Simulink block diagram corresponding to Example 5.4.

$$\ddot{x} = \frac{1}{m}(f - kx - b\dot{x}),$$

which can be represented using the block diagram shown in Figure 5.30. Two `Integrator` blocks are used to form the velocity $\dot{x}$ and the displacement x, both of which are fed back to form the acceleration $\ddot{x}$. Note that a step input causes the motion of the system. Double-click on the block with the name `Step` and type `0` for the `Step time` and `10` for the `Final value` to define the input $f(t) = 10u(t)$.

We can also build a Simscape block diagram to simulate the physical system shown in Figure 5.29a. The mass, translational damper, and translational spring blocks can be found in `Simscape/Foundation Library/Mechanical/Translational Elements`. In the same library, the `Mechanical Translational Reference` block is used to represent a rigidly clamped end. To apply a force input, the `Ideal Force Source` block is included, which can be found in …`/Mechanical/Mechanical Sources`. To obtain a displacement output, the `Ideal Translational Motion Sensor` block is included, which can be found in …`/Mechanical/Mechanical Sensors`. Figure 5.31 is the resulting Simscape block diagram. Note that port symbols (e.g., `C` for "case," `R` for "rod," `P` for "position," `V` for "velocity," `S` for "signal") are useful for making a correct connection.

Define the values of the parameters m, b, and k in the MATLAB Command window. Run both simulations and the same curve as shown in Figure 5.32 can be obtained, which is the resulting displacement output $x(t)$ of the mass–spring–damper system in Figure 5.29a subjected to a step input force. More examples of using Simulink and Simscape to build a mechanical system will be given in Section 5.6.

5.2.4 Static Equilibrium Position and Coordinate Reference

In Example 5.4, we specified the static equilibrium position as the coordinate origin. For this mass–spring–damper system moving only in the horizontal direction, the mass is in equilibrium when the spring is at its free length. Note that it is advantageous to choose the static equilibrium position as the coordinate origin, because this choice can simplify the equation of motion by eliminating static forces. The advantage is obvious when the motion along the vertical direction is involved. The following example shows that the gravity term does not enter into the governing differential equation if the displacement is measured from the static equilibrium.

Consider the mass–spring system shown in Figure 5.33, where the mass is assumed to move only in the vertical direction. The free length of the spring is y_0. Due to gravity, the spring is stretched by δ_{st} when the mass is in static equilibrium and $mg = k\delta_{st}$. Imagine the mass to be displaced downward

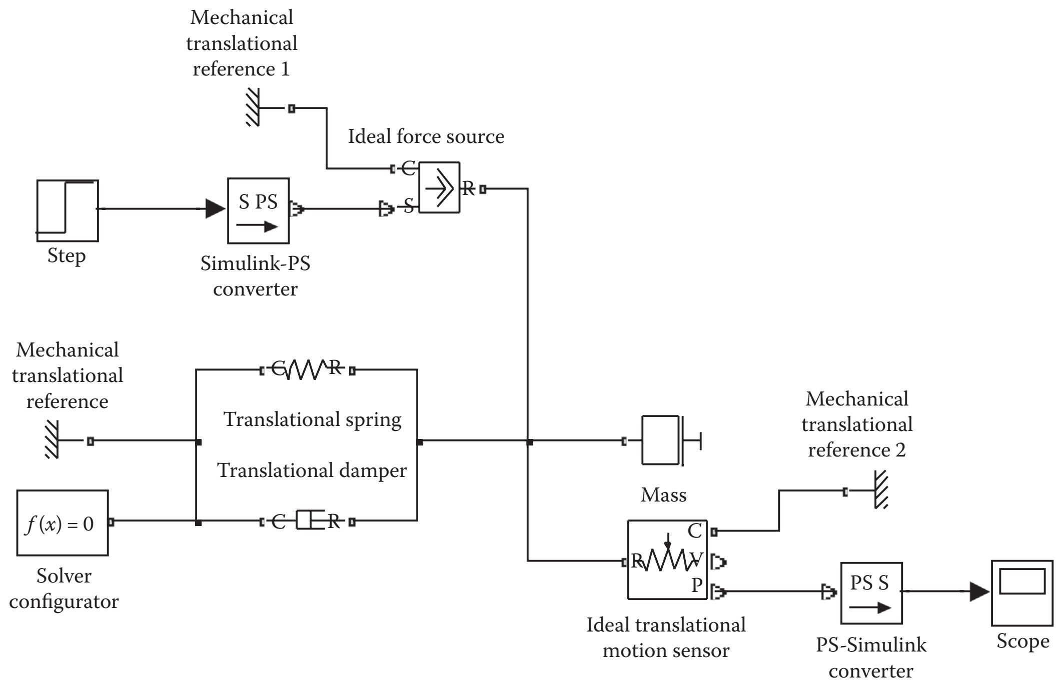


FIGURE 5.31 Simscape block diagram corresponding to Example 5.4.

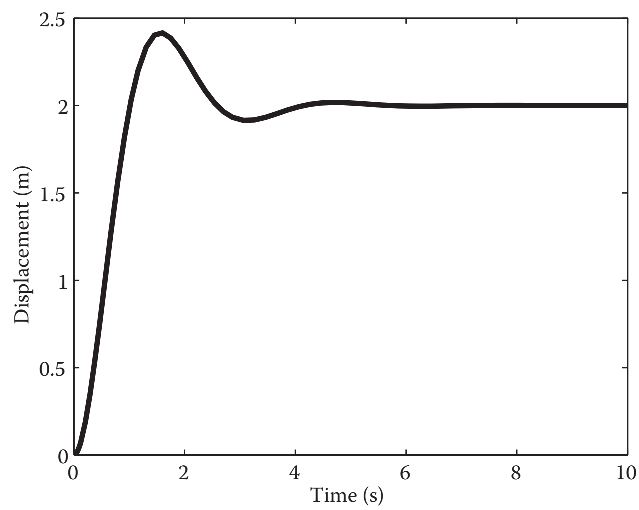


FIGURE 5.32 Displacement output $x(t)$ of the mechanical system in Example 5.4.

by a distance of x. If we choose the undeformed position in Figure 5.33a as the origin of the coordinate y, applying Newton's second law to the free-body diagram in Figure 5.33c gives

$$+\downarrow y: \sum F_y = ma_y,$$

$$mg - ky = m\ddot{y},$$

$$m\ddot{y} + ky = mg.$$

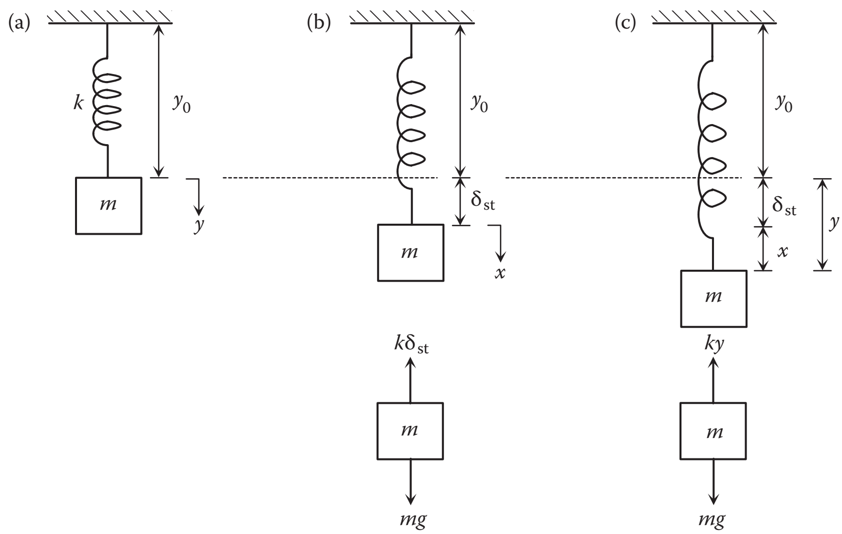


FIGURE 5.33 Choice of coordinate origins for a mass–spring system: (a) undeformed position, (b) static equilibrium position, and (c) dynamic position.

Note that the gravity term mg appears in the equation of motion. Now let us choose the static equilibrium position in Figure 5.33b as the origin of the coordinate x. The equation of motion is

$$+\downarrow x: \sum F_x = ma_x,$$

$$mg - k(x+\delta_{st}) = m\ddot{x},$$

$$m\ddot{x} + kx = 0,$$

where the gravity term mg and the static spring force $k\delta_{st}$ cancel out, resulting in a simpler equation.

Example 5.5: A Two-Degree-of-Freedom Quarter-Car Model

Consider a quarter-car model shown in Figure 5.34a, where m_1 is the mass of one-fourth of the car body and m_2 is the mass of the wheel–tire–axle assembly. The spring k_1 represents the elasticity of the suspension and the spring k_2 represents the elasticity of the tire. $z(t)$ is the displacement input due to the surface of the road. The actuator force, f, applied between the car body and the wheel–tire–axle assembly, is controlled by feedback and represents the active components of the suspension system.

a. Draw the necessary free-body diagrams and derive the differential equations of motion.
b. Determine the state-space representation. Assume that the displacements of the two masses, x_1 and x_2, are the outputs and the state variables are $x_1 = x_1$, $x_2 = x_2$, $x_3 = \dot{x}_1$, and $x_4 = \dot{x}_2$.
c. The parameter values are $m_1 = 290$ kg, $m_2 = 59$ kg, $b_1 = 1000$ N·s/m, $k_1 = 16{,}182$ N/m, and $k_2 = 19{,}000$ N/m. Use MATLAB commands to define the system in the state-space form and then convert it to the transfer function form. Assume that all the initial conditions are zero.

Solution

a. We choose the displacements of the two masses x_1 and x_2 as the generalized coordinates. The static equilibrium positions of m_1 and m_2 are set as the coordinate origins. Assume

$$x_1 > x_2 > z > 0,$$

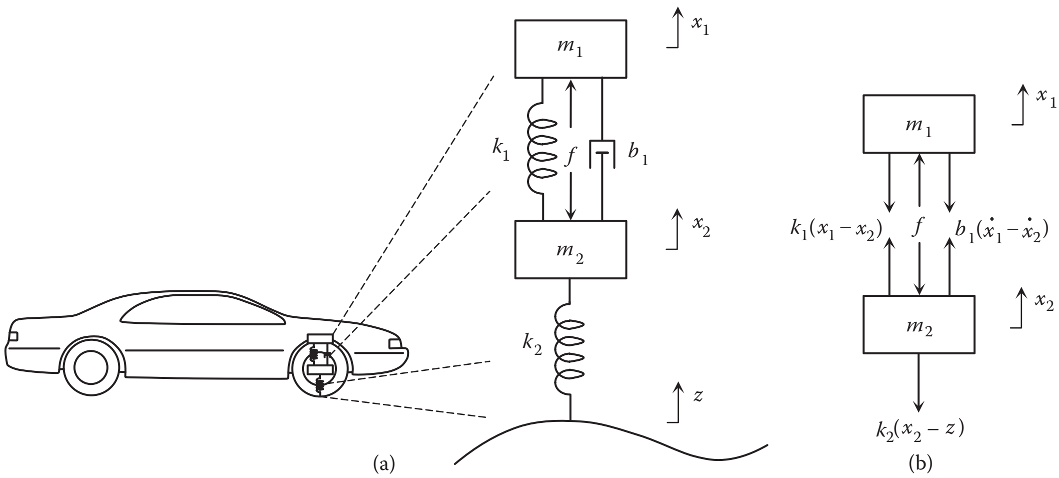


FIGURE 5.34 A quarter-car model: (a) physical system and (b) free-body diagram.

which implies that the springs are in tension and

$$\dot{x}_1 > \dot{x}_2 > \dot{z} > 0.$$

The free-body diagrams of m_1 and m_2 are shown in Figure 5.34b. According to the assumption, the mass m_1 moves faster than the mass m_2, and the elongation of the spring k_1 is $x_1 - x_2$. The force exerted by the spring k_1 on the mass m_1 is downward as it tends to restore to the undeformed position. Because of Newton's third law, the force exerted by the spring k_1 on the mass m_2 has the same magnitude, but opposite in direction. Other spring forces and damping forces can be determined using the same logic. Note that the gravitational forces, m_1g and m_2g, are not included in the free-body diagrams.

Applying Newton's second law to the masses m_1 and m_2, respectively, gives

$$+\uparrow x: \sum F_x = ma_x,$$

$$f - k_1(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2) = m_1\ddot{x}_1,$$

$$-f + k_1(x_1 - x_2) + b_1(\dot{x}_1 - \dot{x}_2) - k_2(x_2 - z) = m_2\ddot{x}_2.$$

Rearranging the equations into the standard input–output form,

$$m_1\ddot{x}_1 + b_1\dot{x}_1 - b_1\dot{x}_2 + k_1x_1 - k_1x_2 = f,$$

$$m_2\ddot{x}_2 - b_1\dot{x}_1 + b_1\dot{x}_2 - k_1x_1 + (k_1 + k_2)x_2 = -f + k_2z,$$

which can be expressed in second-order matrix form (Section 4.1) as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} b_1 & -b_1 \\ -b_1 & b_1 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & k_2 \end{bmatrix}\begin{Bmatrix} f \\ z \end{Bmatrix}.$$

b. Note that the inputs to the system are the actuator force f and the road surface irregularity z. The state, the input, and the output vectors are

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}, \quad \mathbf{u} = \begin{Bmatrix} f \\ z \end{Bmatrix}, \quad \mathbf{y} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}.$$

The state-variable equations are then obtained as

$$\begin{aligned}
\dot{x}_1 &= x_3, \\
\dot{x}_2 &= x_4, \\
\dot{x}_3 &= \ddot{x}_1 = -\frac{k_1}{m_1}x_1 + \frac{k_1}{m_1}x_2 - \frac{b_1}{m_1}\dot{x}_1 + \frac{b_1}{m_1}\dot{x}_2 + \frac{f}{m_1} \\
&= -\frac{k_1}{m_1}x_1 + \frac{k_1}{m_1}x_2 - \frac{b_1}{m_1}x_3 + \frac{b_1}{m_1}x_4 + \frac{1}{m_1}u_1, \\
\dot{x}_4 &= \ddot{x}_2 = \frac{k_1}{m_2}x_1 - \frac{k_1 + k_2}{m_2}x_2 + \frac{b_1}{m_2}\dot{x}_1 - \frac{b_1}{m_2}\dot{x}_2 - \frac{f}{m_2} + \frac{k_2}{m_2}z \\
&= \frac{k_1}{m_2}x_1 - \frac{k_1 + k_2}{m_2}x_2 + \frac{b_1}{m_2}x_3 - \frac{b_1}{m_2}x_4 - \frac{1}{m_2}u_1 + \frac{k_2}{m_2}u_2.
\end{aligned}$$

The output equation is

$$\mathbf{y} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}.$$

Thus, the state-space representation is

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1}{m_1} & \dfrac{k_1}{m_1} & -\dfrac{b_1}{m_1} & \dfrac{b_1}{m_1} \\ \dfrac{k_1}{m_2} & -\dfrac{k_1 + k_2}{m_2} & \dfrac{b_1}{m_2} & -\dfrac{b_1}{m_2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{1}{m_1} & 0 \\ -\dfrac{1}{m_2} & \dfrac{k_2}{m_2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}.$$

c. The following is the MATLAB session:

```
>> m1 = 290;
>> m2 = 59;
>> b1 = 1000;
>> k1 = 16182;
>> k2 = 19000;
```

```
>> A = [0 0 1 0;
   0 0 0 1;
   -k1/m1 k1/m1 -b1/m1 b1/m1;
   k1/m2 -(k1+k2)/m2 b1/m2 -b1/m2];
>> B = [0 0; 0 0; 1/m1 0; -1/m2 k2/m2];
>> C = [1 0 0 0; 0 1 0 0];
>> D = zeros(2,2);
>> sys_ss = ss(A,B,C,D);
>> sys_tf = tf(sys_ss);
```

The command `tf` returns two transfer functions from the input #1 (i.e., f) to the two outputs x_1 and x_2:

$$\frac{X_1(s)}{F(s)} = \frac{0.003448s^2 + 1.11}{s^4 + 20.4s^3 + 652.1s^2 + 1110s + 17{,}970},$$

$$\frac{X_2(s)}{F(s)} = \frac{-0.01695s^2 + 0.04655s + 0.7533}{s^4 + 20.4s^3 + 652.1s^2 + 1110s + 17{,}970},$$

and another two transfer functions from the input #2 (i.e., z) to the outputs x_1 and x_2:

$$\frac{X_1(s)}{Z(s)} = \frac{1110s + 17{,}970}{s^4 + 20.4s^3 + 652.1s^2 + 1110s + 17{,}970},$$

$$\frac{X_2(s)}{Z(s)} = \frac{322s^2 + 1110s + 17{,}970}{s^4 + 20.4s^3 + 652.1s^2 + 1110s + 17{,}970}.$$

Note that the system has two inputs and two outputs. Thus, there are a total of four transfer functions, which can be formed as a 2 × 2 transfer matrix

$$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \begin{bmatrix} \dfrac{X_1(s)}{F(s)} & \dfrac{X_1(s)}{Z(s)} \\ \dfrac{X_2(s)}{F(s)} & \dfrac{X_2(s)}{Z(s)} \end{bmatrix}.$$

Note that the gravity terms in Example 5.5 do not appear in the equations of motion because the static equilibrium positions are chosen as the coordinate origins. Two independent coordinates, x_1 and x_2, are required to specify the system dynamics. Such a system is called a two-degree-of-freedom system, which is a special case of multiple-degree-of-freedom systems.

5.2.5 Massless Junctions

A system of massless junctions is a system of springs and dampers without any masses. The differential equations of motion for such a system can be derived using Newton's second law and simply letting the masses be zero at the junctions.

Example 5.6: A Two-Degree-of-Freedom System with Massless Junctions

Consider the system of massless junctions shown in Figure 5.35a. An external force f is applied to the junction A. Draw the free-body diagrams and derive the differential equations of motion.

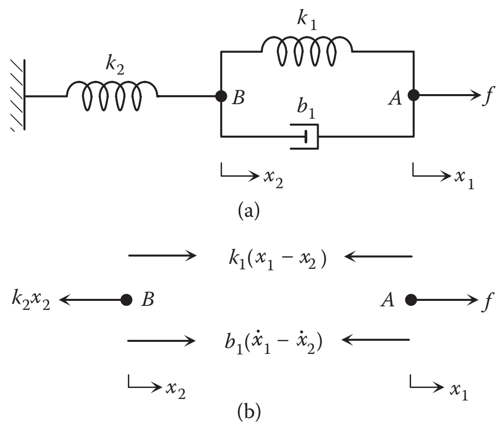


FIGURE 5.35 A two-degree-of-freedom system with massless junctions: (a) physical system and (b) free-body diagram.

Solution

Two massless junctions, A and B, are included in this system. We choose the displacements of the two junctions as the generalized coordinates, which are denoted by x_1 and x_2. Assume that $x_1 > x_2 > 0$. This implies that the two springs are in extension. The free-body diagrams at the two massless junctions are shown in Figure 5.35b. Applying Newton's second law to each massless junction gives

$$+\rightarrow x: \sum F_x = ma_x = 0,$$

$$\text{A}: -k_1(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2) + f = 0,$$

$$\text{B}: k_1(x_1 - x_2) - k_2x_2 + b_1(\dot{x}_1 - \dot{x}_2) = 0.$$

The equations can be rearranged as

$$b_1\dot{x}_1 - b_1\dot{x}_2 + k_1x_1 - k_1x_2 = f,$$

$$-b_1\dot{x}_1 + b_1\dot{x}_2 - k_1x_1 + (k_1 + k_2)x_2 = 0,$$

or in matrix form

$$\begin{bmatrix} b_1 & -b_1 \\ -b_1 & b_1 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix}.$$

Note that the system in Example 5.6 is a two-degree-of-freedom system. The dynamic behavior of the system is described by two first-order differential equations of motion, and thus it is a second-order system. If the two massless junctions are replaced by two masses as shown in Figure 5.36, the resulting system is still a two-degree-of-freedom system, but a fourth-order system. The reader can derive the differential equations of motion for the new system in Figure 5.36 as an exercise.

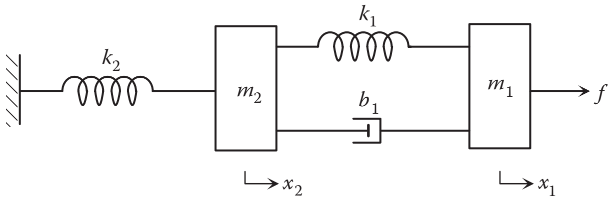


FIGURE 5.36 System obtained by replacing the massless junctions in Figure 5.35 with masses.

In Examples 5.5 and 5.6, the differential equations of motion are also given in second-order matrix form as $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}$. By observation, we find the following:

1. The mass, damping, and stiffness matrices are symmetric.
2. All diagonal entries of these matrices are nonnegative.
3. The off-diagonal elements of both the damping and the stiffness matrices are nonpositive.
4. The off-diagonal elements of the mass matrix are nonnegative.

These results are true for stable mechanical systems with purely translational or rotational motion. The reader can use them as necessary conditions to check the correctness of the differential equations of motion.

5.2.6 D'Alembert's Principle

Newton's second law can be reformulated as

$$\sum \mathbf{F} - m\mathbf{a}_C = 0, \tag{5.24}$$

which is known as D'Alembert's principle. The equation has $\sum \mathbf{F}$ as the sum of all the physical forces and $-m\mathbf{a}_C$ as the inertial force, which is a fictitious force. The minus sign associated with the inertial force indicates that the force acts in the negative direction when $\mathbf{a}_C > 0$. If the inertial force is included with the external forces, the mass can be considered to be in equilibrium. D'Alembert's principle is completely equivalent to the formulation of Newton's second law, although it looks like a classic static force balance. To use D'Alembert's principle correctly, the inertial force must be shown in the free-body diagrams correctly. The following is a simple example that shows the derivation of the differential equation of motion using D'Alembert's principle.

Example 5.7: D'Alembert's Principle

Reconsider the single-degree-of-freedom mass–spring–damper system in Example 5.4. Draw a free-body diagram and derive the differential equation of motion using D'Alembert's principle.

Solution

The displacement x is chosen as the generalized coordinate, and the origin is set at the static equilibrium position. For D'Alembert's principle, the free-body diagram of the mass is shown in Figure 5.37, where the inertial force is shown by a dashed line. Note that we assumed $x > 0$, which

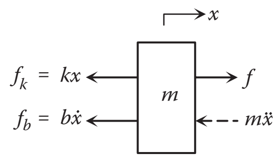


FIGURE 5.37 Free-body diagram of a mass–spring–damper system using D'Alembert's principle.

implies $\ddot{x} > 0$. The inertial force acts in the negative direction with a magnitude of $m\ddot{x}$. Applying D'Alembert's principle in the x direction results in

$$+\rightarrow x: \sum F_x - ma_x = 0,$$

$$f - kx - b\dot{x} - m\ddot{x} = 0,$$

or

$$m\ddot{x} + b\dot{x} + kx = f,$$

which is the same as the one obtained previously in Example 5.4.

PROBLEM SET 5.2

1. For the system shown in Figure 5.38, the input is the force f and the output is the displacement x of the mass.
   a. Draw the necessary free-body diagram and derive the differential equation of motion.
   b. Using the differential equation obtained in Part (a), determine the transfer function. Assume that initial conditions are $x(0) = 0$ and $\dot{x}(0) = 0$.
   c. Using the differential equation obtained in Part (a), determine the state-space representation.

2. Repeat Problem 1 for the system shown in Figure 5.39.
3. Repeat Problem 1 for the system shown in Figure 5.40.
4. Repeat Problem 1 for the system shown in Figure 5.41.
5. For Problems 1 through 4, use Simulink to construct block diagrams to find the displacement output $x(t)$ of the system subjected to an applied force $f(t) = 10u(t)$, where $u(t)$

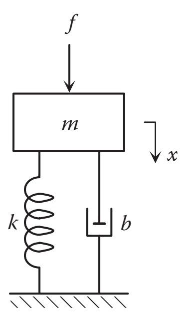


FIGURE 5.38 Problem 1.

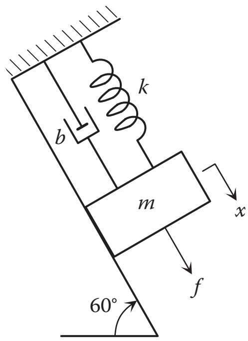


FIGURE 5.39 Problem 2.

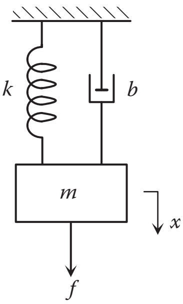


FIGURE 5.40 Problem 3.

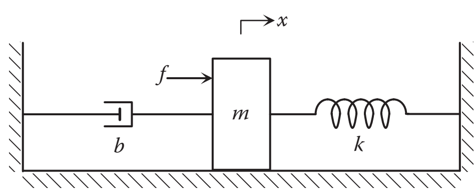


FIGURE 5.41 Problem 4.

is the unit-step function. The parameter values are $m = 1$ kg, $b = 2$ N·s/m, and $k = 5$ N/m. Assume zero initial conditions.

6. The system shown in Figure 5.42 simulates a machine supported by rubbers, which are approximated as four identical spring–damper units. The input is the force f and the output is the displacement x of the mass. The parameter values are $m = 500$ kg, $b = 250$ N·s/m, and $k = 200{,}000$ N/m.

   a. Draw the necessary free-body diagram and derive the differential equation of motion.
   b. Determine the transfer function. Assume zero initial conditions.
   c. Determine the state-space representation.
   d. Find the transfer function from the state-space form and compare with the result obtained in Part (b).

7. Repeat Problem 6 for the system shown in Figure 5.43.
8. The system shown in Figure 5.44 simulates a vehicle traveling on a rough road. The input is the displacement z.
   a. Draw the necessary free-body diagram and derive the differential equation of motion.

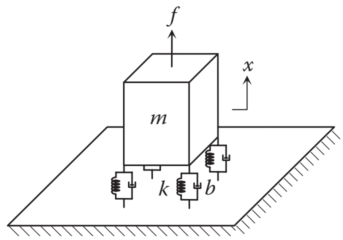


FIGURE 5.42 Problem 6.

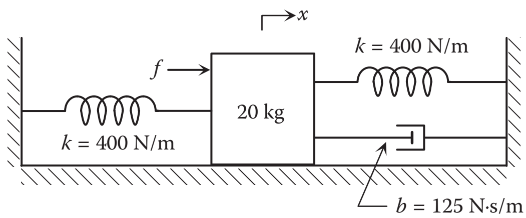


FIGURE 5.43 Problem 7.

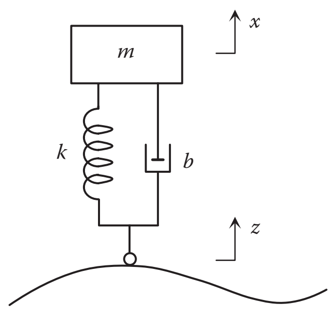


FIGURE 5.44 Problem 8.

b. Assuming zero initial conditions, determine the transfer function for two different cases of output: (1) displacement x and (2) velocity $\dot{x}$.
c. Determine the state-space representation for two different cases of output: (1) displacement x and (2) velocity $\dot{x}$.

9. Repeat Problem 8 for the system shown in Figure 5.45, where the cam and follower impart a displacement z to the lower end of the system.
10. For the system shown in Figure 5.46, the input is the force f and the outputs are the displacements x_1 and x_2 of the masses.
    a. Draw the necessary free-body diagrams and derive the differential equations of motion.
    b. Write the differential equations of motion in the second-order matrix form.
    c. Using the differential equations obtained in Part (a), determine the state-space representation.
11. Repeat Problem 10 for the system shown in Figure 5.47.
12. Repeat Problem 10 for the system shown in Figure 5.48.
13. For Problems 10 through 12, use MATLAB commands to define the systems in the state-space form and then convert to the transfer function form. Assume that the displacements of the two masses, x_1 and x_2, are the outputs, and all initial conditions are zero. The masses are $m_1 = 5$ kg and $m_2 = 15$ kg. The spring constants are $k_1 = 7.5$ kN/m and $k_2 = 15$ kN/m. The viscous damping coefficients are $b_1 = 280$ N·s/m and $b_2 = 90$ N·s/m.

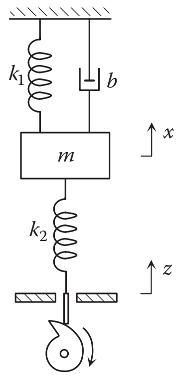


FIGURE 5.45 Problem 9.

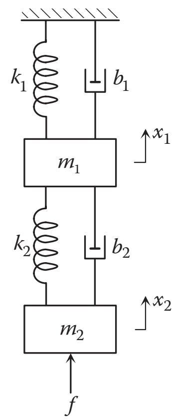


FIGURE 5.46 Problem 10.

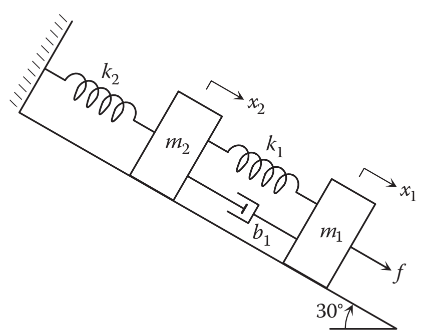


FIGURE 5.47 Problem 11.

14. For the system in Figure 5.49, the inputs are the forces f_1 and f_2 applied to the masses and the outputs are the displacements x_1 and x_2 of the masses.
   a. Draw the necessary free-body diagrams and derive the differential equations of motion.
   b. Write the differential equations of motion in the second-order matrix form.
   c. Using the differential equations obtained in Part (a), determine the state-space representation.

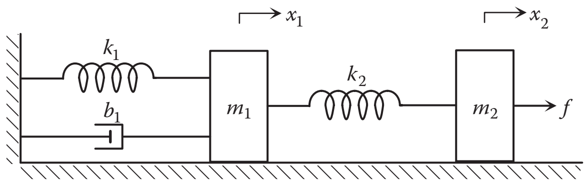


FIGURE 5.48 Problem 12.

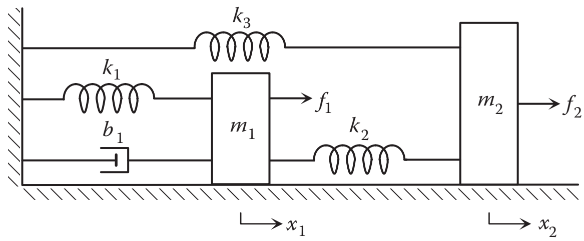


FIGURE 5.49 Problem 14.

15. Repeat Problem 14 for the system shown in Figure 5.50.
16. For Problems 14 and 15, use MATLAB commands to define the systems in the state-space form and then convert to the transfer function form. Assume that the displacements of the two masses, x_1 and x_2, are the outputs, and all initial conditions are zero. The masses are $m_1 = 5$ kg and $m_2 = 15$ kg. The spring constants are $k_1 = 7.5$ kN/m, $k_2 = 15$ kN/m, and $k_3 = 30$ kN/m. The viscous damping coefficients are $b_1 = 280$ N·s/m and $b_2 = 90$ N·s/m.
17. For the system in Figure 5.51, the input is the force f and the outputs are the displacement x_1 of the mass and the displacement x_2 of the massless junction A.
    a. Draw the necessary free-body diagrams and derive the differential equations of motion. Determine the number of degrees of freedom and the order of the system.
    b. Write the differential equations of motion in the second-order matrix form.
    c. Using the differential equation obtained in Part (a), determine the state-space representation.
18. Repeat Problem 17 for the system in Figure 5.52, where the input is the displacement z and the outputs are the displacements x_1 and x_2.

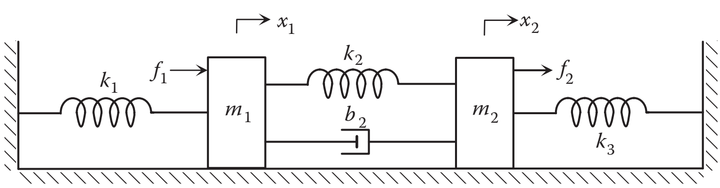


FIGURE 5.50 Problem 15.

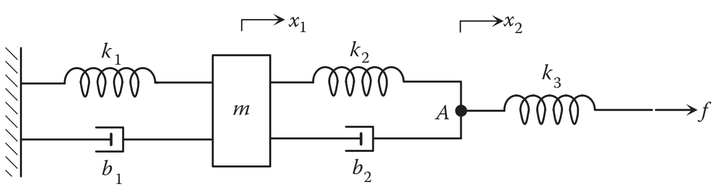


FIGURE 5.51 Problem 17.

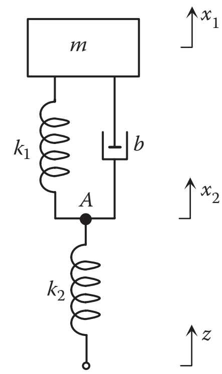


FIGURE 5.52 Problem 18.

5.3 ROTATIONAL SYSTEMS

In this section, we consider the derivation of a mathematical model for a rotational mechanical system. When a rigid body moves arbitrarily in three-dimensional space, the axis of rotation keeps changing. This makes modeling rather complex. Thus, the discussion of three-dimensional rigid body motions is not covered in this text, and we are mainly concerned with two-dimensional or plane motion. The fundamentals of rigid bodies in three dimensions are included only to help the interested reader in further studies.

5.3.1 General Moment Equation

The moment equation is applicable to systems of particles and rigid bodies in three and two dimensions. For a system of particles connected rigidly or a rigid body in arbitrary motion, using Newton's second law leads to the general moment equation given by

$$\sum \mathbf{M}_\mathrm{P} = \dot{\mathbf{H}}_\mathrm{P} + m\mathbf{r}_{\mathrm{C/P}} \times \mathbf{a}_\mathrm{P}, \tag{5.25}$$

where subscript P denotes an arbitrary accelerating point, subscript C denotes the mass center of the system of particles or the rigid body, $\sum \mathbf{M}_\mathrm{P}$ is the sum of all externally applied moments about point P, $\mathbf{H}_\mathrm{P}$ is the angular momentum vector about point P, $\dot{\mathbf{H}}_\mathrm{P}$ is the time rate of change of $\mathbf{H}_\mathrm{P}$, m is the total mass of the system of particles or the mass of the rigid body, $\mathbf{r}_{\mathrm{C/P}}$ is the position vector of the mass center C with respect to point P, and $\mathbf{a}_\mathrm{P}$ is the acceleration vector of point P.

It is usually difficult to obtain $\dot{\mathbf{H}}_\mathrm{P}$ for complex systems. For a rigid body, the angular momentum is related to mass moments of inertia and the angular velocity,

$$\mathbf{H}_\mathrm{P} = \mathbf{I}_\mathrm{P}\omega \tag{5.26}$$

or

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix}_\mathrm{P} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}_\mathrm{P} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}, \tag{5.27}$$

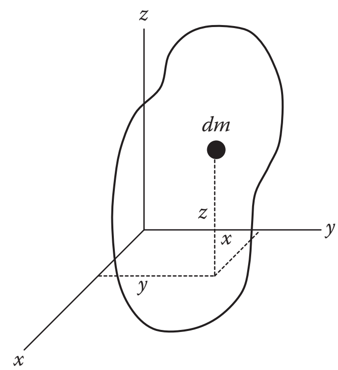


FIGURE 5.53 A differential element of a rigid body.

where $\mathbf{H}_\mathrm{P}$, the angular momentum about point P, and the angular velocity $\boldsymbol{\omega}$ are 3 × 1 vectors. The 3 × 3 matrix $\mathbf{I}_\mathrm{P}$ is known as the mass moment of inertia tensor about point P, where

$$I_{xx} = \int (y^2 + z^2)\,\mathrm{d}m, \quad I_{yy} = \int (x^2 + z^2)\,\mathrm{d}m, \quad I_{zz} = \int (x^2 + y^2)\,\mathrm{d}m, \tag{5.28}$$

$$I_{xy} = I_{yx} = -\int xy\,\mathrm{d}m, \quad I_{yz} = I_{zy} = -\int yz\,\mathrm{d}m, \quad I_{xz} = I_{zx} = -\int xz\,\mathrm{d}m. \tag{5.29}$$

As shown in Figure 5.53, the integrals containing squares represent the mass moments of inertia of the body about the x, y, and z axes, respectively. The integrals containing products of coordinates represent the mass products of inertia of the body.

In general, the time derivatives of the nine elements in the matrix $\mathbf{I}_\mathrm{P}$ are nonzero for a nonsymmetric rigid body, and thus it is difficult to study the dynamics of a rigid body in three dimensions. However, if the motion of the rigid body is restricted to a plane, the complexity is reduced significantly.

5.3.2 Modeling of Rigid Bodies in Plane Motion

In many engineering applications, the motion of rigid bodies is primarily in two dimensions. A rigid body is in plane motion if it translates in two dimensions (plane) and rotates only about an axis that is perpendicular to the plane. For a rigid body in plane motion, the mass moment of inertia is a scalar quantity and the time rate of change of the angular momentum reduces to

$$\dot{\mathbf{H}}_\mathrm{P} = I_\mathrm{P}\boldsymbol{\alpha}, \tag{5.30}$$

where I_P is the mass moment of inertia of the body about an axis through point P and α is the angular acceleration. Then Equation 5.25 becomes

$$\sum \mathbf{M}_\mathrm{P} = I_\mathrm{P}\boldsymbol{\alpha} + m\mathbf{r}_{\mathrm{C/P}} \times \mathbf{a}_\mathrm{P}. \tag{5.31}$$

Note that the rigid body in plane motion is constrained to rotate about only one axis. Therefore, the net moment $\sum \mathbf{M}_P$ and the angular acceleration $\boldsymbol{\alpha}$ are essentially scalars, whose signs signify the directions (e.g., clockwise or counterclockwise). Because the cross product term is $m\mathbf{r}_{C/P} \times \mathbf{a}_P = \mathbf{r}_{C/P} \times (m\mathbf{a}_P)$, it can be considered as the effective moment caused by a fictitious force $m\mathbf{a}_P$, and the direction of the effective moment can be denoted by its sign.

If the rigid body rotates about a fixed axis through point O, Equation 5.31 can be simplified as

$$\sum M_O = I_O \alpha \tag{5.32}$$

with P = O and $a_P = a_O = 0$. I_O is the mass moment of inertia of the body about point O. If the axis of rotation is not fixed, the dynamics model of the rigid body can be derived using Equation 5.31 or

$$\sum M_C = I_C \alpha \tag{5.33}$$

with P = C and $r_{C/P} = 0$. I_C is the mass moment of inertia of the body about the mass center C. Both Equations 5.31 and 5.33 are applicable regardless of whether the axis of rotation is fixed or not.

Example 5.8: A Single-Degree-of-Freedom Rotational Mass–Spring–Damper System

Consider a simple disk–shaft system shown in Figure 5.54a, in which the disk rotates about a fixed axis through point O. A single-degree-of-freedom torsional mass–spring–damper system in Figure 5.54b can be used to approximate the dynamic behavior of the disk–shaft system. I_O is the mass moment of inertia of the disk about point O, K represents the elasticity of the shaft, and B represents torsional viscous damping. Derive the differential equation of motion.

Solution

The free-body diagram of the disk is shown in Figure 5.54c. Because the disk rotates about a fixed axis, we can apply Equation 5.32 about the fixed point O. Assuming that counterclockwise is the positive direction, we have

$$+\curvearrowleft: \sum M_O = I_O \alpha,$$

$$\tau - K\theta - B\dot{\theta} = I_O \ddot{\theta}.$$

Thus,

$$I_O \ddot{\theta} + B\dot{\theta} + K\theta = \tau.$$

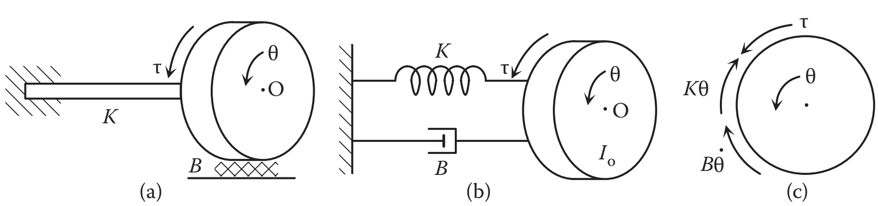


FIGURE 5.54 A rotational mass–spring–damper system: (a) physical system, (b) mass–spring–damper model, and (c) free-body diagram.

Example 5.9: A Two-Degree-of-Freedom Rotational Mass–Spring System

Consider the disk–shaft system shown in Figure 5.55a. The mass moments of inertia of disks are I_1 and I_2, respectively. The shafts can be modeled as massless torsional springs. The torsional mass–spring model is shown in Figure 5.55b.

a. Draw the necessary free-body diagrams and derive the differential equations of motion.
b. Determine the transfer functions $\Theta_1(s)/T(s)$ and $\Theta_2(s)/T(s)$. All initial conditions are assumed to be zero.

Solution

a. We choose the angular displacements θ_1 and θ_2 as the generalized coordinates. Assume that $\theta_1 > \theta_2 > 0$. The free-body diagrams are shown in Figure 5.56. Applying Equation 5.32 about the fixed points O_1 and O_2, respectively, gives

$$+\curvearrowleft: \quad \sum M_O = I_O \alpha,$$

$$\tau - K_1\theta_1 - K_2(\theta_1 - \theta_2) = I_1\ddot{\theta}_1,$$

$$K_2(\theta_1 - \theta_2) = I_2\ddot{\theta}_2.$$

Rearranging the equations results in

$$I_1\ddot{\theta}_1 + (K_1 + K_2)\theta_1 - K_2\theta_2 = \tau,$$

$$I_2\ddot{\theta}_2 - K_2\theta_1 + K_2\theta_2 = 0.$$

b. Taking Laplace transform gives

$$I_1 s^2\Theta_1(s) + (K_1 + K_2)\Theta_1(s) - K_2\Theta_2(s) = T(s),$$

$$I_2 s^2\Theta_2(s) - K_2\Theta_1(s) + K_2\Theta_2(s) = 0.$$

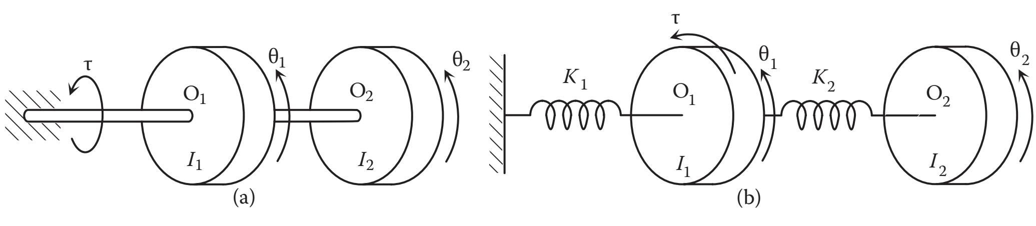


FIGURE 5.55 A disk–shaft system: (a) physical system and (b) mass–spring model.

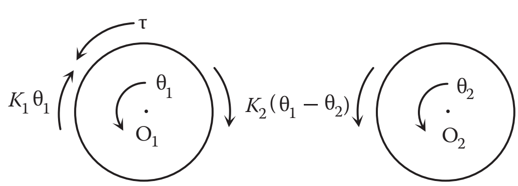


FIGURE 5.56 Free-body diagrams for the disk–shaft system in Figure 5.55.

or in matrix form

$$\begin{bmatrix} I_1 s^2 + K_1 + K_2 & -K_2 \\ -K_2 & I_2 s^2 + K_2 \end{bmatrix} \begin{Bmatrix} \Theta_1(s) \\ \Theta_2(s) \end{Bmatrix} = \begin{Bmatrix} T(s) \\ 0 \end{Bmatrix}.$$

Using Cramer's rule, we can solve for $\Theta_1(s)/T(s)$ and $\Theta_2(s)/T(s)$ as

$$\frac{\Theta_1(s)}{T(s)} = \frac{I_2 s^2 + K_2}{(I_1 s^2 + K_1 + K_2)(I_2 s^2 + K_2) - K_2^2} = \frac{I_2 s^2 + K_2}{I_1 I_2 s^4 + (I_1 K_2 + I_2 K_1 + I_2 K_2)s^2 + K_1 K_2},$$

$$\frac{\Theta_2(s)}{T(s)} = \frac{K_2}{(I_1 s^2 + K_1 + K_2)(I_2 s^2 + K_2) - K_2^2} = \frac{K_2}{I_1 I_2 s^4 + (I_1 K_2 + I_2 K_1 + I_2 K_2)s^2 + K_1 K_2}.$$

5.3.3 Mass Moment of Inertia

As shown in Equation 5.28, the mass moment of inertia of a rigid body about a specified axis of rotation is defined as

$$I = \int r^2 \,\mathrm{d}m, \tag{5.34}$$

where r is the distance between the axis of rotation and the mass element dm. The mass moments of inertia for some rigid bodies with common shapes are given in Table 5.1, in which all masses are assumed to be uniformly distributed and the axes of rotation all pass through the mass centers. If the axis of rotation does not coincide with the axis through the mass center, but is parallel to it, the parallel-axis theorem can be applied to obtain the corresponding moment of inertia,

$$I = I_C + md^2, \tag{5.35}$$

where I_C is the mass moment of inertia of the rigid body about an axis passing through its center of mass, I is the mass moment of inertia about the new axis, and d is the perpendicular distance between the two parallel axes.

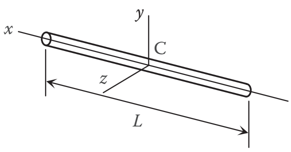
Example 5.10: An Inverted Pendulum–Bob System

Consider the inverted-pendulum system shown in Figure 5.57a, in which a point mass m is attached at the tip of a uniform slender rod of mass M and length L. The mass center of the rod is located at point C. The inverted-pendulum system rotates about an axis through the joint O. The friction at the joint is modeled as a torsional viscous damper of coefficient B.

a. Determine the mass moment of inertia of the system about point O.
b. Draw the free-body diagram for the inverted-pendulum system and obtain the nonlinear equation of motion.
c. Linearize the equation of motion for small angles θ.

TABLE 5.1
Mass Moments of Inertia of Common Geometric Shapes

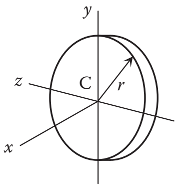
Slender rod



$$I_y = I_z = \frac{1}{12} mL^2$$

$$I_x = 0$$

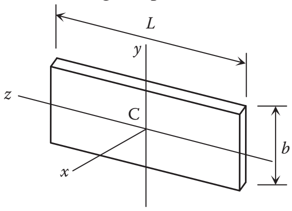
Thin disk



$$I_x = \frac{1}{2} mr^2$$

$$I_y = I_z = \frac{1}{4} mr^2$$

Thin rectangular plate

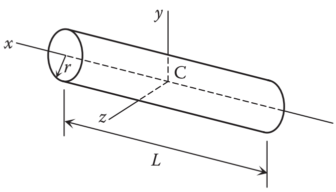


$$I_x = \frac{1}{12} m(L^2 + b^2)$$

$$I_y = \frac{1}{12} mL^2$$

$$I_z = \frac{1}{12} mb^2$$

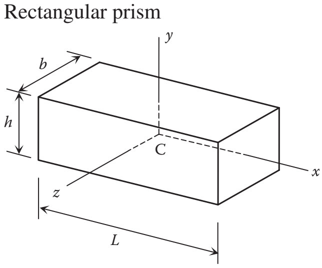
Circular cylinder



$$I_x = \frac{1}{2} mr^2$$

$$I_y = I_z = \frac{1}{12} m(3r^2 + L^2)$$

Rectangular prism

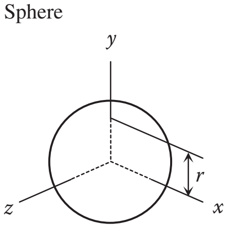


$$I_x = \frac{1}{12} m(b^2 + h^2)$$

$$I_y = \frac{1}{12} m(L^2 + b^2)$$

$$I_z = \frac{1}{12} m(L^2 + h^2)$$

Sphere



$$I_x = I_y = I_z = \frac{2}{5} mr^2$$

Source: Beer, F.P. et al., *Vector Mechanics for Engineers: Dynamics.* 9th ed., McGraw-Hill, 2009; Hibbeler, R.C., *Engineering Mechanics: Dynamics.* 12th ed., Prentice Hall, Upper Saddle River, NJ, 2010.

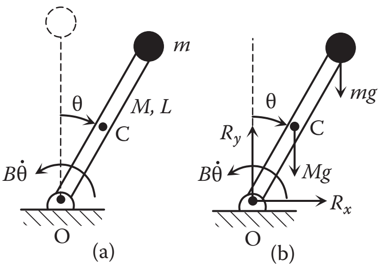


FIGURE 5.57 A pendulum–bob system: (a) physical system and (b) free-body diagram.

Solution

a. The system consists of a point mass and a slender rod. The total mass moment of inertia about point O is

$$I_O = I_{O\_mass} + I_{O\_rod},$$

where

$$I_{O\_mass} = mL^2$$

and I_{O_rod} can be obtained using the parallel-axis theorem,

$$I_{O\_rod} = I_{C\_rod} + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2.$$

Then

$$I_O = \left(m + \frac{M}{3}\right)L^2.$$

b. For the inverted-pendulum system, the free-body diagram is shown in Figure 5.57b, where R_x and R_y are the x and y components of the reaction force at the joint O, respectively. Note that the system rotates about a fixed axis through point O. Applying Equation 5.32 about the fixed point O gives

$$+\curvearrowright: \quad \sum M_O = I_O\alpha,$$

$$\frac{1}{2}L\sin\theta\cdot Mg + L\sin\theta\cdot mg - B\dot{\theta} = I_O\ddot{\theta}.$$

Substituting I_O obtained in Part (a) into the equation, and rearranging it in the input–output differential equation form, we obtain

$$\left(m + \frac{M}{3}\right)L^2\ddot{\theta} + B\dot{\theta} - \left(m + \frac{M}{2}\right)gL\sin\theta = 0.$$

c. For small angles θ, $\sin\theta \approx \theta$. The equation of motion becomes

$$\left(m + \frac{M}{3}\right)L^2\ddot{\theta} + B\dot{\theta} - \left(m + \frac{M}{2}\right)gL\theta = 0.$$

which is a linear equation in terms of θ.

Let us examine the nonlinear differential equation of motion in Part (b). Note that the unknown reaction forces R_x and R_y at the joint O do not appear in the moment equation $\sum M_O = I_O\alpha$. If

an arbitrary nonfixed point P is used, we must apply Equation 5.31. The moments caused by the unknown forces R_x and R_y appear in the equation, and auxiliary equations are required to eliminate the terms related to R_x and R_y. This is the advantage of choosing the fixed point to apply the moment equation if a rigid body rotates about a fixed axis.

Example 5.11: A Coupled Pendulum System

Consider the two-degree-of-freedom system shown in Figure 5.58a, in which two simple pendulums are connected by a translational spring of stiffness k. Each pendulum consists of a point mass m concentrated at the tip of a massless rope of length L. θ_1 and θ_2 are the angular displacements of the pendulums. When $\theta_1 = 0$ and $\theta_2 = 0$, the spring is at its free length. Draw the necessary free-body diagrams and derive the differential equations of motion. Assume small angles for θ_1 and θ_2.

Solution

We choose the angular displacements θ_1 and θ_2 as the generalized coordinates. Assume $\theta_1 > \theta_2 > 0$, which implies that the spring is in tension. The free-body diagrams are shown in Figure 5.58b. Applying Equation 5.32 about the fixed points O_1 and O_2, respectively, gives

$$+\curvearrowright: \quad \sum M_O = I_O\alpha,$$

$$-L\sin\theta_1 \cdot mg - L\cos\theta_1 \cdot f_k = mL^2\ddot{\theta}_1,$$

$$-L\sin\theta_2 \cdot mg + L\cos\theta_2 \cdot f_k = mL^2\ddot{\theta}_2.$$

Note that the spring force f_k is in the horizontal direction due to the small-angle assumption, and its magnitude can be approximated as $k(L\sin\theta_1 - L\sin\theta_2)$. Substituting f_k into the equations and rearranging them results in

$$mL^2\ddot{\theta}_1 + kL^2\cos\theta_1(\sin\theta_1 - \sin\theta_2) + mgL\sin\theta_1 = 0,$$

$$mL^2\ddot{\theta}_2 - kL^2\cos\theta_2(\sin\theta_1 - \sin\theta_2) + mgL\sin\theta_2 = 0.$$

For small angles, we have

$$mL^2\ddot{\theta}_1 + kL^2(\theta_1 - \theta_2) + mgL\theta_1 = 0,$$

$$mL^2\ddot{\theta}_2 - kL^2(\theta_1 - \theta_2) + mgL\theta_2 = 0.$$

The equations of motion can also be derived by applying Newton's second law along the normal and tangential directions. However, the unknown tension forces in the ropes appear in the equations

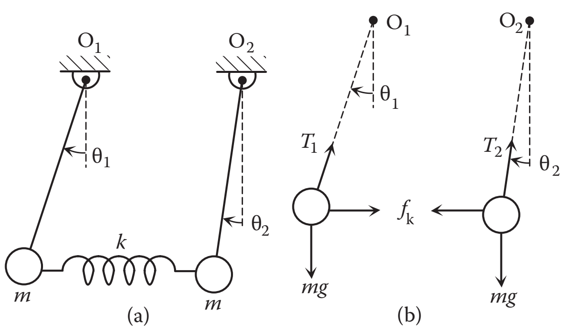


FIGURE 5.58 A coupled pendulum system: (a) physical system and (b) free-body diagram.

and need to be eliminated. In general, for systems involving rotational motion, it may be more efficient to use the moment equation.

5.3.4 Pure Rolling Motion

Wheels are common mechanical systems involving general plane motion. Figure 5.59a shows a uniform disk rolling on a horizontal surface. If there is no slipping between the disk and the surface, the disk undergoes pure rolling motion. The contact point is the instantaneous center (IC), where the velocity is zero. From the results of kinematics, the acceleration of the IC is $r\dot{\theta}^2$, and its direction points from the IC to the mass center C (or for a uniform disk, the geometric center). The problem of finding a_{IC} is left to the reader as an exercise.

The free-body diagram of the disk is shown in Figure 5.59b, where reaction forces at the contact point include the normal force N and the friction force f. Note that a fixed point does not exist for the rolling motion; thus, we cannot use the moment equation $\sum M_O = I_O\alpha$. Also, it is inconvenient to sum the moments about the mass center using the equation $\sum M_C = I_C\alpha$ because the unknown reaction force f inevitably appears in the equation and needs to be eliminated with the help of auxiliary equations. However, if the point IC is used, the moment equation

$$\sum \mathbf{M}_{IC} = I_{IC}\alpha + m\mathbf{r}_{C/IC} \times \mathbf{a}_{IC} \tag{5.36}$$

reduces to

$$\sum M_{IC} = I_{IC}\alpha. \tag{5.37}$$

Because the position vector $\mathbf{r}_{C/IC}$ and the acceleration vector $\mathbf{a}_{IC}$ are parallel, their cross product is zero, $m\mathbf{r}_{C/IC} \times \mathbf{a}_{IC} = 0$. Therefore, we do not need to deal with the moment caused by the friction force at the contact point. This is a special way of using Equation 5.31.

In summary, if there is a point S such that the position vector $\mathbf{r}_{C/S}$ is parallel to the acceleration vector $\mathbf{a}_S$, we can apply

$$\sum M_S = I_S\alpha \tag{5.38}$$

to model the dynamics of the rigid body. I_S is the mass moment of inertia of the body about point S. For the pure rolling disk in Figure 5.59, the IC at the contact point has the property of $\mathbf{r}_{C/IC} \parallel \mathbf{a}_{IC}$.

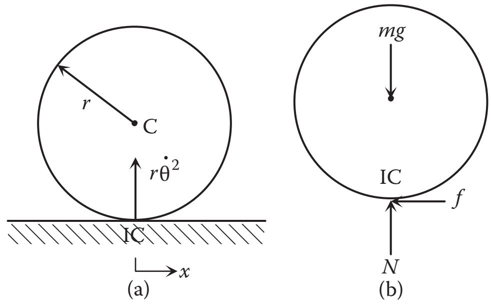


FIGURE 5.59 A pure rolling disk: (a) physical system and (b) free-body diagram.

Example 5.12: A Pure Rolling Disk

Consider the system shown in Figure 5.60a, in which a uniform disk of mass m and radius r rolls on a horizontal surface. A translational spring of stiffness k is attached to the disk. Assuming that there is no slipping between the disk and the surface, derive the differential equation of motion.

Solution

The free-body diagram of the system is shown in Figure 5.60b, in which the normal force N and the friction force f are reaction forces at the contact point. When the disk is at the static equilibrium position, we have $f_k = k\delta_{st}$, where δ_{st} is the static deformation of the spring. Then,

$$+\curvearrowright: \quad \sum M_{IC} = 0,$$

$$r \cdot k\delta_{st} = 0,$$

or $\delta_{st} = 0$. We choose the static equilibrium position as the origin. When the disk rolls, the spring force is $f_k = k(x + \delta_{st}) = kx$. Because of no slipping, the contact point is the IC, and $\mathbf{r}_{C/IC} \| \mathbf{a}_{IC}$. Applying Equation 5.37 gives

$$+\curvearrowright: \quad \sum M_{IC} = I_{IC}\alpha,$$

$$r \cdot kx = I_{IC}\alpha,$$

where

$$I_{IC} = I_C + md^2 = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2.$$

Introducing the assumption of no slipping, $x = r\theta$, we obtain the differential equation of motion

$$\frac{3}{2}mr^2\ddot{\theta} + kr^2\theta = 0.$$

Strictly speaking, the disk in Figure 5.59 or Example 5.12 is not a purely rotational system, and the rolling motion involves both translation and rotation. Such a system can also be modeled using both Newton's second law and the moment equation. We will present the corresponding discussion in the next section.

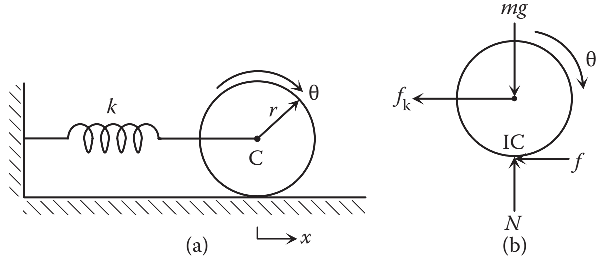


FIGURE 5.60 A pure rolling disk–spring system: (a) physical system and (b) free-body diagram.

PROBLEM SET 5.3

1. Consider the rotational system shown in Figure 5.61. The system consists of a massless shaft and a uniform thin disk of mass m and radius r. The disk is constrained to rotate about a fixed longitudinal axis along the shaft. The shaft is equivalent to a torsional spring of stiffness K. Draw the necessary free-body diagram and derive the differential equation of motion.
2. Repeat Problem 1 for the system shown in Figure 5.62.
3. Consider the torsional mass–spring–damper system in Figure 5.63. The mass moments of inertia of the two disks about their longitudinal axes are I_1 and I_2, respectively. The massless torsional springs represent the elasticity of the shafts and the torsional viscous dampers represent the fluid coupling.
   a. Draw the necessary free-body diagrams and derive the differential equations of motion. Provide the equations in the second-order matrix form.
   b. Determine the transfer functions $\Theta_1(s)/T(s)$ and $\Theta_2(s)/T(s)$. All the initial conditions are assumed to be zero.
   c. Determine the state-space representation with the angular displacements θ_1 and θ_2 as the outputs.
4. Repeat Problem 3 for the system shown in Figure 5.64. The input is the angular displacement ϕ at the end of the shaft.

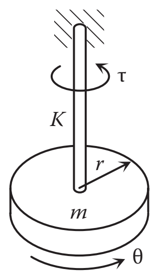


FIGURE 5.61 Problem 1.

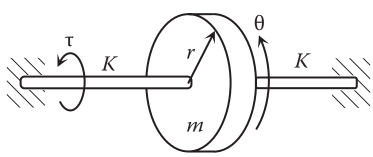


FIGURE 5.62 Problem 2.

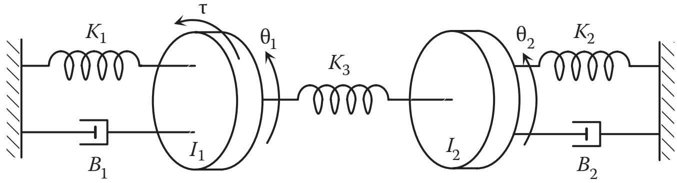


FIGURE 5.63 Problem 3.

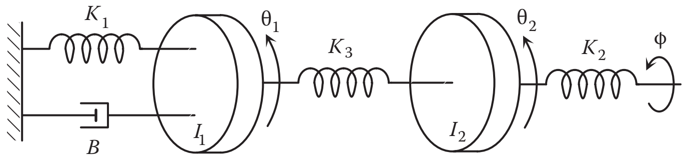


FIGURE 5.64 Problem 4.

5. Consider the pendulum system shown in Figure 5.65. The system consists of a bob of mass m and a uniform rod of mass M and length L. The pendulum pivots at the joint O. Draw the necessary free-body diagram and derive the differential equation of motion. Assume small angles for θ.
6. Repeat Problem 5 for the systems shown in Figure 5.66, in which (a) the mass of the rod is neglected and (b) no bob is attached to the rod.
7. The system shown in Figure 5.67 consists of a uniform rod of mass m and length L and a translational spring of stiffness k at the rod's left tip. The friction at the joint O is modeled as a damper with coefficient of torsional viscous damping B. The input is the force f and the output is the angle θ. The position $\theta = 0$ corresponds to the static equilibrium position when $f = 0$.
   a. Draw the necessary free-body diagram and derive the differential equation of motion for small angles θ.
   b. Using the linearized differential equation obtained in Part (a), determine the transfer function $\Theta(s)/F(s)$. Assume that the initial conditions are $\theta(0) = 0$ and $\dot{\theta}(0) = 0$.
   c. Using the differential equation obtained in Part (a), determine the state-space representation.

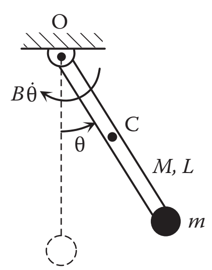


FIGURE 5.65 Problem 5.

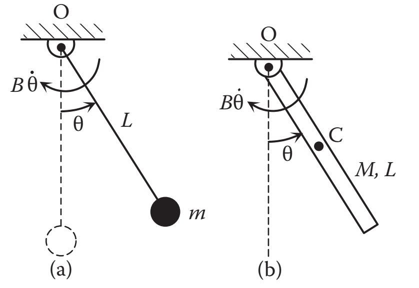


FIGURE 5.66 Problem 6.

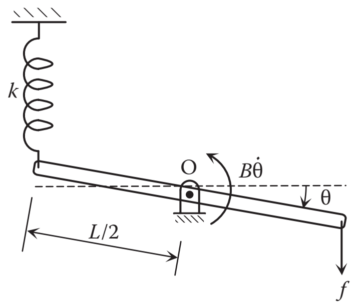


FIGURE 5.67 Problem 7.

8. Repeat Problem 7 for the system shown in Figure 5.68. The position $\theta = 0$ corresponds to the static equilibrium position when $f = 0$.
9. Example 5.4 Part (d) shows how one can represent a linear system in Simulink based on the differential equation of the system. A linear system can also be represented in transfer function or state-space form. The corresponding blocks in Simulink are `Transfer Fcn` and `State-Space`, respectively. Consider Problem 7 and construct a Simulink block diagram to find the output $\theta(t)$ of the system, which is represented using (a) the linearized differential equation of motion, (b) the transfer function, and (c) the state-space form obtained in Problem 7. The parameter values are $m = 0.8$ kg, $L = 0.6$ m, $k = 100$ N/m, $B = 0.5$ N·s/m, and $g = 9.81$ m/s^2. The input force f is the unit-impulse function, which has a magnitude of 10 N and a time duration of 0.1 s.
10. Repeat Problem 9 using the linearized differential equation of motion, the transfer function, and the state-space form obtained in Problem 8. The parameter values are $m = 0.2$ kg, $M = 0.8$ kg, $L = 0.6$ m, $k = 100$ N/m, and $g = 9.81$ m/s^2. The input force f is the unit-impulse function, which has a magnitude of 10 N and a time duration of 0.1 s.
11. Consider the system shown in Figure 5.69, in which the motion of the rod is a small angular rotation. When $\theta = 0$, the springs are at their free lengths.
    a. Determine the mass moment of inertia of the rod about point O. Assume that $a > b$.
    b. Draw the necessary free-body diagram and derive the differential equation of motion for small angles θ.

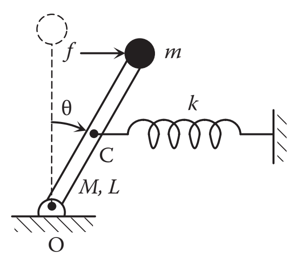


FIGURE 5.68 Problem 8.

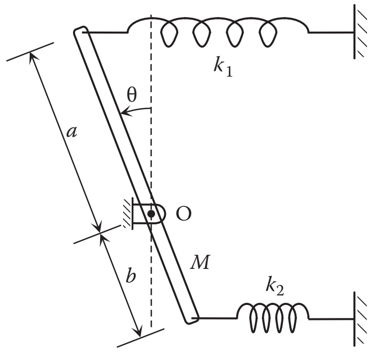


FIGURE 5.69 Problem 11.

12. Consider the system shown in Figure 5.70, in which a lever arm has a spring–damper combination on the other side. When $\theta = 0$, the system is in static equilibrium.
    a. Assuming that the lever arm can be approximated as a uniform slender rod, determine the mass moment of inertia of the rod about point O.
    b. Draw the necessary free-body diagram and derive the differential equation of motion for small angles θ.
13. Consider the two-degrees-of-freedom system shown in Figure 5.71, in which two simple inverted pendulums are connected by a translational spring of stiffness k. Each pendulum consists of a point mass m concentrated at the tip of a massless rod of length L. The inputs are the torques τ_1 and τ_2 applied to the pivot points O_1 and O_2, respectively. The outputs are the angular displacements θ_1 and θ_2 of the pendulums. When $\theta_1 = 0$, $\theta_2 = 0$, $\tau_1 = 0$, and $\tau_2 = 0$, the spring is at its free length.
    a. Draw the necessary free-body diagrams and derive the differential equations of motion. Assume small angles for θ_1 and θ_2. Provide the equations in the second-order matrix form.
    b. Using the differential equations obtained in Part (a), determine the state-space representation with the angular velocities $\dot{\theta}_1$ and $\dot{\theta}_2$ as the outputs.

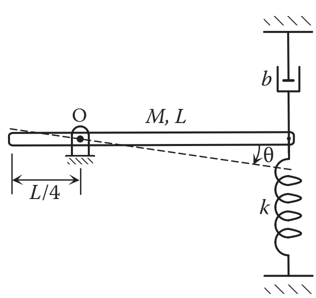


FIGURE 5.70 Problem 12.

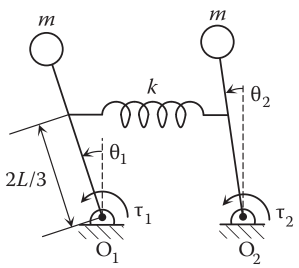


FIGURE 5.71 Problem 13.

14. Repeat Problem 13 for the system shown in Figure 5.72, in which each pendulum is a uniform slender rod of mass m and length L.
15. Consider the system shown in Figure 5.73, in which a uniform sphere of mass m and radius r rolls along an inclined plane of 30°. A translational spring of stiffness k is attached to the sphere. Assuming that there is no slipping between the sphere and the surface, draw the necessary free-body diagram and derive the differential equation of motion.
16. Consider the system shown in Figure 5.74. A uniform solid cylinder of mass m, radius R, and length L is fitted with a frictionless axle along the cylinder's long axis. A spring of stiffness k is attached to a bracket connected to the axle. Assume that the cylinder rolls without slipping on a horizontal surface. Draw the necessary free-body diagram and derive the differential equation of motion.

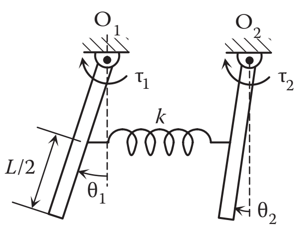


FIGURE 5.72 Problem 14.

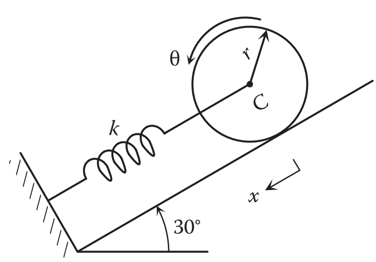


FIGURE 5.73 Problem 15.

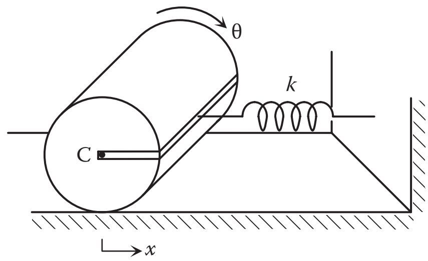


FIGURE 5.74 Problem 16.

5.4 MIXED SYSTEMS: TRANSLATIONAL AND ROTATIONAL

For a system involving both translational and rotational motions, Newton's second law and the moment equation can be used to obtain the model. This section provides only one method to obtain the differential equation of motion using the force and moment equations, which may be applied in different ways.

5.4.1 Force and Moment Equations

Consider a mechanical system in plane motion, which involves translations along the x and y directions and rotation about one axis perpendicular to the x–y plane. For a system of a single mass, applying Newton's second law in the translational directions gives the force equations

$$\sum F_x = ma_{Cx}, \quad \sum F_y = ma_{Cy}. \tag{5.39}$$

The moment equation is in the form of

$$\sum M_C = I_C\alpha \tag{5.40}$$

or

$$\sum M_P = I_C\alpha + M_{\text{eff}\_m\mathbf{a}_C}, \tag{5.41}$$

where the symbol $M_{\text{eff}\_m\mathbf{a}_C}$ is used to represent the effective moment caused by the fictitious force $m\mathbf{a}_C$. Although Equation 5.41 looks different from the general moment Equation 5.31 given in Section 5.3, they are essentially equivalent. Figure 5.75 shows a rigid body in plane motion. Applying Equation 5.31 gives the net moment about point P as $\sum \mathbf{M}_P = I_P\boldsymbol{\alpha} + m\mathbf{r}_{C/P} \times \mathbf{a}_P$. According to the parallel-axis theorem, we have $I_P = I_C + mr_{C/P}^2$. Then, $I_P\alpha = I_C\alpha + mr_{C/P}^2\alpha$ and it follows the direction of α, that is, the counterclockwise direction. Also, from the kinematics of rigid bodies, the acceleration of point P is $\mathbf{a}_P = \mathbf{a}_C + \mathbf{a}_{P/C,t} + \mathbf{a}_{P/C,n}$, as shown in Figure 5.76. Note that $m\mathbf{r}_{C/P} \times \mathbf{a}_C = \mathbf{r}_{C/P} \times (m\mathbf{a}_C)$, which can be considered as the effective moment caused by the fictitious force $m\mathbf{a}_C$ and is denoted as $M_{\text{eff}\_m\mathbf{a}_C}$. The tangential component of the relative acceleration $\mathbf{a}_{P/C,t}$ is perpendicular to the position vector $\mathbf{r}_{C/P}$ with a magnitude of $\alpha r_{C/P}$. Thus, the magnitude of the cross product $m\mathbf{r}_{C/P} \times \mathbf{a}_{P/C,t}$ is $mr_{C/P}^2\alpha$ and its direction is clockwise. The normal component $\mathbf{a}_{P/C,n}$ is parallel to the position vector $\mathbf{r}_{C/P}$, and thus $m\mathbf{r}_{C/P} \times \mathbf{a}_{P/C,n} = 0$. Assuming that the positive direction is counterclockwise, we have $\sum M_P = I_C\alpha + mr_{C/P}^2\alpha + M_{\text{eff}\_m\mathbf{a}_C} - mr_{C/P}^2\alpha$, which is the same as Equation 5.41.

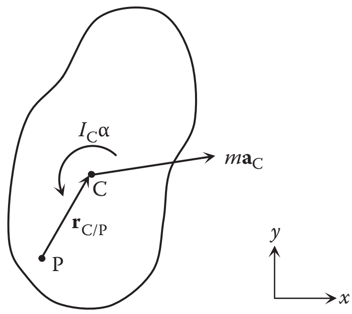


FIGURE 5.75 A rigid body in plane motion.

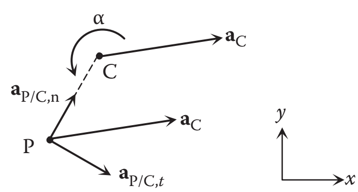


FIGURE 5.76 A kinematic diagram of $\mathbf{a}_P$ and $\mathbf{a}_C$.

For a system of multiple masses, the force equations become

$$\sum F_x = \sum_{i=1}^{n} m_i (a_{Ci})_x, \quad \sum F_y = \sum_{i=1}^{n} m_i (a_{Ci})_y, \tag{5.42}$$

where n is the number of masses and a_{Ci} is the acceleration of the mass center of the ith mass. Because it may be a challenge to find the mass center for the entire system of multiple masses, we use an arbitrary point P to sum the moments,

$$\sum M_P = \sum_{i=1}^{n} I_{Ci}\alpha_i + \sum_{i=1}^{n} M_{\text{eff}\_m_i\mathbf{a}_{Ci}}, \tag{5.43}$$

where I_{Ci} is the moment of inertia of the ith mass about its center of mass and α_i is the angular acceleration of the ith mass.

To derive the differential equation of motion correctly, we recommend drawing two diagrams before applying the force and moment equations. One is the free-body diagram that shows all external forces and moments applied to the system, and the other is the kinematic diagram that indicates the acceleration at the mass center of each mass. The left-hand sides of the force and moment Equations 5.42 and 5.43 are written based on the free-body diagram, and the right-hand sides are written based on the kinematic diagram.

Example 5.12: A Lever Mechanism

Consider the system shown in Figure 5.77, in which a lever arm has a force applied on one side and a spring–damper combination on the other side with a suspended mass. When $\theta = 0$ and $f = 0$, the system is at static equilibrium. Assume that the lever arm can be approximated as a uniform slender rod. Draw the free-body diagram of the lever arm and the suspended mass. Derive the differential equations of motion for small angles θ.

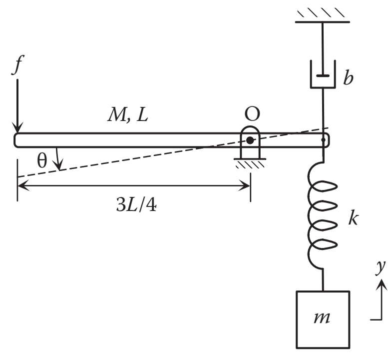


FIGURE 5.77 A lever mechanism.

Solution

This is a mixed system with two masses. The suspended block undergoes translational motion along the vertical direction, and the motion of the rod is pure rotation about point O that is fixed. We choose the displacement of the block y and the angular displacement of the rod θ as the generalized coordinates. Note that y is measured from the static equilibrium position, which is set as the origin. The free-body diagrams and the kinematic diagram of the system are shown in Figure 5.78.

At static equilibrium, for the block, we have

$$+\uparrow: \quad \sum F_y = 0,$$

$$k\delta_{st} - mg = 0,$$

$$k\delta_{st} = mg,$$

and for the arm,

$$+\curvearrowleft: \quad \sum M_O = 0,$$

$$Mg \cdot \frac{L}{4} - k\delta_{st} \cdot \frac{L}{4} = 0,$$

$$Mg = k\delta_{st},$$

where δ_{st} is the static deformation of the spring.

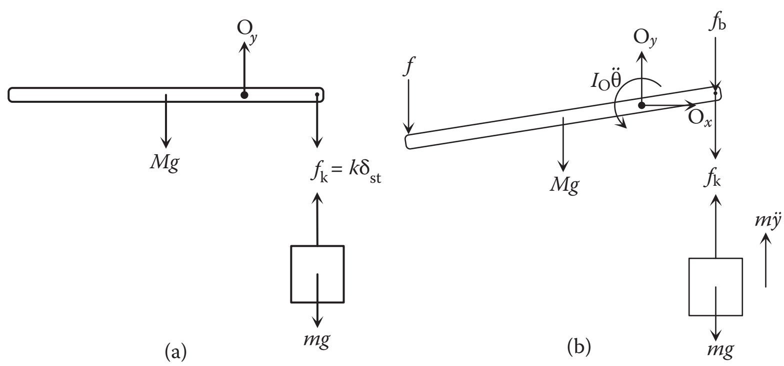


FIGURE 5.78 A lever mechanism: (a) free-body diagram at static equilibrium and (b) free-body diagram and kinematic diagram at dynamic position.

When a force f is applied on one side of the rod, the deformation of the spring caused by the rotation of the rod can be approximated as $L\theta/4$ for small angles θ. Assuming that the block and the rod are displaced in their positive directions and $L\theta/4 > y > 0$, the spring is in tension and the magnitude of the spring force is $f_k = k(L\theta/4 - y + \delta_{st})$. The magnitude of the damping force is $f_b = bL\dot{\theta}/4$. The free-body diagram and the kinematic diagram of the system are shown in Figure 5.78b. Note that the acceleration components $m\ddot{y}$ and $I_O\ddot{\theta}$ are shown together with the forces in the free-body diagram. For complex mechanical systems, we recommend to draw a separate kinematic diagram.

For the block (translation only), applying the force equation in the y direction gives

$$+\uparrow y: \sum F_y = ma_{Cy},$$

$$k\left(\frac{L}{4}\theta - y + \delta_{st}\right) - mg = m\ddot{y}.$$

For the rod (rotation only), applying the moment equation about the fixed point O gives

$$+\curvearrowleft: \quad \sum M_O = I_O\alpha,$$

$$f \cdot \frac{3L}{4}\cos\theta + Mg \cdot \frac{L}{4}\cos\theta - k\left(\frac{L}{4}\theta - y + \delta_{st}\right)\cdot\frac{L}{4}\cos\theta - b\frac{L}{4}\dot{\theta}\cdot\frac{L}{4}\cos\theta = I_O\ddot{\theta},$$

where I_O can be obtained using parallel-axis theorem,

$$I_O = \frac{1}{12}ML^2 + M\left(\frac{L}{4}\right)^2 = \frac{7}{48}ML^2.$$

For small angular motions, $\cos\theta \approx 1$. Introducing the static equilibrium conditions and rearranging the two equations gives

$$m\ddot{y} + ky - \frac{kL}{4}\theta = 0,$$

$$\frac{7}{48}ML^2\ddot{\theta} + \frac{bL^2}{16}\dot{\theta} + \frac{kL^2}{16}\theta - \frac{kL}{4}y = \frac{3fL}{4}.$$

Example 5.13: A Cart–Inverted-Pendulum System

Consider the mechanical system shown in Figure 5.79, in which a uniform rod of mass M and length L is pivoted on a cart of mass m. An external force f is applied to the cart. Assume that the pendulum is constrained to move in a vertical plane, and the cart moves without slipping along a horizontal line. Denote the displacement of the cart as x and the angular displacement of the pendulum as θ.

a. Draw the necessary free-body diagram and kinematic diagram, and derive the nonlinear equations of motion.
b. Linearize the equations of motion for small angular motions, and determine the state-space form with x and θ as the outputs.

Solution

a. This is a mixed system with two masses. The motion of the cart is purely translational, and the inverted pendulum undergoes both translation and rotation. The free-body diagram

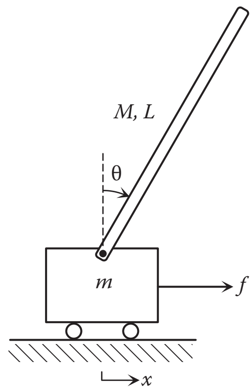


FIGURE 5.79 A cart–inverted-pendulum system.

and the kinematic diagram are given in Figure 5.80. Note that the acceleration of the mass center of the rod consists of three components. On the one hand, the pendulum moves together with the cart along the horizontal line at an acceleration of $\ddot{x}$. On the other hand, it rotates about the pivot P on the cart. The relative rotational accelerations include the tangential component $\frac{L}{2}\ddot{\theta}$ and the normal component $\frac{L}{2}\dot{\theta}^2$.

Applying the force equation to the whole system along the x direction gives

$$+\rightarrow x: \sum F_x = \sum_{i=1}^{2} m_i (a_{Ci})_x,$$

$$f = m\ddot{x} + M\ddot{x} - M\frac{L}{2}\dot{\theta}^2 \cdot \sin\theta + M\frac{L}{2}\ddot{\theta} \cdot \cos\theta.$$

The forces at the pivot are canceled out because they are internal forces between the cart and the pendulum. Applying the moment equation to the pendulum about point P results in

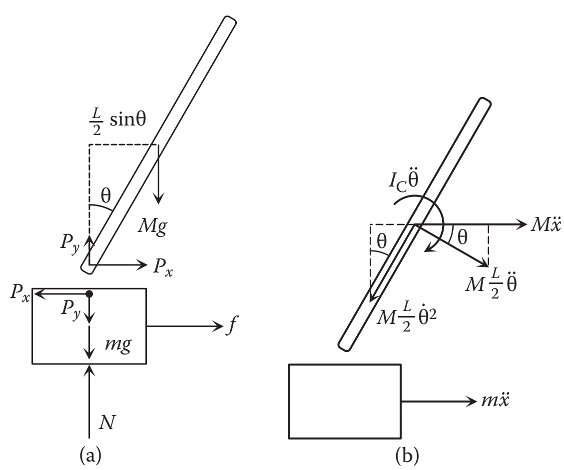


FIGURE 5.80 A cart–inverted-pendulum system: (a) free-body diagram and (b) kinematic diagram.

$$+\curvearrowright: \quad \sum M_{\mathrm{P}} = I_{\mathrm{C}}\alpha + M_{\mathrm{eff}\_m\mathbf{a}_{\mathrm{C}}},$$

$$Mg\cdot\frac{L}{2}\sin\theta = \frac{1}{12}ML^2\ddot{\theta} + M\ddot{x}\cdot\frac{L}{2}\cos\theta + M\frac{L}{2}\ddot{\theta}\cdot\frac{L}{2}.$$

Rearranging the two equations into the standard input–output form

$$(m+M)\ddot{x} + \frac{1}{2}ML\ddot{\theta}\cos\theta - \frac{1}{2}ML\dot{\theta}^2\sin\theta = f,$$

$$\frac{1}{3}ML^2\ddot{\theta} + \frac{1}{2}ML\ddot{x}\cos\theta - \frac{1}{2}MgL\sin\theta = 0.$$

b. For small angular motions, $\cos\theta \approx 1$, $\sin\theta \approx \theta$, $\dot{\theta}^2\theta \approx 0$ (see Section 4.6). The linearized equations are

$$(m+M)\ddot{x} + \frac{1}{2}ML\ddot{\theta} = f,$$

$$\frac{1}{3}ML^2\ddot{\theta} + \frac{1}{2}ML\ddot{x} - \frac{1}{2}MgL\theta = 0.$$

The state, the input, and the output are specified as

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{Bmatrix}, \quad u = f, \quad \mathbf{y} = \begin{Bmatrix} x \\ \theta \end{Bmatrix}.$$

We then take the time derivative of each state variable. For the first two,

$$\dot{x}_1 = \dot{x} = x_3,$$

$$\dot{x}_2 = \dot{\theta} = x_4.$$

Note that the two linearized equations are coupled. The derivatives $\ddot{x}$ and $\ddot{\theta}$ cannot be solved using only one of the equations. To find $\dot{x}_3$ and $\dot{x}_4$, we rewrite the two linearized differential equations as

$$\begin{bmatrix} m+M & \frac{1}{2}ML \\ \frac{1}{2}ML & \frac{1}{3}ML^2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} = \begin{Bmatrix} f \\ \frac{1}{2}MgL\theta \end{Bmatrix},$$

from which $\ddot{x}$ and $\ddot{\theta}$ can be solved using Cramer's rule:

$$\ddot{x} = \frac{(1/3)ML^2 f - (1/4)M^2L^2 g\theta}{(1/12)ML^2(M+4m)},$$

$$\ddot{\theta} = \frac{-(1/2)MLf + (M+m)(1/2)MgL\theta}{(1/12)ML^2(M+4m)}.$$

Simplifying the equations gives

$$\dot{x}_3 = \frac{4}{M+4m}u - \frac{3Mg}{M+4m}x_2,$$

$$\dot{x}_4 = \frac{-6}{L(M+4m)}u + \frac{6(M+m)g}{L(M+4m)}x_2.$$

Thus, the state-space form is

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{3Mg}{M+4m} & 0 & 0 \\ 0 & \dfrac{6(M+m)g}{L(M+4m)} & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{4}{M+4m} \\ -\dfrac{6}{L(M+4m)} \end{bmatrix} u$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u.$$

Note that in Part (a), Equation 5.31 instead of Equation 5.41 can also be applied to the pendulum to derive the second equation of motion. The related free-body diagram and the kinematic diagram are shown in Figure 5.81. Summing all externally applied moments about the pivot P, which moves together with the cart with an acceleration of $\ddot{x}$, gives

$$+\curvearrowright: \quad \sum \mathbf{M}_\text{P} = I_\text{P}\boldsymbol{\alpha} + m\mathbf{r}_{\text{C/P}} \times \mathbf{a}_\text{P}$$

$$Mg \cdot \frac{L}{2}\sin\theta = \frac{1}{3}ML^2\ddot{\theta} + M\frac{L}{2}\ddot{x}\sin\left(\frac{\pi}{2} - \theta\right)$$

$$\frac{1}{3}ML^2\ddot{\theta} + \frac{1}{2}ML\ddot{x}\cos\theta - \frac{1}{2}MgL\sin\theta = 0.$$

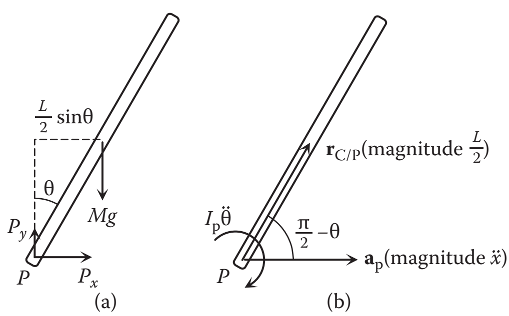


FIGURE 5.81 Applying Equation 5.31 to the pendulum: (a) free-body diagram and (b) kinematic diagram.

5.4.2 Energy Method

The differential equation of motion of a mechanical system can be obtained using the force/moment approach, which is based on Newtonian mechanics. Free-body diagrams are necessary to apply the force and moment equations correctly. An alternative way of obtaining the differential equation of motion is to use the energy method based on analytical mechanics.

For a mass–spring system with negligible friction and damping, the principle of conservation of energy states that

$$T + V = \text{constant} \tag{5.44}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(T+V) = 0, \tag{5.45}$$

where T is the kinetic energy and V is the potential energy, which includes the gravitational potential energy and the elastic potential energy.

The expression for the kinetic energy of a translational or rotational mass element was given in Section 5.1. In general, the kinetic energy of a rigid body in plane motion is

$$T = \frac{1}{2}mv_{\mathrm{C}}^2 + \frac{1}{2}I_{\mathrm{C}}\omega^2, \tag{5.46}$$

where v_{C} is the velocity of the mass center C of the body and ω is the angular velocity of the body. Note that the kinetic energy of a rigid body in plane motion can be separated into two parts: (1) the kinetic energy associated with the translational motion of the mass center C of the body and (2) the kinetic energy associated with the rotation of the body about the mass center C. If a rigid body rotates about a fixed point O with an angular velocity ω, the kinetic energy reduces to $T = (1/2)I_{\mathrm{O}}\omega^2$.

Example 5.14: A Pulley System

Consider the pulley system shown in Figure 5.82. A block of mass m is connected to a translational spring of stiffness k through a cable, which passes by a pulley with mass m_{p} and radius r. The pulley rotates about the fixed mass center O. The moment of inertia of the pulley about its mass center is I_{O}. Use the energy method to derive the equation of motion.

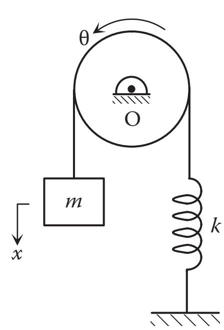


FIGURE 5.82 A pulley system.

Solution

The system has two mass elements, one translational block of mass and one pulley rotating about its fixed mass center. Their motions are related to each other by the geometric constraint, $x = r\theta$. The kinetic energy of the system is

$$T = T_{\text{block}} + T_{\text{pulley}} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_O\dot{\theta}^2.$$

The undeformed position, static equilibrium position, and dynamic position of the mass–spring system are shown in Figure 5.83. Choosing the static equilibrium position as the datum for the gravitational potential energy, we have

$$V_g = -mgx.$$

The elastic potential energy is

$$V_e = \frac{1}{2}k(x + \delta_{st})^2.$$

The total potential energy is

$$V = V_g + V_e = -mgx + \frac{1}{2}kx^2 + \frac{1}{2}k\delta_{st}^2 + kx\delta_{st}.$$

Because of the static equilibrium condition, $mg = k\delta_{st}$, the expression of potential energy becomes

$$V = \frac{1}{2}kx^2 + \frac{1}{2}k\delta_{st}^2.$$

The total energy of the system is

$$T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_O\dot{\theta}^2 + \frac{1}{2}kx^2 + \frac{1}{2}k\delta_{st}^2,$$

for which the time derivative is

$$\frac{d}{dt}(T + V) = m\dot{x}\ddot{x} + I_O\dot{\theta}\ddot{\theta} + kx\dot{x}.$$

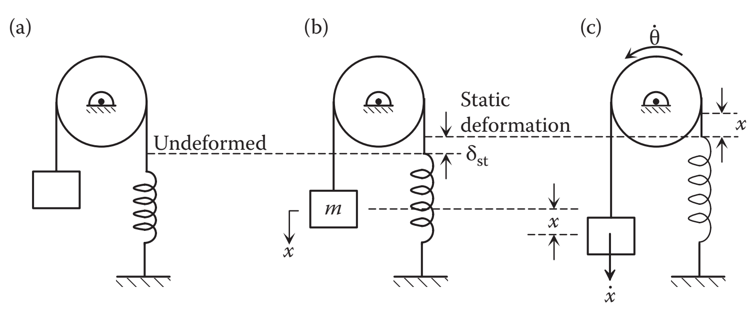


FIGURE 5.83 A pulley system: (a) undeformed position, (b) static equilibrium position, and (c) dynamic position.

Using the geometric constraint, $x = r\theta$, which implies that $\dot{x} = r\dot{\theta}$ and $\ddot{x} = r\ddot{\theta}$, we can rewrite the above equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}(T+V) = mr^2\dot{\theta}\ddot{\theta} + I_O\dot{\theta}\ddot{\theta} + kr^2\theta\dot{\theta}.$$

Applying the principle of conservation of energy, we find

$$(I_O + mr^2)\ddot{\theta} + kr^2\theta = 0.$$

Note that the mechanical system in Example 5.14 is a single-degree-of-freedom system, which requires only one generalized coordinate (e.g., x or θ) to describe the system dynamics. The example shows that if we can obtain the expression of $T + V$ in terms of the generalized coordinate, the equation of motion can be derived by taking the time derivative and setting it equal to zero.

For an n-degree-of-freedom system, n independent equations of motion can be derived using Lagrange's formulation, which is applicable to both conservative and nonconservative systems. The discussion of applying Lagrange's equations to nonconservative systems is beyond the scope of this text, and we are mainly concerned with conservative systems. One of the forms of Lagrange's equations for a conservative system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0, \quad i = 1, 2, \ldots, n, \tag{5.47}$$

where q_i is the ith generalized coordinate and n is the total number of independent generalized coordinates. In general, the kinetic energy is a function of the generalized displacements and the generalized velocities,

$$T = T(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n). \tag{5.48}$$

The potential energy is a function of the generalized displacements,

$$V = V(q_1, q_2, \ldots, q_n). \tag{5.49}$$

Example 5.15: A Double Pendulum

Consider the double pendulum in Figure 5.84, in which two point masses of equal mass m are attached to two rigid links of equal length L. The links are assumed to be massless. The motion of the system is constrained in the vertical plane. Neglecting friction, derive the equations of motion using Lagrange's equations. Assume small angles for θ_1 and θ_2.

Solution

The system is only subjected to gravitational forces, and it is a conservative system. The dynamics of the system can be described using two independent angular displacement coordinates, θ_1 and θ_2. The kinetic energy of the system is

$$T = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2.$$

From kinematics,

$$v_1 = L\dot{\theta}_1$$

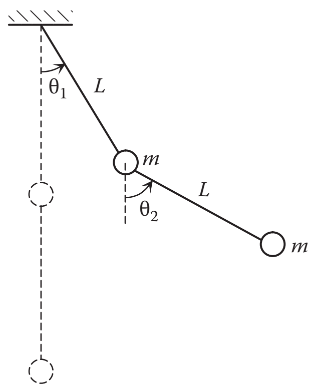


FIGURE 5.84 A double-pendulum system.

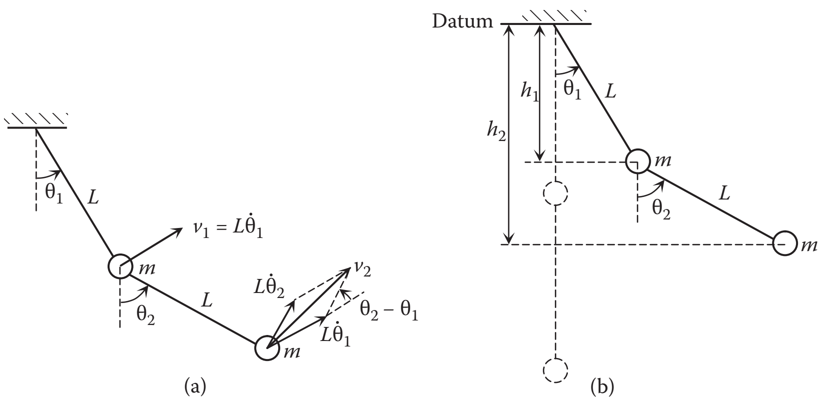


FIGURE 5.85 A double-pendulum: (a) kinematic diagram and (b) positions of the mass centers.

and, as shown in Figure 5.85a,

$$v_2^2 = (L\dot{\theta}_1)^2 + (L\dot{\theta}_2)^2 + 2L^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1),$$

which is obtained by applying the law of cosines. Thus,

$$T = \frac{1}{2}m(L\dot{\theta}_1)^2 + \frac{1}{2}m\left[(L\dot{\theta}_1)^2 + (L\dot{\theta}_2)^2 + 2L^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)\right].$$

Note that no spring elements are involved in the system. This implies that $V_e = 0$ and $V = V_g$. Using the datum defined in Figure 5.85b, we can obtain the gravitational potential energy

$$V_g = -mgh_1 - mgh_2,$$

where

$$h_1 = L\cos\theta_1,$$

$$h_2 = L\cos\theta_1 + L\cos\theta_2.$$

Thus,

$$V = -2mgL\cos\theta_1 - mgL\cos\theta_2.$$

We then apply Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0, \quad i = 1, 2.$$

For $i = 1$, $q_1 = \theta_1$, $\dot{q}_1 = \dot{\theta}_1$,

$$\frac{\partial T}{\partial \dot{\theta}_1} = 2mL^2\dot{\theta}_1 + mL^2\dot{\theta}_2\cos(\theta_2 - \theta_1),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = 2mL^2\ddot{\theta}_1 + mL^2\ddot{\theta}_2\cos(\theta_2 - \theta_1) - mL^2\dot{\theta}_2\sin(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1),$$

$$\frac{\partial T}{\partial \theta_1} = -mL^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2 - \theta_1)(-1),$$

$$\frac{\partial V}{\partial \theta_1} = 2mgL\sin\theta_1.$$

Substituting into Lagrange's equation results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 2mL^2\ddot{\theta}_1 + mL^2\ddot{\theta}_2\cos(\theta_2 - \theta_1) - mL^2\dot{\theta}_2^2\sin(\theta_2 - \theta_1) + 2mgL\sin\theta_1 = 0\,\cdot$$

Similarly, for $i = 2$, $q_2 = \theta_2$, $\dot{q}_2 = \dot{\theta}_2$,

$$\frac{\partial T}{\partial \dot{\theta}_2} = mL^2\dot{\theta}_2 + mL^2\dot{\theta}_1\cos(\theta_2 - \theta_1),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = mL^2\ddot{\theta}_2 + mL^2\ddot{\theta}_1\cos(\theta_2 - \theta_1) - mL^2\dot{\theta}_1\sin(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1),$$

$$\frac{\partial T}{\partial \theta_2} = -mL^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2 - \theta_1),$$

$$\frac{\partial V}{\partial \theta_2} = mgL\sin\theta_2.$$

Substituting into Lagrange's equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = mL^2\ddot{\theta}_2 + mL^2\ddot{\theta}_1\cos(\theta_2 - \theta_1) + mL^2\dot{\theta}_1^2\sin(\theta_2 - \theta_1) + mgL\sin\theta_2 = 0.$$

For small motions ($\theta_1 \approx 0$ and $\theta_2 \approx 0$), the two differential equations of motion are linearized as

$$2mL^2\ddot{\theta}_1 + mL^2\ddot{\theta}_2 + 2mgL\theta_1 = 0,$$

$$mL^2\ddot{\theta}_1 + mL^2\ddot{\theta}_2 + mgL\theta_2 = 0,$$

or in second-order matrix form

$$\begin{bmatrix} 2mL^2 & mL^2 \\ mL^2 & mL^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} 2mgL & 0 \\ 0 & mgL \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

PROBLEM SET 5.4

1. For the pulley system in Example 5.14, draw the free-body diagram and kinematic diagram, and derive the equation of motion using the force/moment approach.
2. The double pulley system shown in Figure 5.86 has an inner radius of r_1 and an outer radius of r_2. The mass moment of inertia of the pulley about point O is I_O. A translational spring of stiffness k and a block of mass m are suspended by cables wrapped around the pulley as shown. Draw the free-body diagram and kinematic diagram, and derive the equation of motion using the force/moment approach.
3. Consider the mechanical system shown in Figure 5.87. A disk–shaft system is connected to a block of mass m through a translational spring of stiffness k. The elasticity of the shaft and the fluid coupling are modeled as a torsional spring of stiffness K and a torsional viscous damper of damping coefficient B, respectively. The radius of the disk is r and its mass moment of inertia about point O is I_O. Assume that the friction between the block and the horizontal surface cannot be ignored and is modeled as a translational viscous damper of damping coefficient b. The input to the system is the force f. Draw the necessary free-body diagram and the kinematic diagram, and derive the equations of motion.

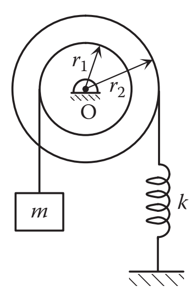


FIGURE 5.86 Problem 2.

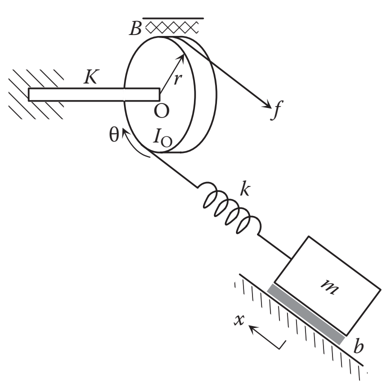


FIGURE 5.87 Problem 3.

4. Consider the mechanical system shown in Figure 5.88, where the motion of the rod is a small angular rotation. When $\theta = 0$ and $f = 0$, the deformation of each spring is zero and the system is at static equilibrium. Assume that the friction between the block of mass m_1 and the horizontal surface cannot be ignored and is modeled as a translational viscous damper of damping coefficient b.
   a. Assuming that $a > c > 0$, determine the mass moment of inertia of the rod about the pivot point O.
   b. Draw the necessary free-body diagram and the kinematic diagram, and derive the equations of motion for small angles.
5. Consider the mechanical system shown in Figure 5.89, in which a simple pendulum is pivoted on a cart of mass m. The pendulum consists of a point mass M concentrated at the tip of a massless rod of length L. Assume that the pendulum is constrained to rotate in a vertical plane, and that the cart moves on a smooth horizontal surface. Denote the displacement of the cart as x and the angular displacement of the pendulum as θ. Draw the necessary free-body diagram and kinematic diagram, and derive the equations of motion for small angles.
6. Consider the mechanical system shown in Figure 5.90. Draw the necessary free-body diagram and kinematic diagram, and derive the equations of motion for small angles.

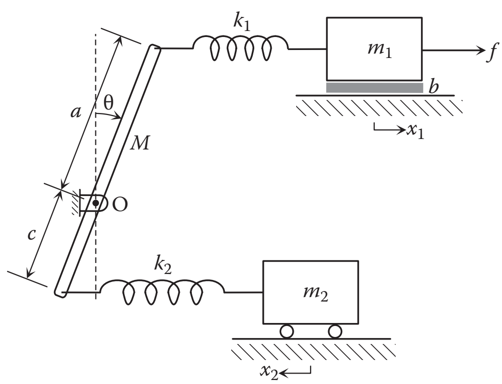


FIGURE 5.88 Problem 4.

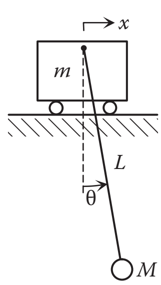


FIGURE 5.89 Problem 5.

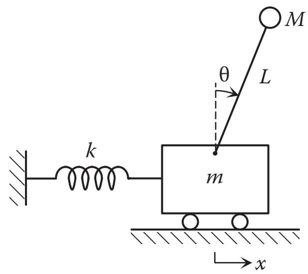


FIGURE 5.90 Problem 6.

7. Consider the mechanical system shown in Figure 5.91. The inputs are the force f_1 applied to the cart and the force f_2 applied at the tip of the rod. The outputs are the displacement x of the cart and the angular displacement θ of the pendulum.
   a. Draw the free-body diagram and kinematic diagram, and derive the equations of motion for small angles.
   b. Using the differential equations obtained in Part (a), determine the state-space representation.
8. Consider the mechanical system shown in Figure 5.92. Draw the necessary free-body diagram and kinematic diagram, and derive the equations of motion for small angles.
9. For the mechanical system in Problem 2, use the energy method to derive the equation of motion.
10. For the mechanical system in Problem 11 of Problem Set 5.3, use the energy method to derive the equation of motion.

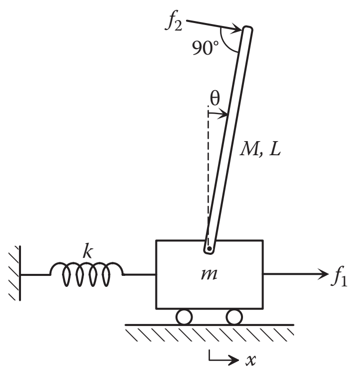


FIGURE 5.91 Problem 7.

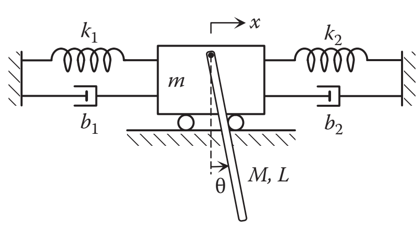


FIGURE 5.92 Problem 8.

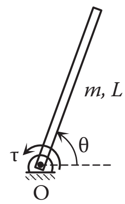


FIGURE 5.93 Problem 13.

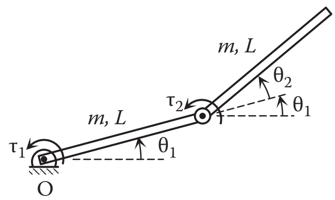


FIGURE 5.94 Problem 14.

11. Repeat Problem 5 using Lagrange's equations.
12. Repeat Problem 6 using Lagrange's equations.
13. A robot arm consists of rigid links connected by joints allowing the relative motion of neighboring links. The dynamic model for a robot arm can be derived using Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right) - \frac{\partial T}{\partial \theta_i} + \frac{\partial V}{\partial \theta_i} = \tau_i, \quad i = 1, 2, \ldots, n,$$

where θ_i is the angular displacement of the ith joint, τ_i is the torque applied to the ith joint, and n is the total number of joints. Consider a single-link planar robot arm as shown in Figure 5.93. Use Lagrange's equations to derive the dynamic model of the robot arm. Assume that the motion of the robot arm is constrained in a vertical plane, and the joint angle varies between 0° and 360°.
14. Repeat Problem 13 for a two-link planar robot arm as shown in Figure 5.94. Assume that the motion of the robot arm is constrained in a horizontal plane, and the joint angles vary between 0° and 360°.

5.5 GEAR–TRAIN SYSTEMS

Gear–train systems are important in many engineering applications. Figure 5.95a shows a pair of ideal gears, which are assumed to be rigid and meshed without backlash. A torque τ_1 produced by a motor is applied to gear 1, which rotates and causes gear 2 to rotate in the opposite direction. The radii of gear 1 and gear 2 are r_1 and r_2, respectively. The relative sizes of the two gears result in a proportionality constant between the angular velocities of the respective shafts.

For the purpose of analysis, a free-body diagram for the rotational gear–train system must be drawn with care. It is convenient to visualize the gears as circles. Figure 5.95b shows the free-body diagram seen from the side of the input shaft. The two gears are tangent at the contact point and rotate without slipping. The torque applied to gear 1 causes an action force F onto gear 2 at the contact point. Because of Newton's third law, gear 1 is subjected to a reaction force at the contact point. Use θ_1 and θ_2 to denote the respective angular displacements. The geometric constraint is

$$r_1\theta_1 = r_2\theta_2, \tag{5.50}$$

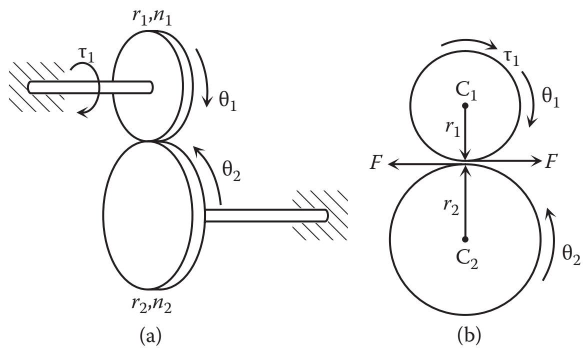


FIGURE 5.95 A gear–train system: (a) physical system and (b) free-body diagram.

which can be rewritten as

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = N, \tag{5.51}$$

where N is called the gear ratio, which also defines the relationship between the numbers of teeth on the two gears:

$$\frac{n_1}{n_2} = N. \tag{5.52}$$

Note that the gear ratio N may be defined differently among authors. Differentiating Equation 5.51 gives

$$\frac{\omega_2}{\omega_1} = N. \tag{5.53}$$

Equations 5.52 and 5.53 show that if the number of teeth on the output gear is larger than the number of teeth on the input gear, then the input gear must rotate faster than the output gear. Thus, the gear pair is a speed reducer for $N < 1$. If the gears have negligible inertia or zero angular acceleration, and if the energy loss due to friction between the gear teeth can be neglected, the input work must be equal to the output work. Under these conditions, the output torque is greater than the input torque for a speed reducer.

To obtain the mathematical model of a gear–train system, the fundamental laws are still applied. We need to draw a free-body diagram and apply the moment equation for each gear. The derivation also requires a consideration of the geometric constraint.

Example 5.16: A Single-Degree-of-Freedom Gear–Train System

For the gear–train system shown in Figure 5.95a, derive the differential equation of motion. The mass moments of inertia of the two gears about their respective fixed centers are I_{C1} and I_{C2}.

Solution

The free-body diagram is shown in Figure 5.95b. The geometric constraint is

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2}.$$

Applying the moment equation to each gear gives

$$+\curvearrowright: \quad \sum M_{C1} = I_{C1}\alpha,$$

$$\tau_1 - r_1 F = I_{C1}\ddot{\theta}_1,$$

and

$$+\curvearrowleft: \quad \sum M_{C2} = I_{C2}\alpha,$$

$$r_2 F = I_{C2}\ddot{\theta}_2.$$

Combining the two equations and eliminating F yields

$$\tau_1 - \frac{r_1}{r_2} I_{C2}\ddot{\theta}_2 = I_{C1}\ddot{\theta}_1.$$

Note that the angular displacements θ_1 and θ_2 are dependent through the geometric constraint. Thus, the gear–train in Figure 5.95 is a single-degree-of-freedom system, which requires only one equation of motion in terms of only one coordinate. Assume that the equation is expressed in terms of θ_1. Introducing the geometric constraint results in the equation of motion

$$\tau_1 - \frac{r_1}{r_2} I_{C2}\left(\frac{r_1}{r_2}\ddot{\theta}_1\right) = I_{C1}\ddot{\theta}_1,$$

simplified to

$$\left(I_{C1} + \frac{r_1^2}{r_2^2} I_{C2}\right)\ddot{\theta}_1 = \tau_1,$$

which is the dynamics seen from the input side.

Example 5.17: A Single-Link Robot Arm

The mechanical model of a single-link robot arm driven by a motor can be represented as a gear–train system, as shown in Figure 5.96, in which two rotational subsystems are coupled with a pair of gears with negligible inertia. The mass moments of inertia of the motor and the load are I_m and I, respectively. The coefficients of torsional viscous damping of the motor and the load are B_m and B, respectively. τ_m is the torque generated by the motor. Assume that the gear ratio is $N = r_1/r_2$. Derive the differential equation of motion in terms of the motor variable θ_m.

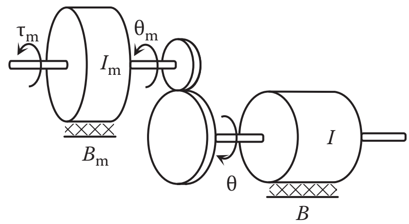


FIGURE 5.96 The mechanical model of a single-link robot arm driven by a motor.

Solution

The free-body diagrams for the motor, the load, and the gear–train are shown in Figure 5.97, where F represents the contact force between the two gears. The moments caused by the contact force on the motor and on the load are r_1F and r_2F, respectively. Applying the moment equation to the motor and the load gives

$$+\curvearrowleft: \quad \sum M_O = I_O\alpha,$$

$$\tau_m - B_m\dot{\theta}_m - r_1F = I_m\ddot{\theta}_m,$$

and

$$+\curvearrowright: \quad \sum M_O = I_O\alpha,$$

$$-B\dot{\theta} + r_2F = I\ddot{\theta}.$$

Solving for F from the equation for the load and substituting it into the equation for the motor results in

$$\tau_m - B_m\dot{\theta}_m - \frac{r_1}{r_2}(I\ddot{\theta} + B\dot{\theta}) = I_m\ddot{\theta}_m.$$

By the geometry of the gears,

$$\dot{\theta} = \frac{r_1}{r_2}\dot{\theta}_m = N\dot{\theta}_m,$$

$$\ddot{\theta} = \frac{r_1}{r_2}\ddot{\theta}_m = N\ddot{\theta}_m,$$

where N is the gear ratio.

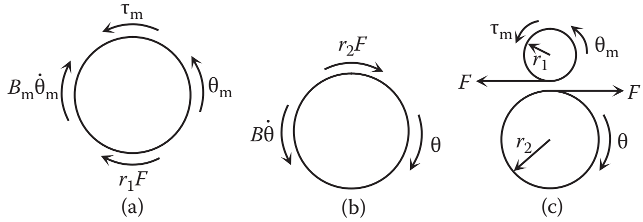


FIGURE 5.97 Free-body diagrams: (a) motor, (b) load, and (c) gear–train.

Substituting these into the previous differential equation gives

$$\tau_m - B_m\dot{\theta}_m - N^2(I\ddot{\theta}_m + B\dot{\theta}_m) = I_m\ddot{\theta}_m,$$

which can be rearranged as

$$(I_m + N^2 I)\ddot{\theta}_m + (B_m + N^2 B)\dot{\theta}_m = \tau_m.$$

The equation is in terms of the angular displacement of the motor.

PROBLEM SET 5.5

1. Repeat Example 5.16, and determine a mathematical model for the simple one-degree-of-freedom system shown in Figure 5.95a in the form of a differential equation of motion in θ_2.
2. Repeat Example 5.17, and determine a mathematical model for the single-link robot arm shown in Figure 5.96 in the form of a differential equation of motion in the load variable θ.
3. Consider the one-degree-of-freedom system shown in Figure 5.98. The system consists of two gears of mass moments of inertia I_1 and I_2 and radii r_1 and r_2, respectively. The applied torque on gear 1 is τ_1. Assume that the gears are connected with flexible shafts, which can be approximated as two torsional springs of stiffnesses K_1 and K_2, respectively.
   a. Draw the necessary free-body diagrams, and derive the differential equation of motion in θ_1.
   b. Using the differential equation obtained in Part (a), determine the transfer function $\Theta_2(s)/T_1(s)$.
   c. Using the differential equation obtained in Part (a), determine the state-space representation with θ_2 as the output.
4. Consider the gear–train system shown in Figure 5.99. The system consists of a rotational cylinder and a pair of gears. The gear ratio is $N = r_1/r_2$. The applied torque on the cylinder is τ_a. Assume that the gears are connected with flexible shafts, which can be approximated as two torsional springs of stiffnesses, K_1 and K_2, respectively.
   a. Draw the necessary free-body diagrams, and derive the differential equations of motion.
   b. Using the differential equations obtained in Part (a), determine the state-space representation. Use θ_a, θ_1, ω_a, and ω_1 as the state variables, and use θ_2 and ω_2 as the output variables.

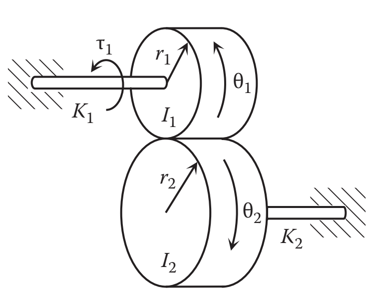


FIGURE 5.98 Problem 3.

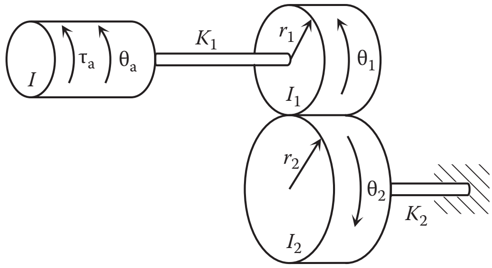


FIGURE 5.99 Problem 4.

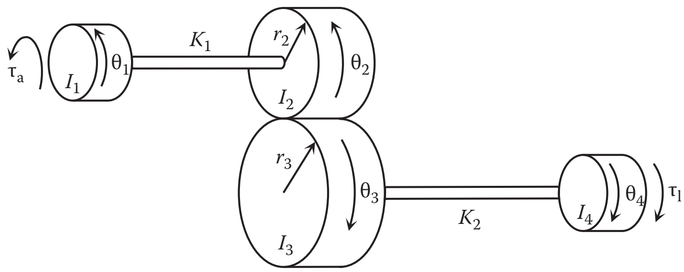


FIGURE 5.100 Problem 5.

5. A three-degree-of-freedom gear–train system is shown in Figure 5.100, which consists of four gears of moments of inertia I_1, I_2, I_3, and I_4. Gears 2 and 3 are meshed and their radii are r_2 and r_3, respectively. Gears 1 and 2 are connected by a relatively long shaft, and gears 3 and 4 are connected in the same way. The shafts are assumed to be flexible, and can be approximated by torsional springs. The applied torque and load torque are τ_a and τ_l on gear 1 and gear 4, respectively. The gears are assumed to be rigid and have no backlash. Derive the differential equations of motion.
6. Repeat Problem 5. Assume that the shaft connecting gears 1 and 2 is relatively short and rigid.

5.6 SYSTEM MODELING WITH SIMULINK AND SIMSCAPE

Simulink is a block diagram environment for multidomain simulation and model-based design. It provides a graphical editor, block libraries, and solvers for modeling, simulating, and analyzing dynamic systems, and even connecting your model to hardware for real-time testing. Simscape is an extension of Simulink for modeling and simulating physical systems spanning mechanical, electrical, hydraulic, and other physical domains. The reader can refer to Sections 1.8 and 1.9 for a brief introduction to Simulink and Simscape. In this section, more examples are included to illustrate mechanical system modeling with Simulink and Simscape. The mathematical models in all examples are given without a detailed derivation and the reader can derive them as an exercise.

5.6.1 Translational Systems

It is known that translational mechanical systems can be modeled as systems with interconnected mass, spring, and damper elements. The dynamics of such systems can be represented by ordinary

differential equations, transfer functions, or in the state-space form. The common inputs are forces or displacements, and the common outputs are displacements, velocities, or accelerations.

To model a translational mechanical system with Simulink and Simscape, three parts of blocks are needed to simulate the input, the system itself, and the output. A Simulink diagram is constructed based on the system's mathematical representation, such as differential equation(s), transfer function(s), or the state-space form. Blocks in the Simulink libraries of `Sources`, `Continuous`, and `Sinks` are commonly used to construct the input, the system itself, and the output, respectively. Whereas a Simscape block diagram is built just as you would assemble a physical system. Basic physical components like mass, spring, damper, etc., are available in the Simscape library of `Translational Elements` for representing one-dimensional translational motion. The blocks in the Simscape libraries of `Mechanical Sources` and `Mechanical Sensors` are used to generate inputs and output measurements.

Example 5.4 in Section 5.2 shows how to build a Simulink or Simscape block diagram of a single-degree-of-freedom mass–spring–damper system with a force input. The mathematical model of the system is given by an ordinary differential equation. In this section, we will consider different numbers of degrees of freedom, different system inputs, and different system representations.

Example 5.18: A Single-Degree-of-Freedom Vehicle Model

The mass–spring–damper system shown in Figure 5.101 represents a vehicle traveling on a rough road. Assume that the surface of the road can be approximated as a sine wave $z = Z_0\sin(\omega t)$, where $Z_0 = 0.01$ m and $\omega = 3.5$ rad/s. The mathematical model of the system is given by an ordinary differential equation

$$m\ddot{x} + b\dot{x} + kx = b\dot{z} + kz,$$

where $m = 3000$ kg, $b = 2000$ N·s/m, and $k = 50$ kN/m.

a. Build a Simulink model of the system based on the mathematical representation and find the displacement output $x(t)$.
b. Convert the ordinary differential equation to a transfer function and repeat Part (a). Assume zero initial conditions.
c. Build a Simscape model of the physical system and find the displacement output $x(t)$.

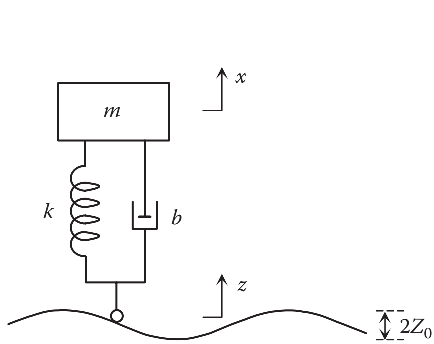


FIGURE 5.101 The mechanical model of a vehicle traveling on a rough road.

Solution

a. Solving for the highest derivative of the output x gives

$$\ddot{x} = \frac{1}{m}(kz + b\dot{z} - kx - b\dot{x}).$$

The corresponding Simulink block diagram is shown in Figure 5.102. Note that the displacement input $z(t)$ is a sine function, which can be defined using a `Sine Wave` block available in the library of `Sources`. Double-click on the block and type `0.01` for the `Amplitude` and `3.5` for the `Frequency` to define the input $z(t) = 0.01\sin(3.5t)$.

b. The transfer function relating the input $z(t)$ to the output $x(t)$ is

$$\frac{X(s)}{Z(s)} = \frac{bs + k}{ms^2 + bs + k}.$$

The Simulink block diagram built based on the transfer function is shown in Figure 5.103, where a `Transfer Fcn` block is used to represent the vehicle system. Double-click on the block and type `[b k]` for the `Numerator coefficient` and `[m b k]` for the `Denominator coefficient` to define the transfer function $X(s)/Z(s)$.

c. The Simscape block diagram corresponding to the physical system is shown in Figure 5.104, which can be created by following these steps:

1. Type `ssc_new` at the MATLAB Command window to open the main Simscape library and create a new model.
2. Open the library of `Simscape/Foundation Library/Mechanical/Translational Elements` and drag the `Mass`, `Translational Damper`, and `Translational Spring` into the model window. Double-click on these blocks to define the parameters `Mass`, `Damping coefficient`, and `Spring rate` as `m`, `b`, and `k`.
3. To add the representation of the displacement input, open the library of `Simscape/Foundation Library/Mechanical/Mechanical Sources` and drag the `Ideal Translational Velocity Source` into the model window. Note that two types of inputs are available in Simscape for translational mechanical systems, and they are used to define either a force or a velocity input. Therefore, the displacement input function in this example must be converted to the velocity by taking the time derivative.

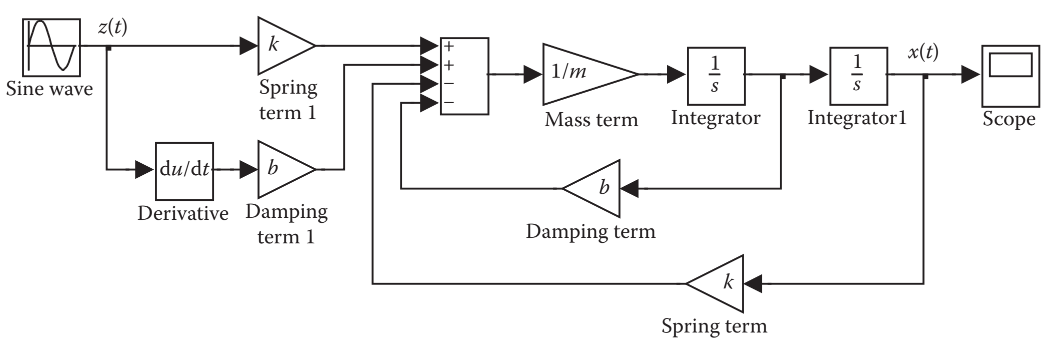


FIGURE 5.102 Simulink block diagram built based on the ordinary differential equation.

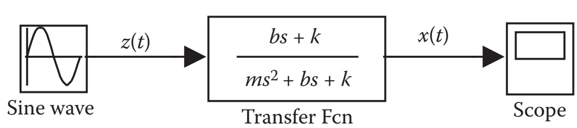


FIGURE 5.103 Simulink block diagram built based on the transfer function.

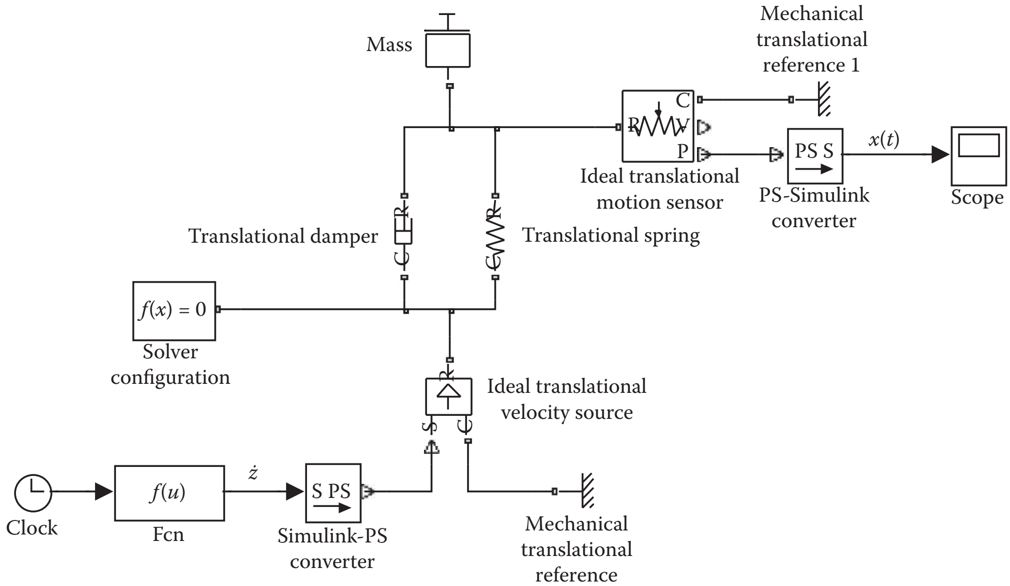


FIGURE 5.104 Simscape block diagram corresponding to Example 5.18.

4. To add the sensor to measure the displacement of the mass, open the library of `Simscape/Foundation Library/Mechanical/Mechanical Sensors` and drag the `Ideal Translational Motion Sensor` into the model window.
5. Now open Simulink libraries to add the source and the scope. As mentioned in Step 3, taking the time derivative of $z(t)$ gives the velocity input function $\dot{z}(t) = 0.035\cos(3.5t)$. Open the library of `Simulink/Sources` to drag the `Clock` block and open the library of `Simulink/User-Defined Functions` to drag the `Fun` block. Double-click on the `Fun` block and type `0.035*cos(3.5*u)` for the `Expression`, where `u` is the default name of the input to the `Fun` block, and here it represents the time t. Note that the `Simulink-PS Converter` and `PS-Simulink Converter` blocks are used to convert Simulink signals into physical signals or vice versa.
6. Orient the blocks and connect them as shown in Figure 5.104.

Define the values of the parameters m, b, and k in the MATLAB Command window. Run all simulations and the same curve as shown in Figure 5.105 can be obtained, which is the resulting displacement output x(t) of the vehicle due to the roughness of the road.

Example 5.19: A Two-Degree-of-Freedom Mass–Spring System

Consider the two-degree-of-freedom mass–spring system shown in Figure 5.106. The mathematical model of the system is given by a set of ordinary differential equations

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0,$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0,$$

where $m_1 = m_2 = 5$ kg, $k_1 = 2000$ N/m, and $k_2 = 4000$ N/m. Assume that initially $\mathbf{x}(0) = [0 \quad 0]^T$ and $\dot{\mathbf{x}}(0) = [1 \quad 0]^T$.

a. Build a Simulink model of the system based on the mathematical representation and find the displacement outputs $x_1(t)$ and $x_2(t)$.
b. Convert the ordinary differential equations to the state-space form and repeat Part (a).
c. Build a Simscape model of the physical system and find the displacement outputs $x_1(t)$ and $x_2(t)$.

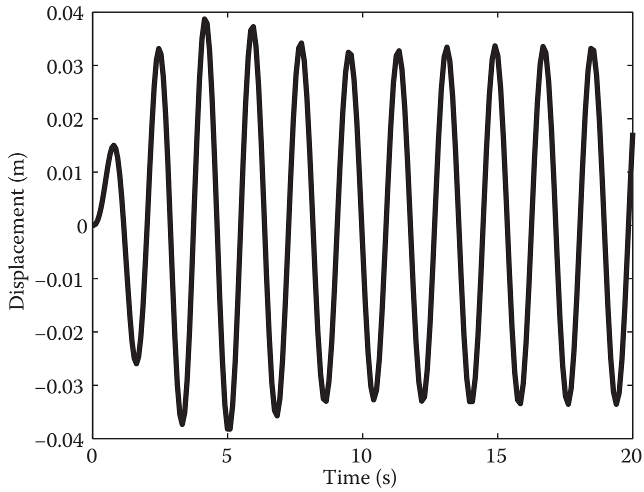


FIGURE 5.105 Displacement output $x(t)$ of the vehicle model in Example 5.18.

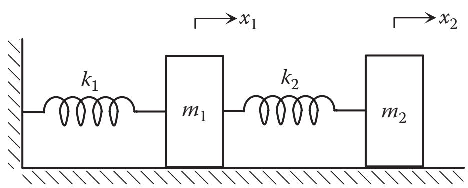


FIGURE 5.106 A two-degree-of-freedom mass–spring system.

Solution

a. Solving for the highest derivatives of the output x_1 and x_2, respectively, gives

$$\ddot{x}_1 = \frac{1}{m_1}\left[-(k_1 + k_2)x_1 + k_2 x_2\right],$$

$$\ddot{x}_2 = \frac{1}{m_2}(k_2 x_1 - k_2 x_2).$$

The corresponding Simulink block diagram is shown in Figure 5.107, where four `Integrator` blocks are included to obtain the signals $\dot{x}_1$, x_1, $\dot{x}_2$, and x_2. The default initial condition of an integrator block is 0. Double-click on the `Integrator` block corresponding to $\dot{x}_1$ and define the `Initial Condition` as `1`, which is the only nonzero initial value.

b. Define the state and the output vectors as

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}, \qquad \mathbf{y} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}.$$

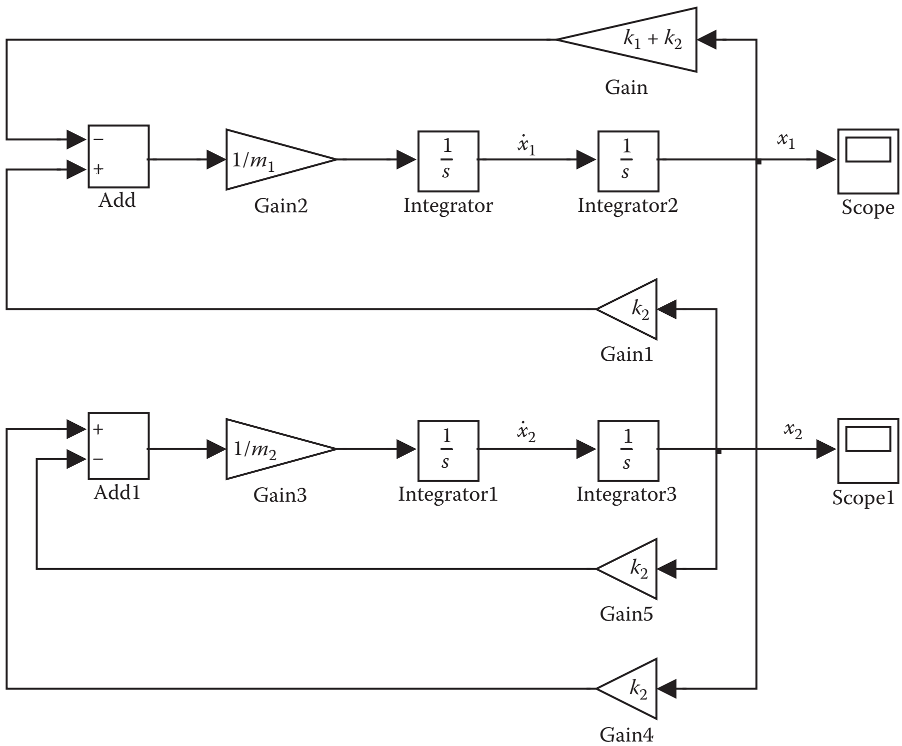


FIGURE 5.107 Simulink block diagram built based on the ordinary differential equations.

The state-space representation is

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & 0 & 0 \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix},$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}.$$

The Simulink block diagram built based on the state-space form is shown in Figure 5.108, in which a `State-Space` block is used to represent the mass–spring system. Note that no external forces are acting on the mass blocks and thus a `Constant` block with a scalar value of 0 is included. Double-click on the `State-Space` block and define the matrices **A**, **B**, **C**, and **D**, where **B** is a 4 × 1 zero matrix and **D** is a 2 × 1 zero matrix. The parameter of `Initial conditions` is `[0; 0; 1; 0]`, which corresponds to $[x_1(0);\ x_2(0);\ \dot{x}_1(0);\ \dot{x}_2(0)]$. The bar-shaped block in Figure 5.108 is called `Demux`, which can be found in the library of `Signal Routing` and is used to split the vector signal **y** into two signals x_1 and x_2.

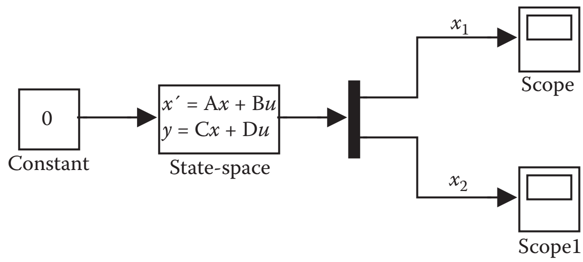


FIGURE 5.108 Simulink block diagram built based on the state-space form.

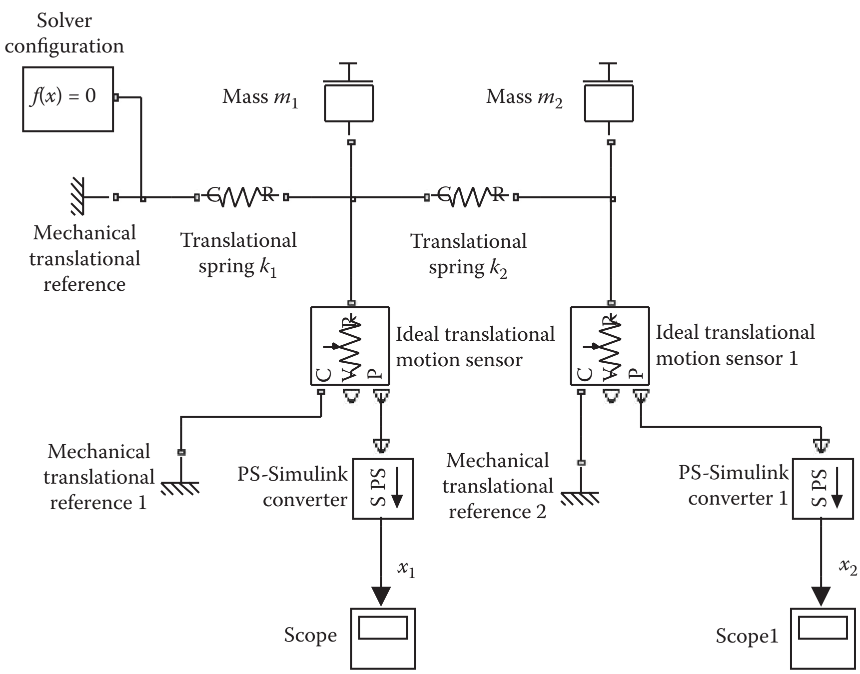


FIGURE 5.109 Simscape block diagram corresponding to Example 5.19.

c. The Simscape block diagram corresponding to the physical system is shown in Figure 5.109, which can be created by following the steps similar to those in Example 5.18. To define the nonzero initial velocity $\dot{x}_1$, double-click on the `Mass` block representing m_1, type `1` for the `Initial velocity`, and choose the unit as `m/s`.

Define the values of the parameters m_1, m_2, k_1, and k_2 in the MATLAB Command window. Run all simulations and the same curves as shown in Figure 5.110 will be obtained, which are the resulting displacement outputs $x_1(t)$ and $x_2(t)$ due to the nonzero initial condition.

The above two examples and Example 5.4 demonstrate basic modeling techniques with Simulink and Simscape. Because a Simulink block diagram of a dynamic system is created based on the system's mathematical model, the modeling techniques discussed in this section can be applied to other types of dynamic systems represented as ordinary differential equations, transfer functions, or in the state-space form. However, a Simscape model describes the physical structure of a dynamic system rather than the underlying mathematics. It is obvious that Simscape diagrams look different for different types of dynamic systems. Therefore, Sections 6.6 and 7.4 will focus on Simscape modeling in various physical domains, such as rotational mechanical, electrical, hydraulic, and thermal.

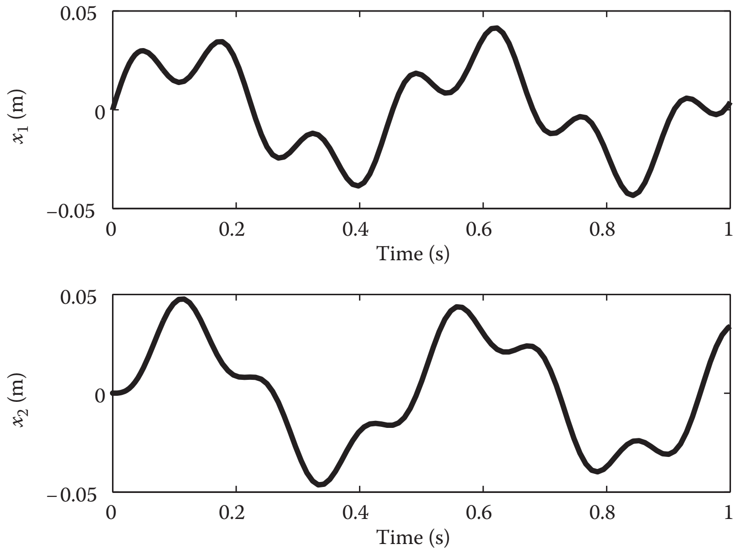


FIGURE 5.110 Displacement outputs $x_1(t)$ and $x_2(t)$ of the system in Example 5.19.

5.6.2 Rotational Systems

The blocks in the library of `Simscape\Foundation Library\Mechanical` can be categorized into two classes: translational and rotational. Basic elements such as inertia, rotational damper, and rotational spring available in the library of `Rotational Elements` are used to model a rotational mechanical system. Two source blocks, `Ideal Torque Source` and `Ideal Angular Velocity Source`, in the library of `Mechanical Sources` are used to generate inputs. Also, two sensors, `Ideal Torque Sensor` and `Ideal Rotational Motion Sensor`, in the library of `Mechanical Sensors` are used to output measurements.

Example 5.20: A Single-Degree-of-Freedom Rotational Mass–Spring–Damper System

Consider the simple disk–shaft system in Example 5.8. A single-degree-of-freedom rotational mass–spring–damper system can be used to approximate the dynamic behavior of the disk–shaft system. The parameter values are $I = 0.01$ kg·m^2, $B = 1.15$ N·m·s/rad, and $K = 4150$ N·m/rad.

a. Assume that a torque $\tau = 10\sin(600t)$ is acting on the disk, which is initially at rest. Build a Simscape model of the physical system and find the angular displacement output $\theta(t)$.
b. Assuming that the external torque is $\tau = 0$ and the initial angular displacement is $\theta(0) = 0.1$ rad, find the angular displacement output $\theta(t)$.

Solution

a. The Simscape block diagram and the angular displacement output of the system are shown in Figures 5.111 and 5.112, respectively. Comparing Figure 5.111 with Figure 5.31, which is the Simscape model of a single-degree-of-freedom translational mass–spring–damper system, reveals the similarity of these two Simscape diagrams. The main difference is that the blocks in this example are all related to rotational motion instead of translational motion.
b. The Simscape model in Figure 5.111 can also be used to simulate the system in Part (b). To specify a zero external torque, we can either define the `Amplitude` of the `Sine Wave` as

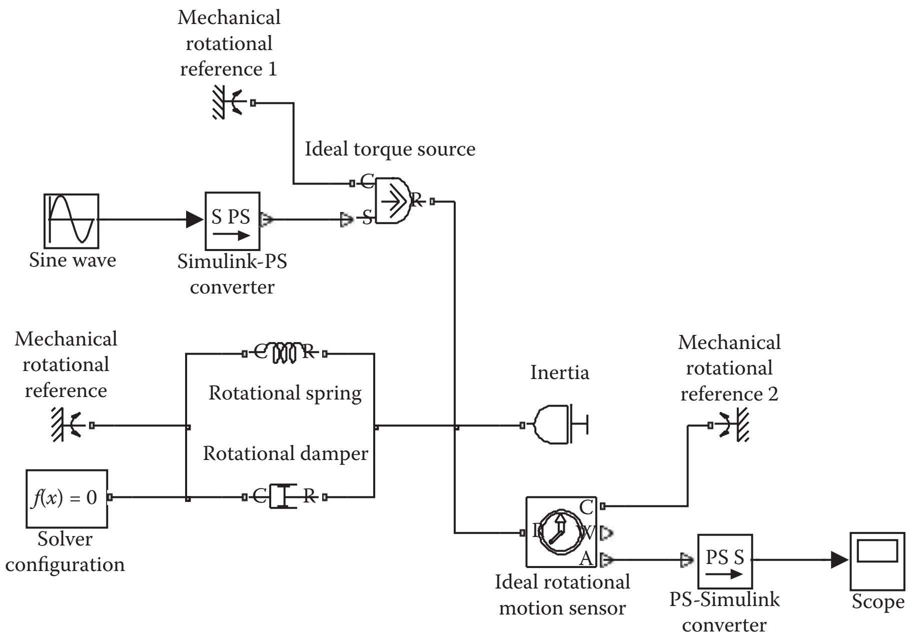


FIGURE 5.111 Simscape block diagram of the physical system in Example 5.20.

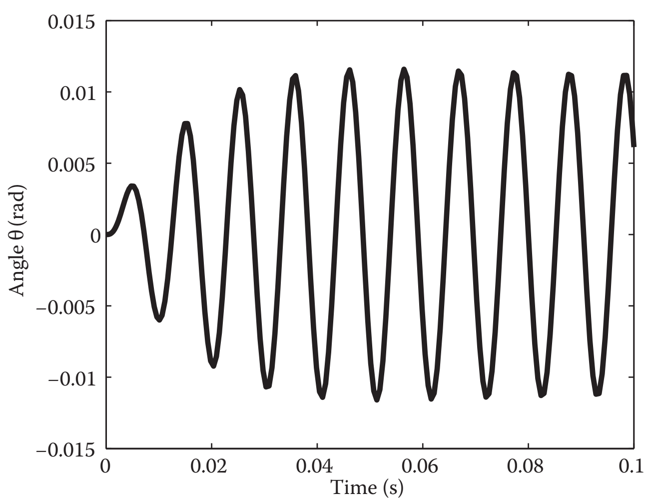


FIGURE 5.112 Angular displacement output θ(t) of the system in Example 5.20 with a torque input.

`0` or delete the blocks related to input generation, including `Sine Wave`, `Simulink-PS Converter`, `Ideal Torque Source`, and `Mechanical Rotational Reference` blocks. To specify a nonzero initial angle, double-click on the `Rotational Spring` block, type `0.1` for the `Initial deformation`, and choose the unit as `rad`. This implies that the spring is initially twisted by 0.1 rad. Also, double-click on the `Ideal Rotational Motion Sensor` block, type `0.1` for the `Initial angle`, and choose the unit as `rad`. The corresponding angular displacement of the system is shown in Figure 5.113.

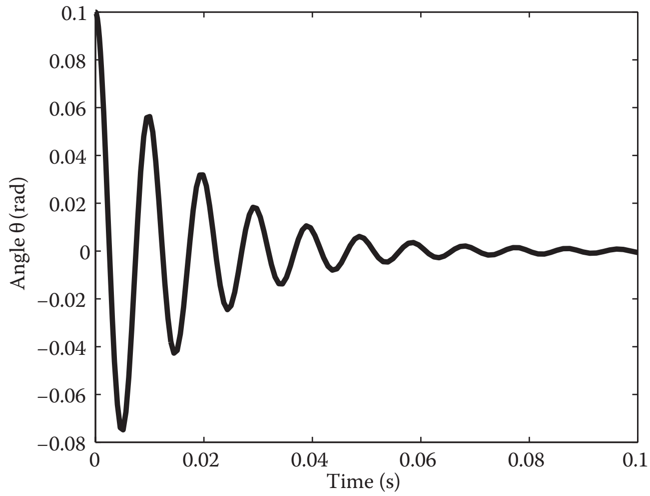


FIGURE 5.113 Angular displacement output $\theta(t)$ of the system in Example 5.20 with a nonzero initial condition.

Example 5.21: A Single-Link Robot Arm with a Gearbox

Consider the single-link robot arm in Example 5.17. It is driven by a DC motor through a gearbox with a gear ratio of $N = 1/70$. The mass moments of inertia of the motor and the load are I_m and I, respectively. The coefficients of torsional viscous damping of the motor and the load are B_m and B, respectively. τ_m is the torque generated by the motor. The parameter values are $I_m = 3.87 \times 10^{-7}$ kg·m², $I = 0.0016$ kg·m², $B_m = 0$ N·m·s/rad, and $B = 0.004$ N·m·s/rad. Assume that the robot arm is initially at rest and the motor torque is $\tau_m(t) = 0.003u(t)$, where $u(t)$ is the unit-step function with a step time of 0 seconds. Build a Simscape model of the physical system and find the angular velocity output $\dot{\theta}(t)$.

Solution

The Simscape model of the system is shown in Figure 5.114. To model the robot arm, two `Inertia` blocks are included, one for the motor and the other for the load. Because the damping of the motor can be ignored, one `Rotational Damper` block is included to represent the damping of the load. The shaft of the motor and that of the load are coupled through a gearbox. Double-click on the `Gear Box` block and type `70` (not 1/70) for the `Gear ratio`, which in Simscape is defined as the ratio of the input shaft angular velocity to that of the output shaft. This is the reciprocal of

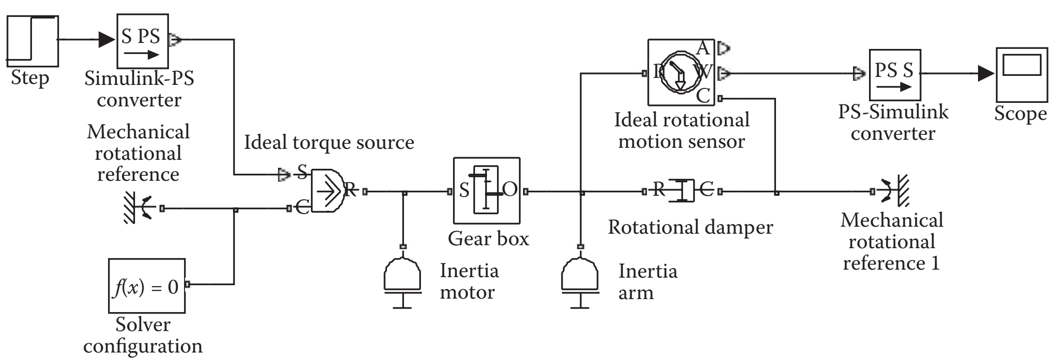


FIGURE 5.114 Simscape block diagram of the single-link robot arm with a gearbox.

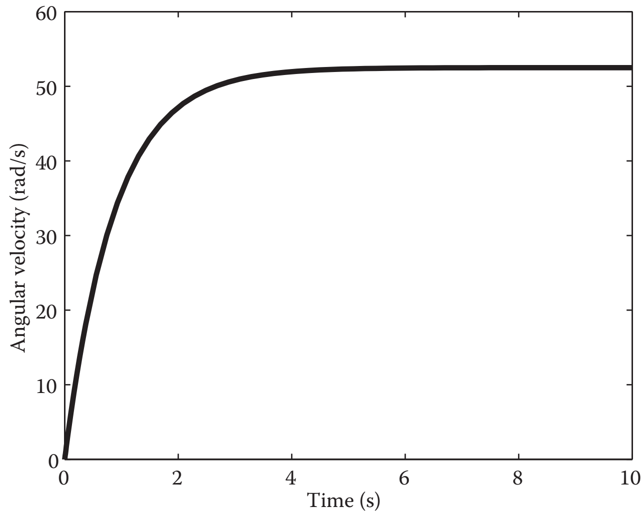


FIGURE 5.115 Angular velocity output $\dot{\theta}(t)$ of the robot arm in Example 5.21.

the gear ratio defined in Equation 5.53. The output of port `W` of the `Ideal Rotation Motion Sensor` is the corresponding angular velocity $\dot{\theta}(t)$, which is shown in Figure 5.115.

Note that only the Simscape block diagrams are given in the above two examples. The corresponding Simulink modeling is left to the reader as an exercise.

PROBLEM SET 5.6

1. Consider the mass–spring–damper system shown in Figure 5.116, in which the force acting on the mass block is a unit-impulse function with a magnitude of 10 N and a duration of 0.1 seconds. The parameter values are $m = 25$ kg, $b = 20$ N·s/m, and $k = 100$ N/m.
   a. Build a Simulink model based on the differential equation of motion of the system and find the displacement output $x(t)$.
   b. Build a Simscape model of the physical system and find the displacement output $x(t)$.
2. Repeat Problem 1 for the mass–spring–damper system shown in Figure 5.117, in which the origin of the coordinate x is set at equilibrium. Assume $x(0) = 0.1$ m and $\dot{x}(0) = 0$ m/s. The parameter values are $m = 20$ kg, $b = 125$ N·s/m, and $k = 400$ N/m.
3. Consider the mass–spring–damper system shown in Figure 5.118. The mass block m_1 and the spring k_1 represent a rotating machine, which is subjected to a harmonic disturbance

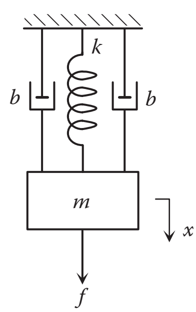


FIGURE 5.116 Problem 1.

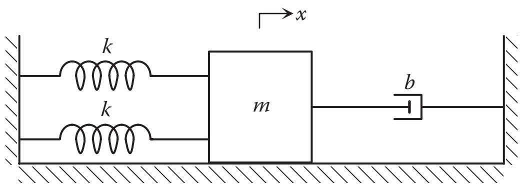


FIGURE 5.117 Problem 2.

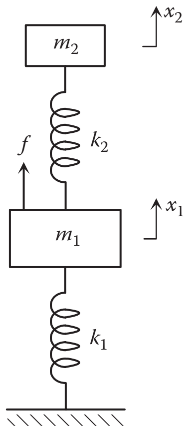


FIGURE 5.118 Problem 3.

force $f = 40\sin(7\pi t)$ N due to a rotating unbalanced mass. The mass block m_2 and the spring k_2 represent a vibration absorber (see Section 9.3 for more details), which is designed to reduce the displacement of the machine. The parameter values are m_1 = 6 kg, k_1 = 6000 N/m, m_2 = 1.65 kg, and k_2 = 800 N/m.

a. Build a Simulink model based on the differential equations of motion of the system and find the displacement outputs $x_1(t)$ and $x_2(t)$.
b. Build a Simscape model of the physical system and find the displacement outputs $x_1(t)$ and $x_2(t)$.

4. Repeat Problem 3 for the two-degree-of-freedom quarter-car model in Example 5.5. Assume that the surface of the road can be approximated as a sine wave $z = Z_0\sin(2\pi vt/L)$, where Z_0 = 0.01, L = 10 m, and the speed v = 20 km/h. If the car moves at a speed of 100 km/h, rerun the simulations and compare the results with those obtained in the case of 20 km/h. Ignore the control force f for both cases.
5. Consider the disk–shaft system in Problem 2 of Problem Set 5.3. The system is approximated as a single-degree-of-freedom rotational mass–spring system, where m = 10 kg, r = 0.05 m, and K = 1000 N·m/rad.
   a. Assume that a torque $\tau = 50u(t)$ N·m is acting on the disk, which is initially at rest. Build a Simscape model of the physical system and find the angular displacement output $\theta(t)$.
   b. Assuming that the external torque is $\tau = 0$ and the initial angular displacement is $\theta(0)$ = 0.1 rad, find the angular displacement output $\theta(t)$.
6. Consider the pendulum–bob system in Problem 5 of Problem Set 5.3. The parameter values are m = 0.1 kg, M = 1.2 kg, L = 0.6 m, and B = 0.5 N·s/m. The initial angular displacement is $\theta(0)$ = 0.1 rad and the initial angular velocity is $\dot{\theta} = 0.1$ rad/s.
   a. Build a Simulink block diagram based on the nonlinear mathematical model of the system and find the angular displacement output $\theta(t)$.

b. Build a Simscape model of the nonlinear physical system and find the angular displacement output $\theta(t)$.

5.7 SUMMARY

This chapter was devoted to modeling mechanical systems. Because real systems are usually quite complicated, simplifying assumptions must be made to reduce the system to an idealized model consisting of interconnected mass, damper, and spring elements. The relations between the external forces or moments applied to the elements and the associated element variables are given by

- Mass translation: $f = m\ddot{x}$ rotation about fixed O: $\tau = I_O\ddot{\theta}$
- Damper translation: $f = b\dot{x}_{\text{rel}}$ rotation: $\tau = B\dot{\theta}_{\text{rel}}$
- Spring translation: $f = kx_{\text{rel}}$ rotation: $\tau = K\theta_{\text{rel}}$

Here, the spring force is dependent on the relative displacement between the two ends of the spring, and the damping force depends on the relative velocity between the two ends of the damper.

For a system of interconnected mechanical elements, the dynamic equations of motion can be obtained by applying Newton's second law or the moment equation (or both). The number of equations of motion is determined by the number of degrees of freedom of the system. The number of degrees of freedom of a dynamic system is defined as the number of independent generalized coordinates that specify the configuration of the system. The static equilibrium position of a mechanical system is usually chosen as the coordinate origin. This choice can simplify the equation of motion by eliminating static forces.

To apply Newton's second law or the moment equation to a mechanical system, it is useful to draw a free-body diagram for each mass in the system, showing all external forces or moments. The noninput forces can be described in terms of displacements or velocities using the expressions associated with the basic spring or damper elements. Drawing correct free-body diagrams is the most important step in analyzing mechanical systems using the force/moment approach.

In this chapter, we were mainly concerned with the modeling of mechanical systems in plane motion, which involves translations along the x and y directions and rotation about one axis perpendicular to the x–y plane. Newton's second law is used for modeling translational mechanical systems, whereas the moment equation is used for rotational mechanical systems. Newton's second law and the moment equation are used together for mixed translational and rotational systems.

The general form of Newton's second law for a rigid body (or a particle) is given by

$$\sum \mathbf{F} = m\mathbf{a}_C$$

or

$$\sum F_x = ma_{Cx}, \quad \sum F_y = ma_{Cy}.$$

For a rigid body rotating about a fixed axis through point O, the moment equation is given by

$$\sum M_O = I_O\alpha.$$

If the axis of rotation is not fixed, the moment equation can be set about the mass center C as

$$\sum M_C = I_C\alpha$$

or an arbitrary point P as

$$\sum \mathbf{M}_{\mathrm{P}} = I_{\mathrm{P}}\alpha + m\mathbf{r}_{\mathrm{C/P}} \times \mathbf{a}_{\mathrm{P}},$$

which is equivalent to

$$\sum M_{\mathrm{P}} = I_{\mathrm{C}}\alpha + M_{\mathrm{eff}\_m\mathbf{a}_{\mathrm{C}}}.$$

The symbol $M_{\mathrm{eff}\_m\mathbf{a}_{\mathrm{C}}}$ represents the effective moment caused by the fictitious force $m\mathbf{a}_{\mathrm{C}}$.

The mass moments of inertia for some rigid bodies with common shapes were given in Table 5.1, in which all masses are assumed to be uniformly distributed and the axes of rotation all pass through the mass centers. If the axis of rotation does not coincide with the axis through the mass center, but is parallel to it, the parallel-axis theorem can be applied to obtain the corresponding moment of inertia,

$$I = I_{\mathrm{C}} + md^2,$$

where d is the distance between the two parallel axes.

For a system of multiple masses, the force equations become

$$\sum F_x = \sum_{i=1}^{n} m_i (a_{\mathrm{C}i})_x, \quad \sum F_y = \sum_{i=1}^{n} m_i (a_{\mathrm{C}i})_y,$$

and the moment equation becomes

$$\sum M_{\mathrm{P}} = \sum_{i=1}^{n} I_{\mathrm{C}i}\alpha_i + \sum_{i=1}^{n} M_{\mathrm{eff}\_m_i\mathbf{a}_{\mathrm{C}i}}.$$

The force/moment approach is based on Newtonian mechanics. An alternative way of obtaining the system's equations of motion is to use the energy method based on analytical mechanics. For a single-degree-of-freedom mass–spring system with negligible friction and damping, the principle of conservation of energy states that

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V) = 0.$$

The kinetic energy of a rigid body in plane motion can be separated into two parts: (1) the kinetic energy associated with the translational motion of the mass center C of the body, and (2) the kinetic energy associated with the rotation of the body about C. Expressions for the kinetic energy of a rigid body in plane motion are given as

- Translation only: $T = \frac{1}{2}mv^2$
- Rotation about a fixed point O: $T = \frac{1}{2}I_{\mathrm{O}}\omega^2$
- Mixed translation and rotation: $T = \frac{1}{2}mv_{\mathrm{C}}^2 + \frac{1}{2}I_{\mathrm{C}}\omega^2$

The potential energy includes

- Gravitational potential energy: $V_g = mgh$
- Elastic potential energy: $V_e = \frac{1}{2}kx^2$ or $V_e = \frac{1}{2}K\theta^2$

For an n-degree-of-freedom system, n independent equations of motion can be derived using Lagrange's formulation. One of the forms of Lagrange's equations for a conservative system is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0, \quad i = 1, 2, \ldots, n,$$

where q_i is the ith generalized coordinate and n is the total number of independent generalized coordinates. In contrast to the force/moment approach with Newton's second law and the moment equation, Lagrange's equations do not require a free-body diagram. However, velocity analysis is essential to Lagrange's equations.

REVIEW PROBLEMS

1. Determine the equivalent spring constant for the system shown in Figure 5.119.
2. Determine the equivalent spring constant for the system shown in Figure 5.120.
3. Consider the system shown in Figure 5.121, in which a mass–spring system is hung from the middle of a massless beam. Assume that the beam can be modeled as a spring and the equivalent stiffness at the midspan is 192 EI_A/L^3, where E is the modulus of elasticity of beam material and I_A is the area moment of inertia about the beam's longitudinal axis.
   a. Derive the differential equation of motion for the system.
   b. Using the differential equation obtained in Part (a), determine the transfer function $X(s)/F(s)$. Assume that the initial conditions are $x(0) = 0$ and $\dot{x}(0) = 0$.
   c. Using the differential equation obtained in Part (a), determine the state-space representation. Assume that the output is the displacement x of the mass.
4. An accelerometer attached to an object can be modeled as a mass–damper–spring system, as shown in Figure 5.122. Denote the displacement of the mass relative to the object, the absolute displacement of the mass, and the absolute displacement of the object as $x(t)$, $y(t)$, and $z(t)$, respectively, where $x(t) = y(t) - z(t)$ and $x(t)$ is measured electronically.
   a. Draw the necessary free-body diagram and derive the differential equation in terms of $x(t)$.
   b. Using the differential equation obtained in Part (a), determine the transfer function $X(s)/Z(s)$. Assume that the initial conditions are $x(0) = 0$ and $\dot{x}(0) = 0$.
   c. Using the differential equation obtained in Part (a), determine the state-space representation. The input is the absolute displacement of the object $z(t)$ and the output is the displacement of the mass relative to the object $x(t)$.

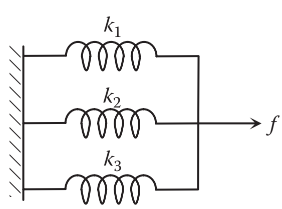


FIGURE 5.119 Problem 1.

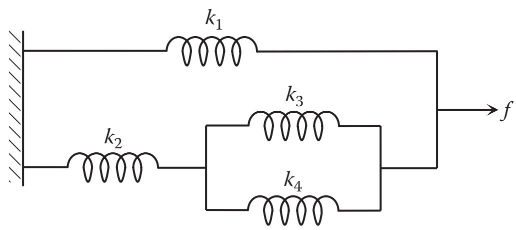


FIGURE 5.120 Problem 2.

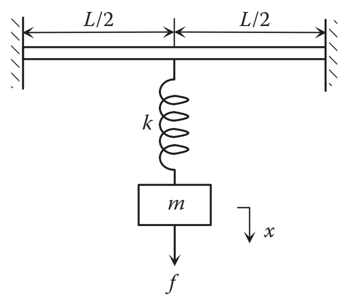


FIGURE 5.121 Problem 3.

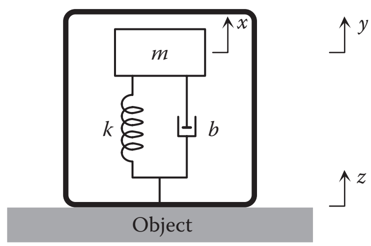


FIGURE 5.122 Problem 4.

5. For the system shown in Figure 5.123, the inputs are the forces f_1 and f_3, and the outputs are the displacements x_1, x_2, and x_3. Draw the necessary free-body diagrams and derive the differential equations of motion. Write the differential equations of motion in the second-order matrix form.
6. Consider a quarter-car model shown in Figure 5.124, where m_1 is the mass of the seats including passengers, m_2 is the mass of one-fourth of the car body, and m_3 is the mass of the wheel–tire–axle assembly. The spring k_1 represents the elasticity of the seat supports, k_2 represents the elasticity of the suspension, and k_3 represents the elasticity of the tire. $z(t)$ is the displacement input due to the surface of the road. Draw the necessary free-body diagrams and derive the differential equations of motion. Write the differential equations of motion in the second-order matrix form.

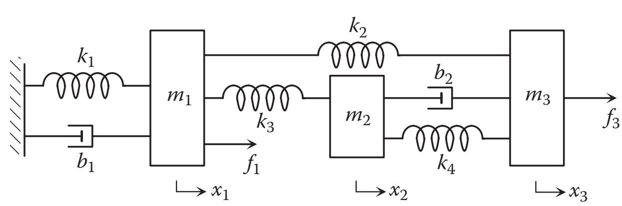


FIGURE 5.123 Problem 5.

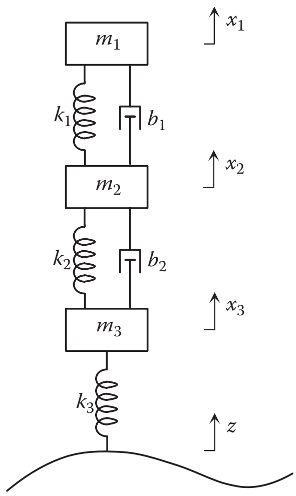


FIGURE 5.124 Problem 6.

7. The system shown in Figure 5.125 consists of a uniform rod of mass m and length L and a translational spring of stiffness k at the rod's tip. The friction at the joint O is modeled as a damper with coefficient of torsional viscous damping B. The input is the force f and the output is the angle θ. The position $\theta = 0$ corresponds to the static equilibrium position when $f = 0$.
   a. Draw the necessary free-body diagram and derive the differential equation of motion for small angles θ.
   b. Using the linearized differential equation obtained in Part (b), determine the transfer function $\Theta(s)/F(s)$. Assume that the initial conditions are $\theta(0) = 0$ and $\dot{\theta}(0) = 0$·
   c. Using the differential equation obtained in Part (b), determine the state-space representation.
8. Consider the inverted-pendulum system shown in Figure 5.126. The system consists of a bob of mass m and a massless rod of length L. Two springs of stiffness k are connected to the middle of the rod. The position $\theta = 0$ corresponds to the static equilibrium position.
   a. Draw the necessary free-body diagram and derive the differential equation of motion for small angles θ.
   b. Using the differential equation obtained in Part (a), determine the state-space representation. Assume that the outputs are the angular displacement θ and the angular velocity $\dot{\theta}$.

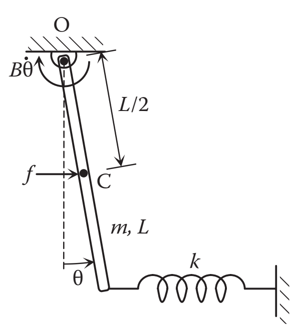


FIGURE 5.125 Problem 7.

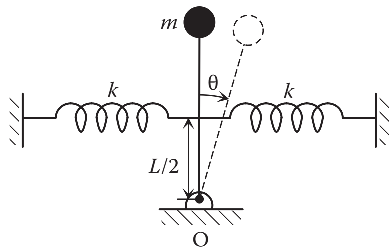


FIGURE 5.126 Problem 8.

9. Consider the system shown in Figure 5.127. Assume that a cylinder of mass m rolls without slipping. Draw the necessary free-body diagram and derive the differential equation of motion for small angles θ.
10. The pulley of mass M shown in Figure 5.128 has a radius of r. The mass moment of inertia of the pulley about the point O is I_O. A translational spring of stiffness k and a block of mass m are connected to the pulley as shown. Assume that the pulley rolls without slipping. Derive the equation of motion using (a) the force/moment approach, and (b) the energy approach.
11. Consider the mechanical system shown in Figure 5.129, in which a uniform rod of mass m and length L is attached to a massless rigid link of equal length. Assume that the system is constrained to move in a vertical plane. Denote the angular displacement of the link as θ_1, and the angular displacement of the rod as θ_2. Derive the equations of motion for small angles using (a) the force/moment approach, and (b) the energy approach.

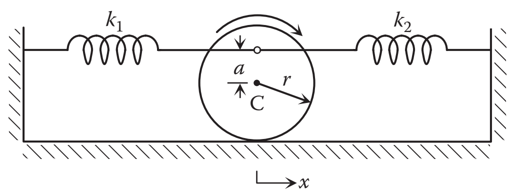


FIGURE 5.127 Problem 9.

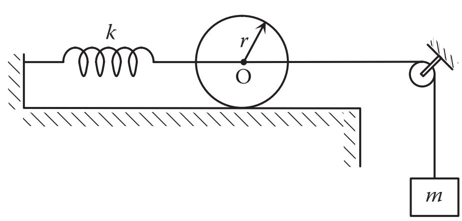


FIGURE 5.128 Problem 10.

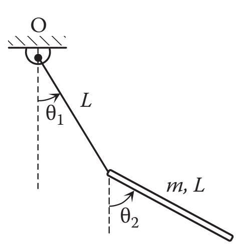


FIGURE 5.129 Problem 11.

12. Consider a half-car model shown in Figure 5.130, in which I_C is the mass moment of inertia of the car body about the pitch axis, m_b is the mass of the car body, m_f is the mass of the front wheel–tire–axle assembly, and m_r is the mass of the rear wheel–tire–axle assembly. Each of the front and rear wheel–tire–axle assemblies is represented by a mass–spring–damper system. The input is the force f, and the car undergoes vertical and pitch motion. Derive the equations of motion using the force/moment approach.
13. A rack and pinion is a pair of gears that convert rotational motion into translation. As shown in Figure 5.131, a torque τ is applied to the shaft. The pinion rotates and causes the rack to translate. The mass moment of inertia of the pinion is I and the mass of the rack is m. Draw the free-body diagram and derive the differential equation of motion.
14. Consider the mass–spring–damper system shown in Problem 9 of Problem Set 5.2, in which the cam and follower impart a displacement $z(t)$ in the form of a periodic sawtooth

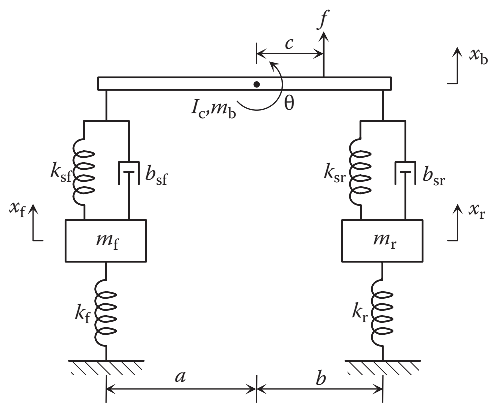


FIGURE 5.130 Problem 12.

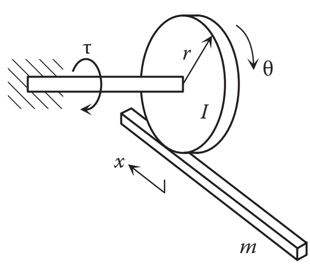


FIGURE 5.131 Problem 13.

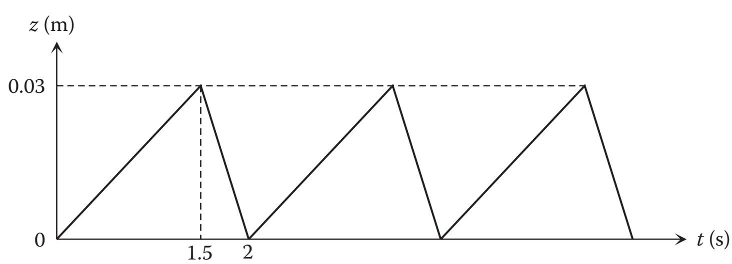


FIGURE 5.132 Problem 14.

function (Figure 5.132) to the lower end of the system. The values of the system parameters are m = 12 kg, b = 200 N·s/m, k_1 = 4000 N/m, and k_2 = 2000 N/m.

a. Build a Simulink model based on the differential equation of motion of the system and find the displacement output $x(t)$.
b. Build a Simscape model of the physical system and find the displacement output $x(t)$.

6 Electrical, Electronic, and Electromechanical Systems

Many engineering systems have electrical, electronic, or electromechanical subsystems as important components, such as power supplies, motors, sensors, or controllers. In this chapter, we discuss the modeling techniques for these systems. We first introduce the fundamentals of electrical elements, which include resistors, inductors, and capacitors. The two main physical laws, Kirchhoff's voltage law and Kirchhoff's current law, are then reviewed and applied to develop mathematical models of electrical circuits. For electronic systems, we take a look at simple operational amplifiers (op-amps), and the op-amp equation is presented, which is useful for obtaining models of amplifiers. This is followed by the modeling of electromechanical systems. The coupling between electrical and mechanical subsystems is established and applied to motor modeling. To simplify the modeling of electrical systems, the concept of impedance is introduced, which provides an alternative way of obtaining mathematical models of systems. The chapter concludes with a simulation of electrical, electronic, and electromechanical systems using MATLAB®, Simulink®, and Simscape™ computer tools.

6.1 ELECTRICAL ELEMENTS

Electrical systems, or electrical circuits, can usually be considered as interconnections of lumped elements, such as sources, resistors, inductors, and capacitors. Sources are active electrical elements, which can provide energy to the circuit and serve as the inputs. Resistors, inductors, and capacitors can store or dissipate energy available in the circuit; however, they cannot produce energy. They are referred to as passive electrical elements.

Two primary variables used to describe the dynamic behavior of an electrical circuit are current and voltage. Current is the time rate of change of charge,

$$i = \frac{\mathrm{d}q}{\mathrm{d}t}, \tag{6.1}$$

where q is charge in coulomb (C) and i is current in ampere (A). For a two-terminal electrical element, the current entering one end of the element must be equal to the current leaving the other end. As shown in Figure 6.1, an arrow is used to denote the direction in which the positive current (or charge) flows.

The voltage at a point in a circuit is a measure of the electrical potential difference between that point and a reference point called the ground. The unit of voltage is volt (V). If a point has the same electrical potential as the ground, it has a voltage of zero. For a two-terminal electrical element, the voltages at both ends are different. As shown in Figure 6.1, v_1 and v_2 denote the terminal voltages with respect to the ground, and

$$v = v_1 - v_2, \tag{6.2}$$

where v is the voltage across the element. The sense of the voltage v is indicated by plus and minus signs. The terminal with the plus sign has a higher voltage than that with the minus sign. The

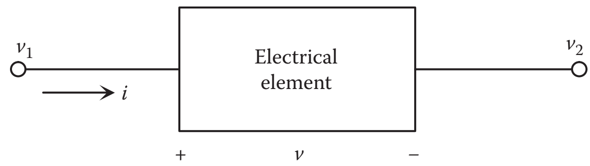


FIGURE 6.1 A two-terminal electrical element.

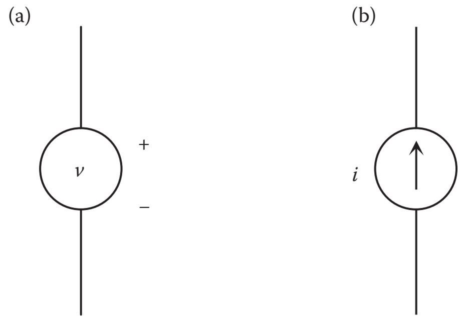


FIGURE 6.2 Active electrical elements: (a) ideal voltage source and (b) ideal current source.

positive sense of the current associated with an electrical element is defined such that within the element, the positive current is assumed to flow from the high-voltage terminal to the low-voltage terminal. If the current flow has the same direction as the voltage drop, then the energy is supplied to the element. The supplied energy is either stored in the element or dissipated by the element. This type of electrical element is passive. For an active electrical element, current flows in the direction opposite to the voltage drop, and it supplies energy to the rest of the circuit. The power supplied to a passive element or generated by an active element is

$$P = vi. \tag{6.3}$$

An active electrical element can be modeled as an ideal current source or an ideal voltage source, as shown in Figure 6.2. An ideal current source provides specified current no matter how much voltage is required by the circuit. An ideal voltage source provides specified voltage, no matter how much current flows in the circuit.

To derive the dynamic model of an electrical circuit, it is important to understand the voltage–current relations for all passive electrical elements.

6.1.1 Resistors

The voltage–current relation for a linear resistor as shown in Figure 6.3 is an algebraic relationship,

$$v = Ri, \tag{6.4}$$

where R is the resistance in units of ohm (Ω). Equation 6.4 is known as Ohm's law, which states that the voltage and current of a linear resistor are directly proportional to each other. It is an empirical formula and can be obtained by a series of measurements.

A resistor dissipates energy by converting it into heat. The power dissipated by a linear resistor is given by

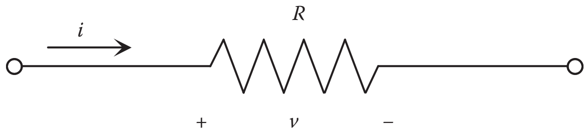


FIGURE 6.3 A resistor and its variables.

$$P = Ri^2 = \frac{v^2}{R}. \tag{6.5}$$

In many electrical circuits, multiple resistors are used. They are arranged in different ways, such as series connections, parallel connections, or both. The equivalent resistance for several resistors arranged in any of these configurations can be obtained to simplify the modeling procedure.

Figure 6.4 shows a circuit with two resistors in series. It is known that the current (or charge) remains unchanged when crossing an electrical element. Thus, for the series connection, the current is the same through each resistor. Ohm's law gives $v_1 = R_1 i$ and $v_2 = R_2 i$, where the voltages v_1 and v_2 also represent a measure of the energy required to move a charge through resistor R_1 or resistor R_2, respectively. The total voltage required to move a charge across the two resistors is $v = v_1 + v_2 = (R_1 + R_2)i$. Comparing this result with the equivalent circuit, where $v = R_{eq} i$, we have

$$R_{eq} = R_1 + R_2. \tag{6.6}$$

A circuit with parallel resistors is shown in Figure 6.5. Note that the parallel resistors share the same terminals. Thus, the voltage across each resistor must be the same, $v_1 = v_2$. Ohm's law gives $v = R_1 i_1$ and $v = R_2 i_2$, where i_1 and i_2 are currents through resistors R_1 and R_2, respectively. Because of the conservation of charge, $i = i_1 + i_2 = v(1/R_1 + 1/R_2)$. Comparing this result with the equivalent circuit, where $i = v/R_{eq}$, we have

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{6.7}$$

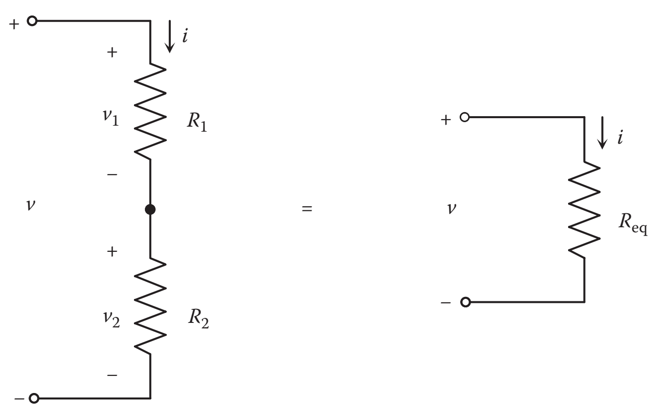


FIGURE 6.4 Equivalence for two resistors in series.

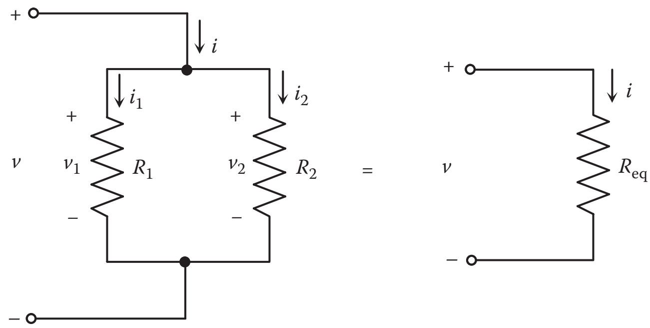


FIGURE 6.5 Equivalence for two resistors in parallel.

or

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}. \tag{6.8}$$

The results for series or parallel resistance can be extended to n resistors. For a circuit of n resistors in series, the equivalent resistance is equal to the sum of all the individual resistances R_i:

$$R_{eq} = R_1 + R_2 + \cdots + R_n. \tag{6.9}$$

For a system of n resistors in parallel, the reciprocal of the equivalent resistance R_{eq} is equal to the sum of all the reciprocals of the individual resistances R_i:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}. \tag{6.10}$$

6.1.2 Inductors

Figure 6.6 shows the symbol for an inductor. The voltage–current relation for a linear inductor is

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t} \tag{6.11}$$

or

$$i = \frac{1}{L}\int v\,\mathrm{d}t \tag{6.12}$$

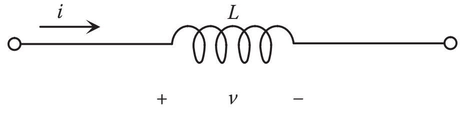


FIGURE 6.6 An inductor and its variables.

where L is the inductance, and the unit is henry (H).

The energy supplied to an inductor is stored in its magnetic field. The stored energy can be derived by performing integration of the power

$$P = vi = \left(L\frac{\mathrm{d}i}{\mathrm{d}t}\right)i \tag{6.13}$$

as follows:

$$E(t) = \int_0^t P(t)\,\mathrm{d}t = \int_0^t Li(t)\,\mathrm{d}i = \frac{1}{2}Li^2\Big|_0^t = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(0) = \frac{1}{2}Li^2(t). \tag{6.14}$$

In Equation 6.14, it is assumed that the current through the inductor is zero at $t = 0$. Note that the energy stored in an inductor is dependent on the square of the current through the inductor and is independent of how the current is established.

6.1.3 Capacitors

Figure 6.7 shows the symbol for a capacitor, where C is the capacitance in units of farad (F). The capacitance is a measure of how much charge can be stored for a given voltage difference across the capacitor, and the mathematical description is $q = Cv$ or $v = q/C$. Note that the charge q is related to the current, $i = \mathrm{d}q/\mathrm{d}t$ or $q = \int i\,\mathrm{d}t$. Thus, the voltage–current relation for a capacitor is expressed as

$$v = \frac{1}{C}\int i\,\mathrm{d}t \tag{6.15}$$

or

$$i = C\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{6.16}$$

A capacitor is also designed to store energy. The energy supplied to the capacitor is stored in its electrical field and can be derived by performing integration of the power

$$P = vi = v\left(C\frac{\mathrm{d}v}{\mathrm{d}t}\right) \tag{6.17}$$

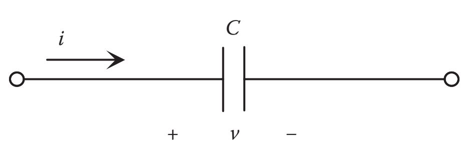


FIGURE 6.7 A capacitor and its variables.

as follows:

$$E(t)=\int_0^t P(t)\,\mathrm{d}t=\int_0^t Cv(t)\,\mathrm{d}v=\frac{1}{2}Cv^2\Big|_0^t=\frac{1}{2}Cv^2(t)-\frac{1}{2}Cv^2(0)=\frac{1}{2}Cv^2(t). \quad (6.18)$$

In Equation 6.18, it is assumed that the voltage across the capacitor is zero at $t = 0$. Note that the energy stored in a capacitor is dependent on the square of the voltage across the capacitor and is independent of how the voltage is acquired.

PROBLEM SET 6.1

1. Determine the equivalent resistance R_{eq} for the circuit shown in Figure 6.8.
2. Determine the equivalent resistance R_{eq} for the circuit shown in Figure 6.9.
3. Determine the equivalent resistance R_{eq} for the circuit shown in Figure 6.10. Assume that all resistors have the same resistance of R.
4. Determine the equivalent resistance R_{eq} for the circuit shown in Figure 6.11. Assume that all resistors have the same resistance of R.
5. A potentiometer is a variable resistor with three terminals. Figure 6.12a shows a potentiometer connected to a voltage source. The two end terminals are labeled as 1 and 2, and the adjustable terminal is labeled as 3. The potentiometer acts as a voltage divider, and

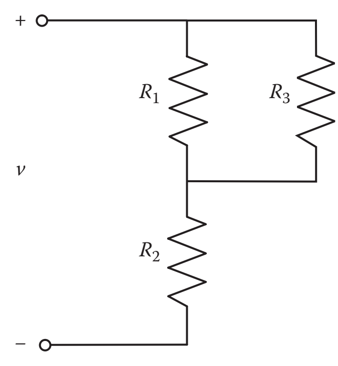


FIGURE 6.8 Problem 1.

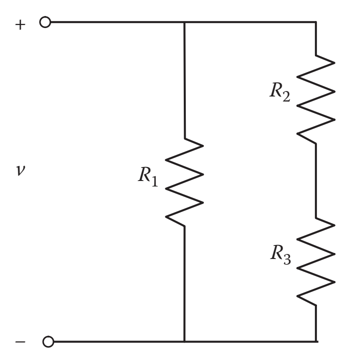


FIGURE 6.9 Problem 2.

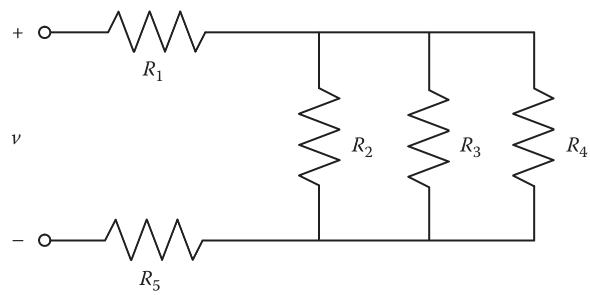


FIGURE 6.10 Problem 3.

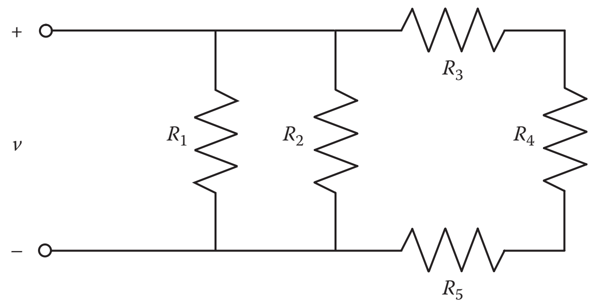


FIGURE 6.11 Problem 4.

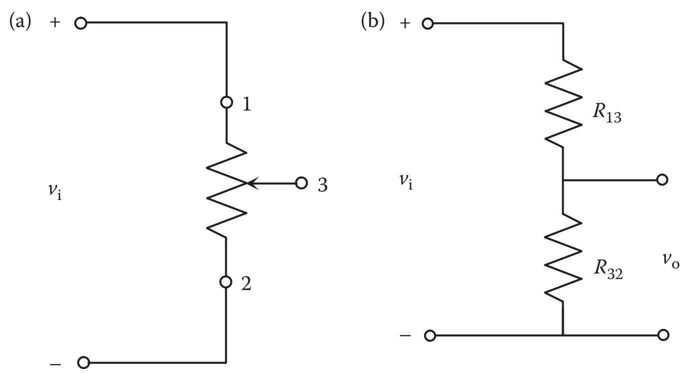


FIGURE 6.12 Problem 5. (a) Potentiometer and (b) voltage divider.

the total resistance is separated into two parts as shown in Figure 6.12b, where R_{13} is the resistance between terminal 1 and terminal 3, and R_{32} is the resistance between terminal 3 and terminal 2. Determine the relationship between the input voltage v_i and the output voltage v_o.

6. The output voltage of a voltage divider is not fixed but varies according to the load.
   a. Find the output voltage v_o in Figure 6.13 for two different values of load resistance: (1) $R_L = 10\ \text{k}\Omega$ and (2) $R_L = 100\ \text{k}\Omega$.
   b. If the output voltage v_o must be greater than 3.8 V, determine the minimum value of the load resistance.
7. Consider an inductive divider shown in Figure 6.14. For an alternating current (AC) input v_i, prove that the output voltage of the inductive voltage divider is $v_o = \frac{L_2}{L_1 + L_2} v_i$.

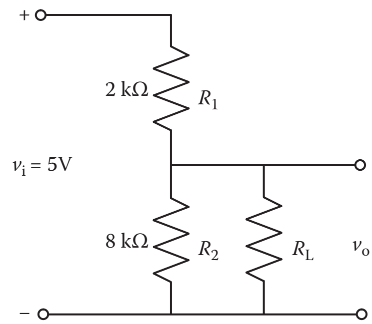


FIGURE 6.13 Problem 6.

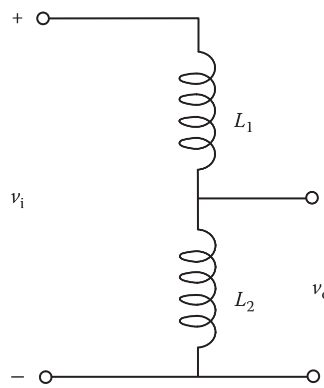


FIGURE 6.14 Problem 7.

8. Consider a capacitive divider shown in Figure 6.15. For an AC input v_i, prove that the output voltage of the capacitive voltage divider is $v_o = \frac{C_1}{C_1 + C_2} v_i$.
9. Consider a circuit of two inductors, L_1 and L_2, in series. Prove that the equivalent inductance of the circuit is $L_{eq} = L_1 + L_2$.
10. Consider a circuit of two inductors, L_1 and L_2, in parallel. Prove that the equivalent inductance of the circuit is $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}$.
11. Consider a circuit of two capacitors, C_1 and C_2, in series. Prove that the equivalent capacitance of the circuit is $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$.

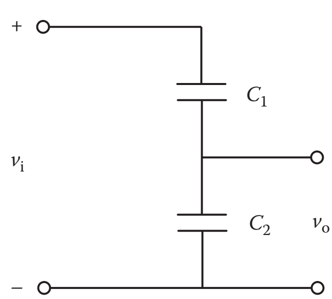


FIGURE 6.15 Problem 8.

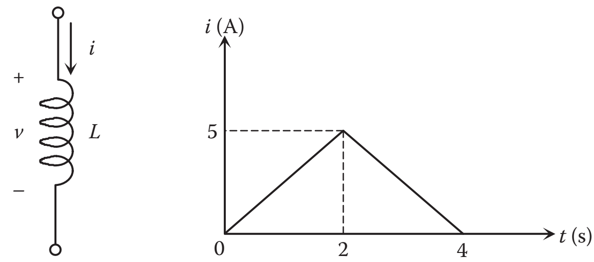


FIGURE 6.16 Problem 13.

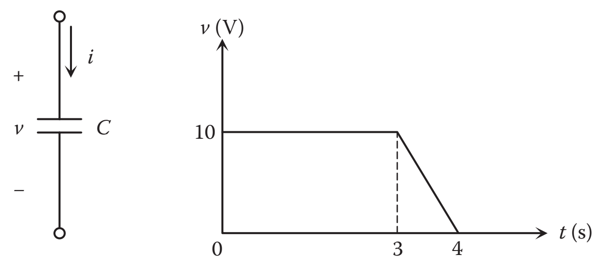


FIGURE 6.17 Problem 14.

12. Consider a circuit of two capacitors, C_1 and C_2, in parallel. Prove that the equivalent capacitance of the circuit is $C_{eq} = C_1 + C_2$.
13. The current through an inductor of 10 mH is shown in Figure 6.16. Find the voltage across the inductor. What is the energy stored in the inductor when (1) $t = 2$ s and (2) $t = 4$ s?
14. The voltage across a capacitor of 100 μF is shown in Figure 6.17. Find the current through the capacitor. What is the energy stored in the capacitor when (1) $t = 3$ s and (2) $t = 4$ s?

6.2 ELECTRIC CIRCUITS

When electrical elements are interconnected to form an electrical circuit, the dynamics model of the circuit can be developed using the voltage–current relations for electrical elements along with two main physical laws. The two laws are known as Kirchhoff's voltage law and Kirchhoff's current law.

6.2.1 Kirchhoff's Voltage Law

For a closed path, or a loop, in a circuit, Kirchhoff's voltage law states that the algebraic sum of voltages around the loop must be zero,

$$\sum_j v_j = 0, \tag{6.19}$$

where v_j is the voltage across the jth element in the loop.

As an example, let us consider a series RLC circuit shown in Figure 6.18, in which a resistor, an inductor, and a capacitor are connected in series. An ideal voltage source provides the desired voltage to the circuit. The current flows from the high-voltage terminal of the source, crosses three passive elements, and enters the low-voltage terminal of the source. If we sum the voltages around the loop in a clockwise direction, a voltage drop occurs for each of the three passive elements, and a voltage gain occurs for the source. Assigning a positive sign to a voltage drop, and a negative sign to a voltage gain, Kirchhoff's voltage law gives $v_R + v_L + v_C - v_a = 0$ or $v_a = v_R + v_L + v_C$, which implies that the voltage of the source must equal the sum of the voltages across the resistor, the inductor, and the capacitor.

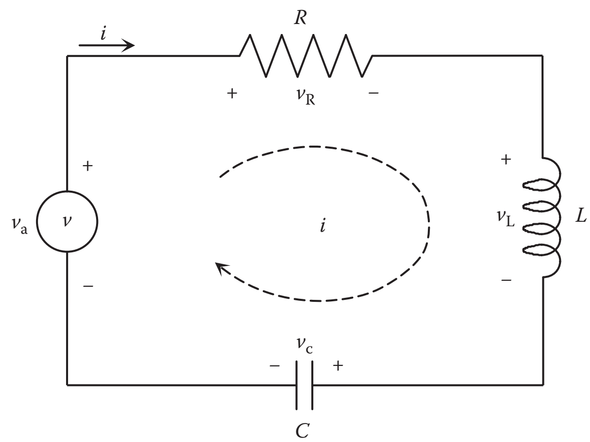


FIGURE 6.18 A series RLC circuit.

Example 6.1: A Series RLC Circuit

Consider the series RLC circuit shown in Figure 6.18.

a. Derive the differential equation in terms of the loop current i.
b. Determine the transfer function $I(s)/V_a(s)$, which relates the source voltage $v_a(t)$ to the loop current $i(t)$. Assume that all the initial conditions are zero.
c. Determine the transfer function $V_C(s)/V_a(s)$, which relates the source voltage $v_a(t)$ to the capacitor voltage $v_C(t)$. Assume that all the initial conditions are zero.

Solution

a. Applying Kirchhoff's voltage law to the single loop along the clockwise direction gives

$$v_R + v_L + v_C - v_a = 0.$$

For the series loop, the same current flows through each element. The expressions for v_R, v_L, and v_C are

$$v_R = Ri,$$

$$v_L = L\frac{di}{dt},$$

$$v_C = \frac{1}{C}\int i\,dt.$$

We then have

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = v_a.$$

Note that the above equation is an integral–differential equation, not a differential equation. To eliminate the integral term, we take the time derivative of both sides of the equation. Rearranging the terms results in

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=\frac{dv_a}{dt},$$

which is a second-order differential equation for current i with the time derivative of the applied voltage as the forcing function.

b. Taking the Laplace transform of the above differential equation yields

$$Ls^2I(s)+RsI(s)+\frac{1}{C}I(s)=sV_a(s).$$

The transfer function relating the input voltage $v_a(t)$ and the output current $i(t)$ is

$$\frac{I(s)}{V_a(s)}=\frac{s}{Ls^2+Rs+(1/C)}.$$

c. Note that the capacitor voltage v_C does not appear explicitly in the differential equation. To determine the transfer function $V_C(s)/V_a(s)$, we use the result of the transfer function $I(s)/V_a(s)$ and apply the voltage–current relation for a capacitor

$$v_C=\frac{1}{C}\int i\,dt,$$

which gives

$$V_C(s)=\frac{1}{Cs}I(s).$$

Thus, the transfer function relating the input voltage $v_a(t)$ and the capacitor voltage $v_C(t)$ is

$$\frac{V_C(s)}{V_a(s)}=\frac{1}{Cs}\frac{I(s)}{V_a(s)}=\frac{1}{LCs^2+RCs+1}.$$

The circuit in Figure 6.18 can also be modeled using a differential equation for charge q. Recall that current is the time rate of change of charge, $i = dq/dt$ or $q = \int i\,dt$. Rewriting the current-related terms in the equation $Ri + L(di/dt) + (1/C)\int i dt = v_a$ in terms of q gives

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=v_a,$$

which is a second-order differential equation for the charge $q(t)$ with the applied voltage as the forcing function.

6.2.2 Kirchhoff's Current Law

When the terminals of two or more circuit elements are connected together, the common junction is referred to as a node. For a node in a circuit, Kirchhoff's current law states that the sum of the currents entering the node must be equal to the sum of the currents leaving that node. If we assign a positive sign to the current entering the node and a negative sign to the current leaving the node, the algebraic sum of the currents at the node must be zero,

$$\sum_j i_j = 0, \tag{6.20}$$

where i_j is the current of the jth element at the node.

Example 6.2: A Parallel RLC Circuit

Consider the parallel RLC circuit shown in Figure 6.19, in which an ideal current source supplies the desired current to the circuit.

a. Derive the differential equation relating the input current i_a to the output voltage v_o.
b. Determine the transfer function $V_o(s)/I_a(s)$, which relates the input current $i_a(t)$ to the output voltage $v_o(t)$. Assume that all the initial conditions are zero.
c. Determine the transfer function $I_L(s)/I_a(s)$, which relates the input current $i_a(t)$ to the current through the inductor $i_L(t)$. Assume that all the initial conditions are zero.

Solution

a. The three currents i_R, i_L, and i_C are defined in Figure 6.19. Each passive element has one terminal connected to the ground and the other terminal connected to a common node. We can apply Kirchhoff's law to either the ground or node 1. Applying Kirchhoff's current law to node 1 gives

$$i_a - i_R - i_L - i_C = 0.$$

For the parallel connection, the voltages across all three elements are the same. The expressions for i_R, i_L, and i_C are

$$i_R = \frac{v_o}{R},$$

$$i_L = \frac{1}{L}\int v_o \, dt,$$

$$i_C = C\frac{dv_o}{dt}.$$

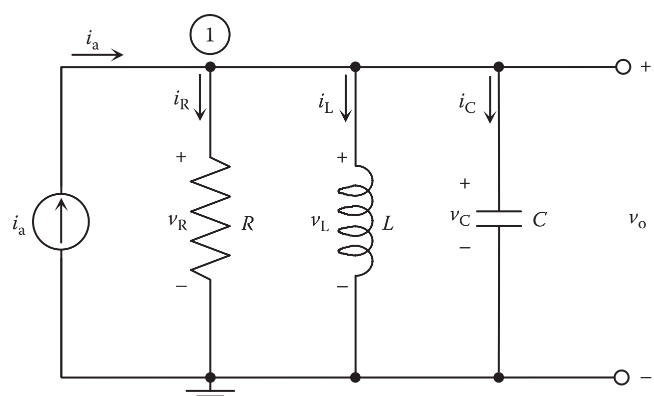


FIGURE 6.19 A parallel RLC circuit.

We then have

$$\frac{v_o}{R} + \frac{1}{L}\int v_o \, dt + C\frac{dv_o}{dt} = i_a,$$

which is an integral–differential equation. To eliminate the integral term, we take the time derivative of both sides of the equation. Rearranging the terms results in

$$C\frac{d^2v_o}{dt^2} + \frac{1}{R}\frac{dv_o}{dt} + \frac{1}{L}v_o = \frac{di_a}{dt},$$

which is a second-order differential equation for the output voltage $v_o(t)$.

b. Taking the Laplace transform of the above differential equation yields

$$Cs^2V_o(s) + \frac{1}{R}sV_o(s) + \frac{1}{L}V_o(s) = sI_a(s).$$

The transfer function relating the input current $i_a(t)$ and the output voltage $v_o(t)$ is

$$\frac{V_o(s)}{I_a(s)} = \frac{s}{Cs^2 + (1/R)s + (1/L)}.$$

c. To find the transfer function $I_L(s)/I_a(s)$, note that

$$i_L = \frac{1}{L}\int v_o \, dt,$$

which gives

$$I_L(s) = \frac{1}{Ls}V_o(s).$$

Thus,

$$\frac{I_L(s)}{I_a(s)} = \frac{1}{Ls}\frac{V_o(s)}{I_a(s)} = \frac{1}{LCs^2 + (L/R)s + 1}.$$

The above two simple examples illustrate how one can develop a differential equation for a series or parallel RLC circuit using Kirchhoff's voltage law or Kirchhoff's current law. However, if circuit components are series connected in some parts and parallel in others, we must selectively apply Kirchhoff's voltage law and Kirchhoff's current law to obtain the desired differential equation.

Example 6.3: An RC Circuit

Consider the circuit shown in Figure 6.20. Derive the differential equation relating the output voltage $v_o(t)$ to the input voltage $v_a(t)$.

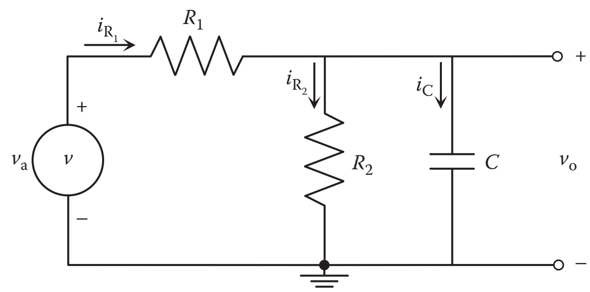


FIGURE 6.20 An RC circuit.

Solution

The currents through the three passive elements (i_{R_1}, i_{R_2}, and i_C) are defined in Figure 6.20. Applying Kirchhoff's current law gives

$$i_{R_1} = i_{R_2} + i_C.$$

Applying voltage–current relations and Kirchhoff's voltage law gives

$$i_{R_1} = \frac{v_{R_1}}{R_1} = \frac{v_a - v_o}{R_1},$$

$$i_{R_2} = \frac{v_{R_2}}{R_2} = \frac{v_o}{R_2},$$

$$i_C = C\frac{\mathrm{d}v_C}{\mathrm{d}t} = C\frac{\mathrm{d}v_o}{\mathrm{d}t}.$$

Substituting the above expressions into Kirchhoff's current law, we have

$$\frac{v_a - v_o}{R_1} = \frac{v_o}{R_2} + C\frac{\mathrm{d}v_o}{\mathrm{d}t}.$$

Rearranging the terms results in

$$C\frac{\mathrm{d}v_o}{\mathrm{d}t} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_o = \frac{1}{R_1}v_a.$$

As shown in Example 6.3, the procedure for deriving the differential equation of a circuit consists of applying Kirchhoff's laws and voltage–current relations for the components of the circuit. However, obtaining a set of equations for more complicated circuits is not that easy. Instead, a formal method is needed that produces a small, simple set of equations leading directly to the input–output relation. The two commonly used methods are the node method (which relies on Kirchhoff's current law) and the loop method (which relies on Kirchhoff's voltage law).

6.2.3 Node Method

If a node in a circuit is chosen as the reference, any other node can be assigned a voltage, which is defined between this node and the reference. This common reference node is usually referred to as the

ground. To apply the node method to a circuit, we start by labeling all currents at each node whose voltage is unknown. The current through each passive circuit element is expressed in terms of the node voltages using the voltage–current relations given in Section 6.1. We then apply Kirchhoff's current law to each node, and the resulting set of equations can be combined to obtain the complete model of the circuit. The following two examples show details of the node-voltage method.

Example 6.4: Circuit Modeling Using the Node Method: One Node

Consider the circuit shown in Figure 6.21. Derive the differential equation relating the output voltage $v_o(t)$ to the input voltage $v_a(t)$.

Solution

Note that the voltage at node 1 is unknown and we denote it as v_1. All currents entering or leaving node 1 are labeled as shown in Figure 6.21. Applying Kirchhoff's current law to node 1 gives

$$i_L - i_R - i_C = 0.$$

Expressing the current through each element in terms of the node voltage, we have

$$\frac{1}{L}\int (v_a - v_1)\mathrm{d}t - \frac{v_1 - 0}{R} - C\frac{\mathrm{d}}{\mathrm{d}t}(v_1 - 0) = 0.$$

Differentiating the above equation with respect to time results in

$$\frac{v_a - v_1}{L} - \frac{\dot{v}_1}{R} - C\ddot{v}_1 = 0.$$

Because the node voltage v_1 is essentially the output voltage v_o, the above equation can be rewritten as

$$RLC\ddot{v}_o + L\ddot{v}_o + Rv_o = Rv_a,$$

which is the input–output equation relating the applied voltage v_a and the output voltage v_o.

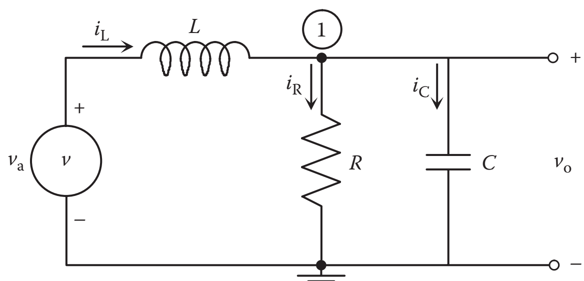


FIGURE 6.21 A circuit containing one node.

Example 6.5: Circuit Modeling Using the Node Method: Two Nodes

Consider the circuit shown in Figure 6.22. Using the node method, derive the differential equations of the system in terms of node voltages.

Solution

Note that the voltages at node 1 and node 2 are unknown. Denote the voltage at node 1 as v_1 and the voltage at node 2 as v_2. All currents entering or leaving node 1 and node 2 are labeled as shown in Figure 6.22.

At node 1, applying Kirchhoff's current law gives

$$i_{L_1} - i_{L_2} - i_{R_1} = 0.$$

Expressing the current through each element in terms of the node voltages, we have

$$\frac{1}{L_1}\int (v_a - v_1)\mathrm{d}t - \frac{1}{L_2}\int (v_1 - v_2)\mathrm{d}t - \frac{v_1 - 0}{R_1} = 0.$$

Differentiating the above equation with respect to time results in

$$\frac{v_a - v_1}{L_1} - \frac{v_1 - v_2}{L_2} - \frac{\dot{v}_1}{R_1} = 0,$$

which can be rearranged to give the first differential equation

$$\frac{1}{R_1}\dot{v}_1 + \left(\frac{1}{L_1} + \frac{1}{L_2}\right)v_1 - \frac{1}{L_2}v_2 = \frac{1}{L_1}v_a.$$

Similarly, applying Kirchhoff's current law to node 2 yields

$$i_{L_2} - i_{R_2} - i_C = 0.$$

Expressing the current through each element in terms of the node voltages, we have

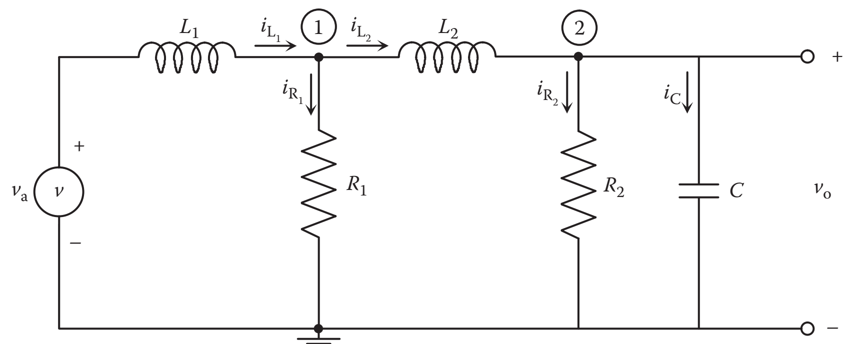


FIGURE 6.22 A circuit containing two nodes.

$$\frac{1}{L_2}\int (v_1 - v_2)\mathrm{d}t - \frac{v_2 - 0}{R_2} - C\frac{\mathrm{d}}{\mathrm{d}t}(v_2 - 0) = 0.$$

Differentiating the above equation with respect to time results in

$$\frac{v_1 - v_2}{L_2} - \frac{\dot{v}_2}{R_2} - C\ddot{v}_2 = 0.$$

which can be rearranged to give the second differential equation

$$C\ddot{v}_2 + \frac{1}{R_2}\dot{v}_2 - \frac{1}{L_2}v_1 + \frac{1}{L_2}v_2 = 0.$$

Note that the differential equation obtained at node 1 is first-order, and that obtained at node 2 is second-order. Thus, the circuit shown in Figure 6.22 is a third-order system. In second-order matrix form, we have

$$\begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}\begin{Bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{Bmatrix} + \begin{bmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_2} \end{bmatrix}\begin{Bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{Bmatrix} + \begin{bmatrix} \frac{1}{L_1} + \frac{1}{L_2} & -\frac{1}{L_2} \\ -\frac{1}{L_2} & \frac{1}{L_2} \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{L_1}v_a \\ 0 \end{Bmatrix}.$$

As presented in Sections 4.3 and 4.4, the above input–output equation can be converted to transfer functions or the state-space form. The problem of finding the transfer function $V_o(s)/V_a(s)$ is left to the reader as an exercise.

6.2.4 Loop Method

In a circuit carrying some current, there exists at least one loop. Starting from any loop, a current circulating that loop can be assigned. For any additional loop containing at least one new element that is not in any previous loop, a new loop current can be assigned. To apply the loop method to a circuit, we start by assigning each loop current. Generally, assume that all unknown currents flow in the clockwise direction, and all known currents follow the directions of the current sources. The voltage across each passive element is expressed in terms of loop currents using the voltage–current relations given in Section 6.1. We then apply Kirchhoff's voltage law to each loop with unknown current, and the resulting set of equations can be combined to obtain the complete model of the circuit. The following example shows the details of the loop-current method.

Example 6.6: Circuit Modeling Using the Loop Method: Two Loops

Reconsider the circuit in Figure 6.21 and solve Example 6.4 using the loop method.

Solution

Assign loop currents as shown in Figure 6.23. Note that there are two loops with unknown currents.

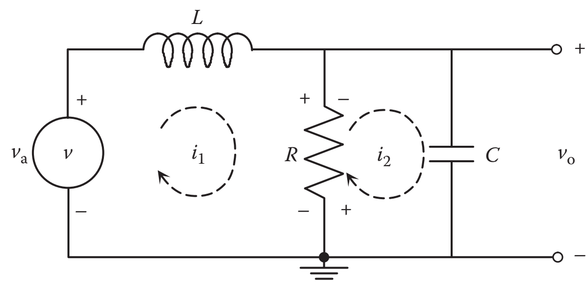


FIGURE 6.23 A circuit containing two loops.

For loop 1, applying Kirchhoff's voltage law gives

$$v_L + v_R - v_a = 0.$$

Expressing the voltage across each element in terms of the loop currents, we have

$$L\frac{di_1}{dt} + R(i_1 - i_2) - v_a = 0.$$

Similarly, applying Kirchhoff's voltage law to loop 2 gives

$$v_C + v_R = 0,$$

$$\frac{1}{C}\int i_2\,dt + R(i_2 - i_1) = 0.$$

Differentiating the above equation with respect to time results in

$$\frac{1}{C}i_2 + R\frac{di_2}{dt} - R\frac{di_1}{dt} = 0.$$

The differential equation obtained in each loop is first-order; thus, the circuit is a second-order system. The two first-order differential equations can be written in matrix form as follows:

$$\begin{bmatrix} L & 0 \\ -R & R \end{bmatrix}\begin{Bmatrix} di_1/dt \\ di_2/dt \end{Bmatrix} + \begin{bmatrix} R & -R \\ 0 & \frac{1}{C} \end{bmatrix}\begin{Bmatrix} i_1 \\ i_2 \end{Bmatrix} = \begin{Bmatrix} v_a \\ 0 \end{Bmatrix}.$$

Note that the output voltage v_o is related to the current through the capacitor, that is, $v_o = \frac{1}{C}\int i_2\,dt$ or $V_o(s) = \frac{1}{Cs}I_2(s)$. Taking the Laplace transform of the system of differential equations gives

$$\begin{bmatrix} Ls + R & -R \\ -Rs & Rs + \frac{1}{C} \end{bmatrix}\begin{Bmatrix} I_1(s) \\ I_2(s) \end{Bmatrix} = \begin{Bmatrix} V_a(s) \\ 0 \end{Bmatrix}.$$

Using Cramer's rule to solve for $I_2(s)$, we have

$$I_2(s) = \frac{Rs}{RLs^2 + \frac{L}{C}s + \frac{R}{C}}V_a(s).$$

Thus,

$$V_o(s) = \frac{1}{Cs} I_2(s) = \frac{R}{RLCs^2 + Ls + R} V_a(s).$$

Taking the inverse Laplace transform gives the same input–output equation obtained previously in Example 6.4,

$$RLC\ddot{v}_o + L\dot{v}_o + Rv_o = Rv_a.$$

Example 6.6 shows that the loop method is similar to the node method. The choice between the two methods is often made based on the circuit at hand. For example, considering the circuit in Figure 6.22, there are two independent nodes, but three independent loops. Therefore, the node method is expected to be easier to apply (see Example 6.5). The reader can try the loop method to solve the same problem. We will emphasize the node method to obtain mathematical models of circuits in this book.

6.2.5 State Variables of Circuits

To represent a circuit model in state-space form, we need to choose an appropriate set of state variables whose time derivatives are expressed in terms of the state variables and inputs. Because the choice of state variables is not unique, it is difficult to identify the appropriate states for expressing a circuit in state-space form. Here, we introduce a customary choice of state variables by identifying the energy storage elements. As stated in Section 6.1, both inductors and capacitors can store energy. In a given circuit, knowledge of the voltage signals across capacitors and of the current signals through inductors is sufficient enough to calculate other circuit variables using only algebraic equations. Generally, inductor currents and capacitor voltages are continuous in nature and are often chosen as the state variables. To determine the state-space form of an electric circuit, we need to find the expression of $\mathrm{d}i_L/\mathrm{d}t$ or $\mathrm{d}v_C/\mathrm{d}t$ for each inductor or capacitor. Based on the voltage–current relations, we have

$$\frac{\mathrm{d}i_L}{\mathrm{d}t} = \frac{v_L}{L} \tag{6.21}$$

and

$$\frac{\mathrm{d}v_C}{\mathrm{d}t} = \frac{i_C}{C}, \tag{6.22}$$

where v_L is the inductor voltage and i_C is the capacitor current. Thus, the problem is converted to expressing v_L and i_C in terms of state variables and inputs using Kirchhoff's laws and voltage–current relations for electrical elements.

Example 6.7: State-Variable Model of the Circuit in Example 6.4

Reconsider the circuit shown in Figure 6.21.

a. Derive the state-variable model with inductor currents and capacitor voltages as states. The input is the applied voltage v_a, and the output is the voltage across the capacitor C.
b. Based on the state-space form obtained in Part (a), determine the differential equation relating the output voltage $v_o(t)$ to the input voltage $v_a(t)$.

Solution

a. We first label the nodes and currents as we did in Example 6.4. Note that the circuit has two independent energy storage elements, L and C. This implies that two state variables are needed, and they are

$$x_1 = i_L, \quad x_2 = v_C.$$

Their time derivatives are

$$\dot{x}_1 = \frac{di_L}{dt} = \frac{1}{L}v_L,$$

$$\dot{x}_2 = \frac{dv_C}{dt} = \frac{1}{C}i_C.$$

We need to express the voltage across the inductor, v_L, and the current through the capacitor, i_C, in terms of the state variables and the input. Note that

$$v_L = v_a - v_1 = v_a - v_C = u - x_2.$$

Applying Kirchhoff's current law to node 1 gives

$$i_C = i_L - i_R = i_L - \frac{v_C}{R} = x_1 - \frac{1}{R}x_2.$$

Thus, the complete set of two state-variable equations is

$$\dot{x}_1 = \frac{1}{L}v_L = -\frac{1}{L}x_2 + \frac{1}{L}u,$$

$$\dot{x}_2 = \frac{1}{C}i_C = \frac{1}{C}x_1 - \frac{1}{RC}x_2.$$

The output equation is

$$y = v_o = v_C = x_2.$$

The state equation and the output equation can be written in matrix form as follows:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{RC} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + 0 \cdot u.$$

b. Note that $V_o(s) = Y(s)$ and $V_a(s) = U(s)$. As presented in Section 4.4, the state-space form can be converted to a transfer function using

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D.$$

Substituting **A**, **B**, **C**, and D matrices obtained in Part (a) gives

$$\frac{V_o(s)}{V_a(s)} = [0 \ \ 1]\begin{bmatrix} s & \dfrac{1}{L} \\ -\dfrac{1}{C} & s+\dfrac{1}{RC} \end{bmatrix}^{-1}\begin{bmatrix} 1/L \\ 0 \end{bmatrix} = \frac{R}{RLCs^2 + Ls + R},$$

which returns the same input–output equation as the one obtained in Examples 6.4 and 6.6.

PROBLEM SET 6.2

1. Consider the first-order RC circuit shown in Figure 6.24.
   a. Derive the input–output differential equation relating v_C and v_a.
   b. Determine the transfer function $I(s)/V_a(s)$, which relates the loop current $i(t)$ to the applied voltage $v_a(t)$. Assume that all the initial conditions are zero.
2. Consider the first-order RL circuit shown in Figure 6.25.
   a. Derive the input–output differential equation relating i_L and v_a.
   b. Determine the transfer function $V_L(s)/V_a(s)$, which relates the voltage across the inductor $v_L(t)$ to the applied voltage $v_a(t)$. Assume that all the initial conditions are zero.
3. Consider the circuit shown in Figure 6.26. Use the node method to derive the input–output differential equation relating v_o and v_a.
4. Repeat Problem 3 using the loop method.
5. Consider the circuit shown in Figure 6.27. Use the node method to derive the input–output differential equation relating i and v_a.
6. Repeat Problem 5 using the loop method.

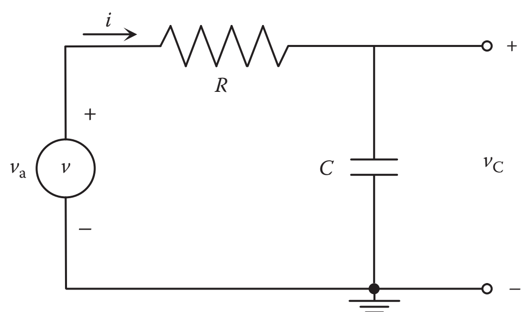


FIGURE 6.24 Problem 1.

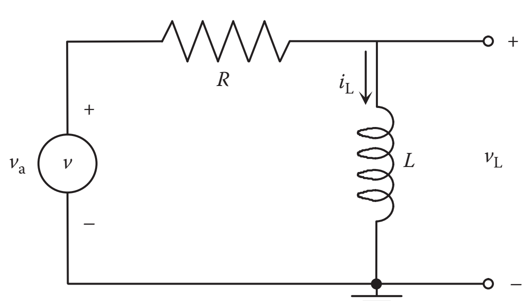


FIGURE 6.25 Problem 2.

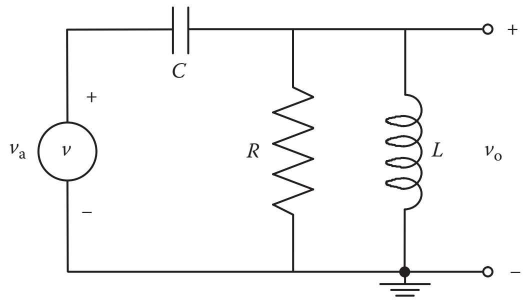


FIGURE 6.26 Problem 3.

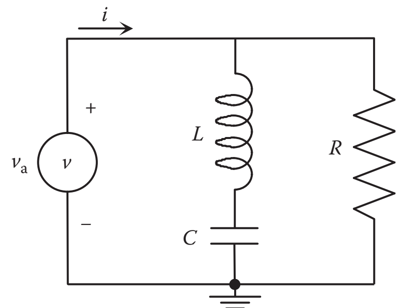


FIGURE 6.27 Problem 5.

7. Consider the circuit shown in Figure 6.28. Use the node method to derive the input–output differential equation relating v_o and v_a.
8. Repeat Problem 7 using the loop method.
9. Consider the circuit shown in Figure 6.29. Use the node method to derive the differential equations for node voltages. Determine the transfer function $V_o(s)/V_a(s)$. Assume that all initial conditions are zero.
10. Reconsider the circuit shown in Figure 6.29. Use the loop method to derive the differential equations for loop currents. Determine the transfer function $V_o(s)/V_a(s)$. Assume that all initial conditions are zero.
11. Consider the circuit shown in Figure 6.26. Determine a suitable set of state variables and obtain the state-space representation with v_o as the output.
12. Repeat Problem 11 for the circuit shown in Figure 6.27 with i as the output.

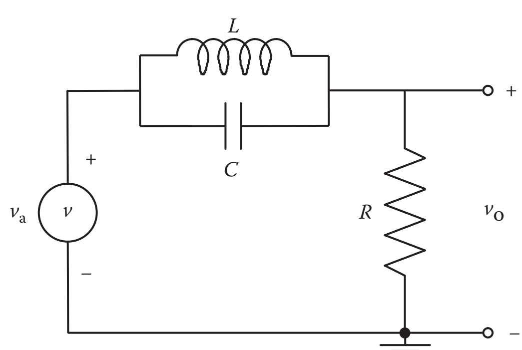


FIGURE 6.28 Problem 7.

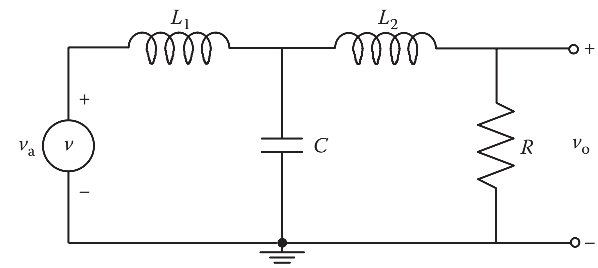


FIGURE 6.29 Problem 9.

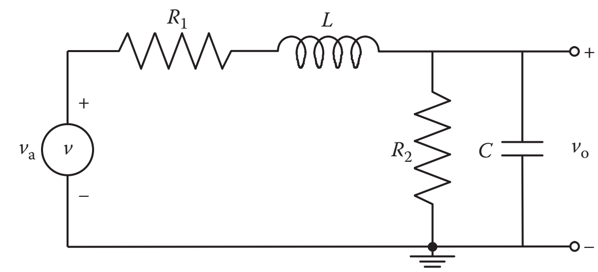


FIGURE 6.30 Problem 15.

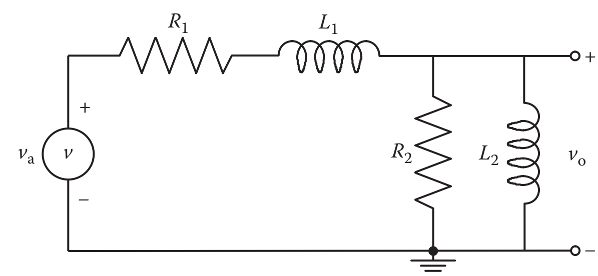


FIGURE 6.31 Problem 16.

13. Repeat Problem 11 for the circuit shown in Figure 6.28.
14. Repeat Problem 11 for the circuit shown in Figure 6.29.
15. Repeat Problem 11 for the circuit shown in Figure 6.30.
16. Repeat Problem 11 for the circuit shown in Figure 6.31.

6.3 OPERATIONAL AMPLIFIERS

An op-amp is an electronic element that is used to amplify electrical signals and drive physical devices. Figure 6.32 shows the schematic diagram of an op-amp, which is a voltage amplifier with a high gain K. Unlike the electrical elements discussed in earlier sections, op-amps have more than two terminals. The diagram in Figure 6.32 does not show all of the terminals connected to the physical devices. It only shows a pair of input terminals and one output terminal. The output voltage is

$$v_o = K(v_+ - v_-), \tag{6.23}$$

where K is a very large positive number, typically 10^5 to 10^6. Because the output voltage v_o must be a finite number and K is very large, the voltage difference between the input terminals must approach zero. Thus,

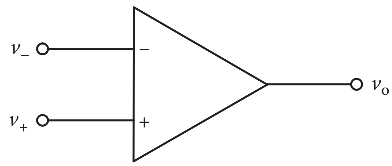


FIGURE 6.32 An op-amp.

$$v_+ \approx v_-, \tag{6.24}$$

which is considered to be the op-amp equation.

Note that the diagram in Figure 6.32 is a simple symbol for an op-amp, which typically contains many resistors, inductors, and capacitors built on an integrated chip.

Example 6.8: An Op-Amp Multiplier

Consider the op-amp circuit shown in Figure 6.33, in which one resistor R_2 is in parallel connection with an op-amp, and the resulting parallel circuit is in series connection with another resistor R_1. Determine the relation between the input voltage v_i and the output voltage v_o. Assume that the current drawn by the op-amp is very small.

Solution

Label the currents at the nodes with unknown voltages. The system has only one significant node: node 1. Applying Kirchhoff's current law to node 1 gives

$$i_1 - i_2 - i_3 = 0.$$

Because the current drawn by the op-amp is very small, that is, $i_3 \approx 0$, we have

$$i_1 \approx i_2.$$

Using the voltage–current relation for each resistor yields

$$\frac{v_i - v_1}{R_1} = \frac{v_1 - v_o}{R_2}.$$

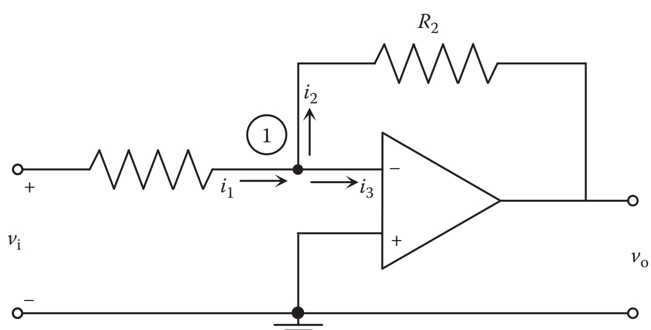


FIGURE 6.33 An op-amp multiplier circuit.

Note that the input terminal marked with the plus sign is connected to the ground. From the op-amp equation $v_+ \approx v_-$, we have

$$v_1 = v_- \approx v_+ = 0.$$

Thus, the relation between the input voltage v_i and the output voltage v_o is

$$\frac{v_i}{R_1} = -\frac{v_o}{R_2}$$

or

$$v_o = -\frac{R_2}{R_1} v_i.$$

This circuit is known as an op-amp multiplier and is widely used in control systems. Op-amps can also be used for integrating and differentiating signals.

Example 6.9: An Op-Amp Differentiator

Consider the op-amp circuit shown in Figure 6.34. Derive the differential equation relating the input voltage v_i and the output voltage v_o.

Solution

Note that the current drawn by the op-amp is very small. Applying Kirchhoff's current law to node 1 gives

$$i_1 = i_2,$$

$$C\frac{d}{dt}(v_i - v_1) = \frac{v_1 - v_o}{R}.$$

Because the input terminal marked with the plus sign is connected to the ground, the op-amp equation yields $v_1 = v_- \approx v_+ = 0$. Thus, the differential equation for the op-amp circuit in Figure 6.34 is

$$C\dot{v}_i = -\frac{v_o}{R}$$

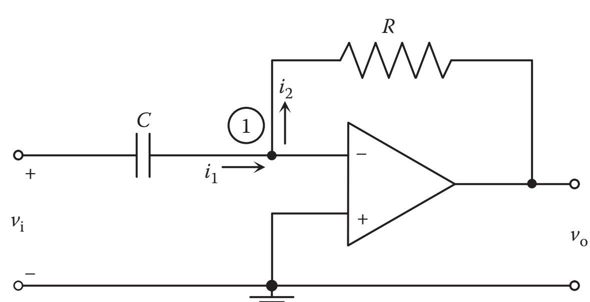


FIGURE 6.34 An op-amp differentiator circuit.

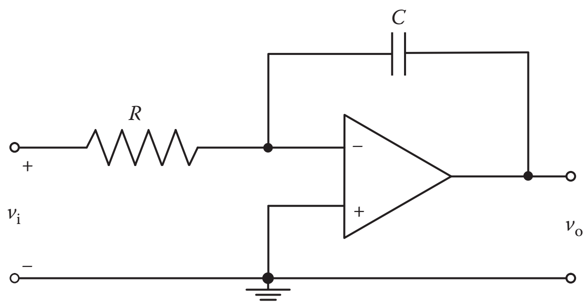


FIGURE 6.35 An op-amp integrator circuit.

or

$$v_o = -RC\dot{v}_i.$$

This implies that the output voltage v_o is proportional to the time derivative of the input voltage v_i. The circuit in Figure 6.34 is therefore called a differentiator. Switching the resistor and the capacitor in Figure 6.34 results in an op-amp integrator shown in Figure 6.35. The relation between the output voltage and the input voltage is $\dot{v}_o = -v_i/(RC)$. We leave the derivation as an exercise for the reader.

Example 6.10: An Op-Amp Circuit

Consider the op-amp circuit shown in Figure 6.36. Derive the differential equation relating the output voltage $v_o(t)$ and the input voltage $v_i(t)$.

Solution

Note that the current flowing into the input terminal of the op-amp is very small. Applying Kirchhoff's current law to node 1 yields

$$i_{R_1} + i_{C_1} - i_{R_2} - i_{C_2} = 0.$$

Using the voltage–current relations for electrical elements to express each term in the equation, we obtain

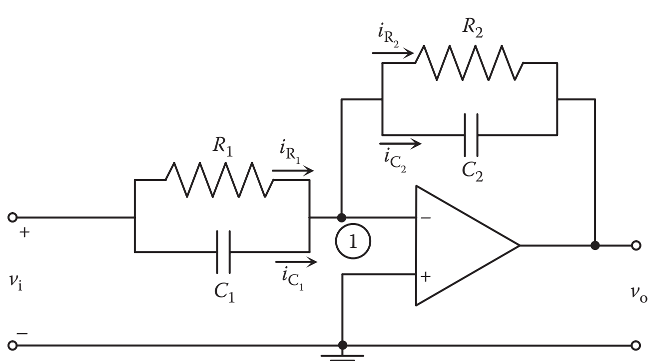


FIGURE 6.36 An op-amp circuit.

$$\frac{v_i - v_1}{R_1} + C_1 \frac{d}{dt}(v_i - v_1) - \frac{v_1 - v_o}{R_2} - C_2 \frac{d}{dt}(v_1 - v_o) = 0.$$

Because $v_+ = 0$, the op-amp equation yields $v_1 = v_- \approx v_+ = 0$. Substituting this into the previous equation results in

$$\frac{v_i}{R_1} + C_1 \dot{v}_i - \frac{-v_o}{R_2} - C_2(-\dot{v}_o) = 0,$$

which can be rearranged into

$$C_2 \dot{v}_o + \frac{1}{R_2} v_o = -C_1 \dot{v}_i - \frac{1}{R_1} v_i.$$

PROBLEM SET 6.3

1. The op-amp circuit shown in Figure 6.37 is a summing amplifier. Determine the relation between the input voltage v_i and the output voltage v_o.
2. The op-amp circuit shown in Figure 6.38 is a difference amplifier. Determine the relation between the input voltage v_i and the output voltage v_o.
3. The op-amp circuit shown in Figure 6.39 is a noninverting amplifier. Determine the relation between the input voltage v_i and the output voltage v_o.
4. Consider the op-amp integrator circuit shown in Figure 6.35. Derive the differential equation relating the input voltage v_i and the output voltage v_o.
5. Consider the op-amp circuit shown in Figure 6.40. Derive the differential equation relating the input voltage v_i and the output voltage v_o.
6. Repeat Problem 5 for the op-amp circuit shown in Figure 6.41.

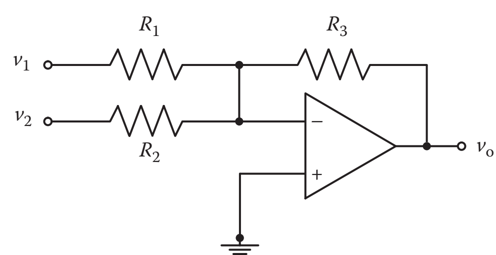


FIGURE 6.37 Problem 1.

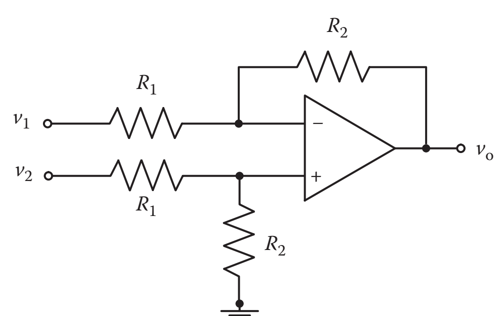


FIGURE 6.38 Problem 2.

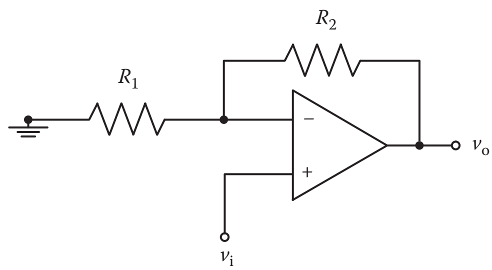


FIGURE 6.39 Problem 3.

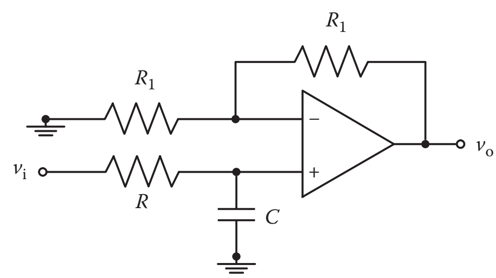


FIGURE 6.40 Problem 5.

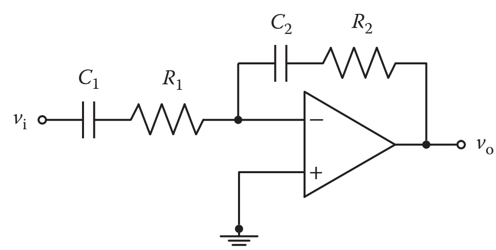


FIGURE 6.41 Problem 6.

6.4 ELECTROMECHANICAL SYSTEMS

Many useful devices, such as motors, generators, speakers, microphones, and accelerometers, are constructed by combining electrical elements and mechanical elements. For such electromechanical systems, we must apply electrical principles (e.g., Kirchhoff's laws) and mechanical principles (e.g., Newton's second law) to develop the dynamics model of the system. In this section, we discuss the modeling of direct current (DC) motors, which can generate forces or torques using electrical subsystems, and are essential actuators in control systems.

6.4.1 Elemental Relations of Electromechanical Systems

In a variety of electromechanical systems, electrical and mechanical subsystems are coupled by a magnetic field. Figure 6.42 shows a DC motor, which consists of basic elements (including the stator, the rotor, the armature, and the commutator). The stator provides a magnetic field across the rotor. The current is conducted to coils attached to the rotor via brushes, and the rotor is free to rotate. The combined unit of coils attached to the rotor is called the armature. The brushes are in contact with the rotating commutator, which causes the current to always be in the proper conductor windings so as to produce a torque and keep it in the proper direction. The magnetic coupling

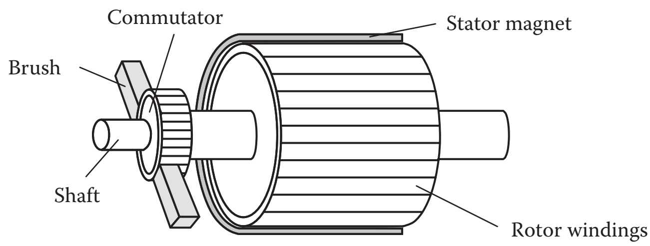


FIGURE 6.42 A DC brush motor.

relations between the electrical and mechanical subsystems in a DC motor can be derived using fundamental electromagnetic laws in introductory physics textbooks [4].

For simplicity, let us first consider a wire carrying a current within a magnetic field. Assume that the wire is either a straight conductor perpendicular to a uniform magnitude field or a circular conductor in a radial magnetic field. These are two common situations in many applications. Then a force will be exerted on the wire, and the relation between the force f and the current i is

$$f = BLi, \tag{6.25}$$

where B is the magnetic flux density in tesla (1 T = 1 Wb/m^2) and L is the length of the conductor in the magnetic field. The direction of the force can be determined using the right-hand rule as shown in Figure 6.43a. Curl four fingers from the positive current direction to the positive direction of the magnetic field, and the thumb will point to the positive direction of the force.

If the conductor moves relative to the magnetic field, then a voltage will be induced in the conductor. Figure 6.43b shows a straight conductor moving upward in a magnetic field. Assume that the direction of the motion is perpendicular to the direction of the magnetic field. Then the scalar relation between the induced voltage e_b and the velocity v of the conductor is

$$e_b = BLv. \tag{6.26}$$

To avoid confusion, we use e to denote the voltage instead of v as we did in the previous sections. Again, using the right-hand rule, curl four fingers from the positive direction of the velocity to the positive field direction, and the thumb will point to the positive direction of the induced voltage e_b. Note that the induced voltage e_b opposes the current, and is known as back electromotive force (emf, an old term for voltage).

For the DC motor shown in Figure 6.42, assume that the armature current is i and the number of armature coils is n. By Equation 6.25, the force generated on the armature due to the magnetic field is $f = nBLi$. If the radius of the armature is r, then the torque produced by the motor is

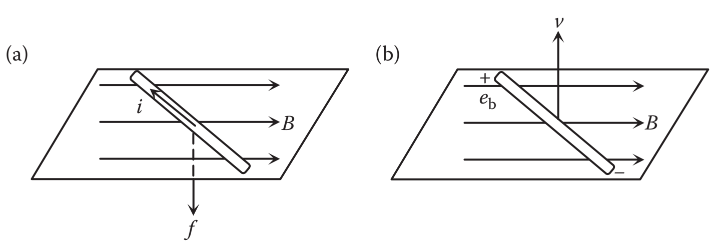


FIGURE 6.43 The direction of (a) the force on a conductor and (b) the voltage induced in a moving conductor.

$$\tau_m = fr = nBLir = K_t i. \tag{6.27}$$

where $K_t = nBLr$ is the torque constant of the motor. Note that the linear velocity of the coils is proportional to the angular velocity, $v = \omega r$. Then, by Equation 6.26, the back emf generated in the armature due to the rotating motion is

$$e_b = nBLv = nBL\omega r = K_e \omega, \tag{6.28}$$

where $K_e = nBLr$ is the back emf constant of the motor. The two constants, K_t and K_e, have the same expression and they will have the same numerical value if expressed in the same system of units. Equations 6.27 and 6.28 are used to model the coupling between the electrical and mechanical subsystems in a DC motor. Two primary types of DC motors, armature-controlled DC motors and field-controlled DC motors, are discussed next.

6.4.2 Armature-Controlled Motors

Figure 6.44 shows an electromechanical system with an armature-controlled DC motor. The electrical system is represented by an armature circuit, in which v_a is applied armature voltage, R_a is armature resistance, L_a is armature inductance, and e_b is back emf generated in the armature. The mechanical part is represented by a rotational system, in which I is the mass moment of inertia due to the rotor and the load, B is the viscous rotational damping associated with the load, τ_m is the torque produced by the motor, and τ_L is an additional torque applied to the load. In general, the load torque acts in the direction opposite to the motor torque. The differential equations of the system can be derived by using Kirchhoff's voltage law, the moment equation, and the electromechanical coupling relations.

For the electrical circuit, applying Kirchhoff's voltage law gives

$$\sum_j v_j = 0,$$

$$R_a i_a + L_a \frac{di_a}{dt} + e_b - v_a = 0. \tag{6.29}$$

For the mechanical part, applying the moment equation gives

$$+\curvearrowright: \quad \sum M_C = I_C \alpha,$$

$$\tau_m - \tau_L - B\dot{\theta} = I\ddot{\theta}. \tag{6.30}$$

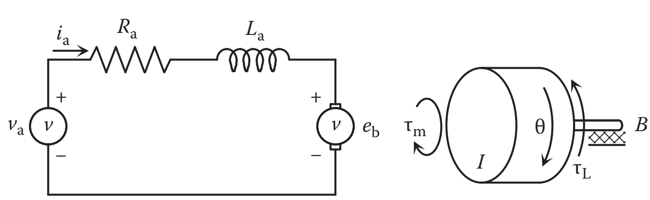


FIGURE 6.44 An electromechanical system with an armature-controlled DC motor.

Substituting the coupling relations between the electrical and mechanical subsystems, $e_b = K_e\omega = K_e\dot{\theta}$ and $\tau_m = K_t i_a$, into Equations 6.29 and 6.30, we obtain

$$L_a \frac{di_a}{dt} + R_a i_a + K_e\dot{\theta} = v_a, \tag{6.31}$$

$$I\ddot{\theta} + B\dot{\theta} - K_t i_a = -\tau_L. \tag{6.32}$$

Note that the stiffness terms associated with the variable θ do not appear in Equations 6.31 and 6.32. Thus, the system dynamics can also be expressed in terms of ω instead of θ, as

$$L_a \frac{di_a}{dt} + R_a i_a + K_e\omega = v_a, \tag{6.33}$$

$$I\dot{\omega} + B\omega - K_t i_a = -\tau_L. \tag{6.34}$$

Assume that all the initial conditions are set to zero. Taking the Laplace transform of Equations 6.33 and 6.34 results in

$$L_a sI_a(s) + R_a I_a(s) = V_a(s) - K_e\Omega(s), \tag{6.35}$$

$$Is\Omega(s) + B\Omega(s) = -T_L(s) + K_t I_a(s). \tag{6.36}$$

Figure 6.45 shows a block diagram of the above system. Using the basic rules for block diagram operation and reduction (see Section 4.5), we obtain the transfer function relating the armature voltage $v_a(t)$ and angular velocity $\omega(t)$, with $\tau_L(t) = 0$,

$$\frac{\Omega(s)}{V_a(s)} = \frac{(1/(L_a s + R_a))\cdot K_t\cdot(1/(Is+B))}{1+(1/(L_a s + R_a))\cdot K_t\cdot(1/(Is+B))\cdot K_e} = \frac{K_t}{L_a Is^2 + (L_a B + R_a I)s + R_a B + K_t K_e}. \tag{6.37}$$

The transfer function relating the load torque $\tau_L(t)$ and $\omega(t)$, with $v_a(t) = 0$, is

$$\frac{\Omega(s)}{T_L(s)} = \frac{-(1/(Is+B))}{1-(1/(Is+B))\cdot(-K_e)\cdot(1/(L_a s + R_a))\cdot K_t} = -\frac{L_a s + R_a}{L_a Is^2 + (L_a B + R_a I)s + R_a B + K_t K_e}. \tag{6.38}$$

Note that the two transfer functions have the same denominator, which is characteristic of the system. The order of the characteristic polynomial implies that the system is second-order. The

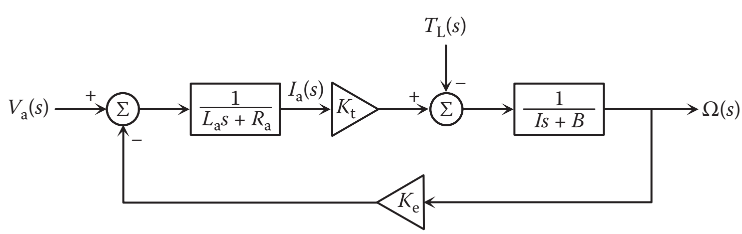


FIGURE 6.45 Block diagram of an armature-controlled DC motor.

transfer functions $\Omega(s)/V_a(s)$ and $\Omega(s)/T_L(s)$ can also be obtained by solving Equations 6.35 and 6.36 using Cramer's rule, and this is left as an exercise for the reader.

The motor model can be represented in state-space form by choosing appropriate states. We choose the armature current i_a and the angular velocity ω as the states. As shown in Figure 6.44, the armature voltage and the load torque are the two inputs. Let $u_1 = v_a$ and $u_2 = \tau_L$. Solving for the time derivatives di_a/dt and $\dot{\omega}$ from Equations 6.33 and 6.34 yields

$$\dot{x}_1 = \frac{di_a}{dt} = -\frac{R_a}{L_a} i_a - \frac{K_e}{L_a}\omega + \frac{1}{L_a} v_a, \tag{6.39}$$

$$\dot{x}_2 = \dot{\omega} = \frac{K_t}{I} i_a - \frac{B}{I}\omega - \frac{1}{I}\tau_L. \tag{6.40}$$

Thus, the state equation is

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_e}{L_a} \\ \frac{K_t}{I} & -\frac{B}{I} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & -\frac{1}{I} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}. \tag{6.41}$$

If we select ω as the output, then the output equation is

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}. \tag{6.42}$$

Example 6.11: A Single-Link Robot Arm Driven by a DC Motor

Consider the dynamic system shown in Figure 6.46, in which a single-link robot arm is driven by a DC motor. The differential equation of the robot arm in terms of the motor variable θ_m was determined in Example 5.17 to be

$$(I_m + N^2 I)\ddot{\theta}_m + (B_m + N^2 B)\dot{\theta}_m = \tau_m,$$

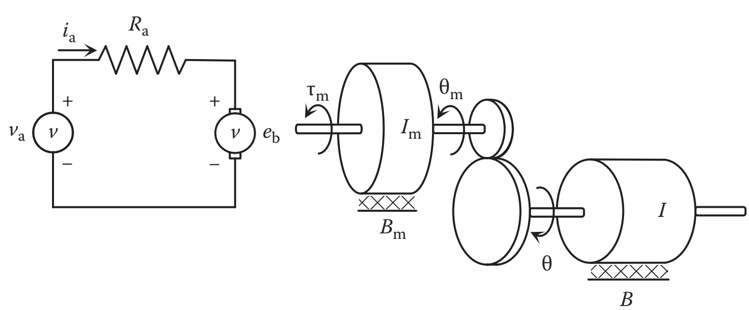


FIGURE 6.46 The model of a single-link robot arm driven by an armature-controlled DC motor.

where I_m and I are the mass moments of inertia of the motor and the load, respectively, B_m and B are the coefficients of the torsional viscous damping of the motor and the load, respectively, τ_m is the torque generated by the motor, and N is the gear ratio. Assume that the armature inductance is negligibly small, that is, $L_a \approx 0$. The torque and the back emf constants of the motor are K_t and K_e, respectively.

a. Derive the differential equation relating the applied voltage v_a and the link angular displacement θ.
b. Determine the transfer function $\Theta(s)/V_a(s)$ using the differential equation obtained in Part (a). Assume that all initial conditions are zero.

Solution

a. For the electrical circuit, applying Kirchhoff's voltage law gives

$$R_a i_a + e_b - v_a = 0,$$

where $e_b = K_e\dot{\theta}_m$. With the given gear ratio,

$$\theta = N\theta_m,$$

we have

$$R_a i_a = v_a - K_e \frac{1}{N}\dot{\theta}.$$

Thus, the current i_a can be expressed as

$$i_a = \frac{1}{R_a} v_a - \frac{K_e}{R_a}\frac{1}{N}\dot{\theta}.$$

The model of the mechanical part in terms of θ is given by

$$(I_m + N^2 I)\frac{1}{N}\ddot{\theta} + (B_m + N^2 B)\frac{1}{N}\dot{\theta} = \tau_m,$$

where $\tau_m = K_t i_a$. Substituting the expression of the current i_a, we obtain

$$(I_m + N^2 I)\frac{1}{N}\ddot{\theta} + (B_m + N^2 B)\frac{1}{N}\dot{\theta} = K_t\left(\frac{1}{R_a} v_a - \frac{K_e}{R_a}\frac{1}{N}\dot{\theta}\right).$$

Rearranging the equation yields

$$(I_m + N^2 I)\ddot{\theta} + \left(B_m + N^2 B + \frac{K_t K_e}{R_a}\right)\dot{\theta} = \frac{N K_t}{R_a} v_a.$$

b. Taking the Laplace transform results in

$$\frac{\Theta(s)}{V_a(s)} = \frac{N K_t / R_a}{(I_m + N^2 I)s^2 + (B_m + N^2 B + K_t K_e / R_a)s}.$$

This electromechanical system can be modeled using Simulink and Simscape. The details will be discussed in Section 6.6.

6.4.3 Field-Controlled Motors

In all but the smallest motors, the magnetic field is established by a current in separate field windings on the stator. For an armature-controlled DC motor, a constant current source is supplied to the field windings and the applied armature voltage, v_a, varies. Another way of controlling a DC motor is to keep the armature current i_a constant while varying the voltage applied to the field windings. A simple model of a field-controlled DC motor is shown in Figure 6.47, in which the shaft in the mechanical subsystem is assumed to be massless, rigid, and undamped. The electrical part is represented by a field circuit, where R_f is field resistance, L_f is field inductance, v_f is field voltage, and i_f is field current. Note that there is no back emf created in the field circuit. The torque generated by the motor is proportional to the field current,

$$\tau_m = K_t i_f. \tag{6.43}$$

The system under consideration has two inputs, v_f and τ_L. Two independent variables, i_f and θ, can be used to describe the system dynamics. For the electrical part, we apply Kirchhoff's voltage law to the field circuit,

$$L_f \frac{di_f}{dt} + R_f i_f = v_f. \tag{6.44}$$

For the mechanical part, introducing the motor equation (Equation 6.43) and applying the moment equation gives

$$I\ddot{\theta} + B\dot{\theta} - K_t i_f = -\tau_L \tag{6.45}$$

or

$$I\dot{\omega} + B\omega - K_t i_f = -\tau_L. \tag{6.46}$$

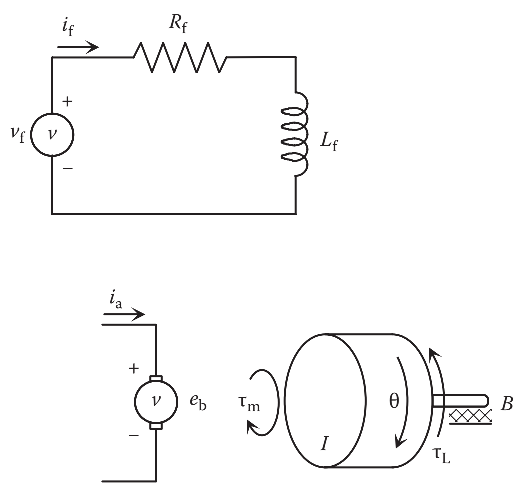


FIGURE 6.47 An electromechanical system with a field-controlled DC motor.

Equations 6.44 and 6.45, or Equations 6.44 and 6.46, are the system differential equations of the field-controlled DC motor.

Assuming zero initial conditions and taking the Laplace transform of Equations 6.44 and 6.46, we have

$$L_f sI_f(s) + R_f I_f(s) = V_f(s), \tag{6.47}$$

$$Is\Omega(s) + B\Omega(s) = K_t I_f(s) - T_L(s). \tag{6.48}$$

A block diagram of the system is shown in Figure 6.48. The transfer functions $\Omega(s)/V_f(s)$ and $\Omega(s)/T_L(s)$ can be easily derived from the diagram,

$$\frac{\Omega(s)}{V_f(s)} = \frac{K_t}{(L_f s + R_f)(Is + B)} = \frac{K_t}{L_f Is^2 + (L_f B + R_f I)s + R_f B}, \tag{6.49}$$

$$\frac{\Omega(s)}{T_L(s)} = -\frac{1}{Is + B}. \tag{6.50}$$

If we choose the field current and the angular velocity as the state variables ($x_1 = i_f$ and $x_2 = \omega$), the field voltage and the load torque as the inputs ($u_1 = v_f$ and $u_2 = \tau_L$), and the angular velocity as the output ($y = \omega$), then the state-space form is

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -\dfrac{R_f}{L_f} & 0 \\ \dfrac{K_t}{I} & -\dfrac{B}{I} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} \dfrac{1}{L_f} & 0 \\ 0 & -\dfrac{1}{I} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}. \tag{6.51}$$

PROBLEM SET 6.4

1. Reconsider the armature-controlled motor in Figure 6.44. Equations 6.31 and 6.32 represent the dynamics of the system in terms of the variables i_a and θ.
   a. Assuming the angle θ to be the output, draw a block diagram to represent the dynamics of the armature-controlled motor.
   b. Derive the transfer functions $\Theta(s)/V_a(s)$ and $\Theta(s)/T_L(s)$. All of the initial conditions are assumed to be zero.
   c. Determine the state-space form.
2. Reconsider the field-controlled motor in Figure 6.50. Equations 6.44 and 6.45 represent the dynamics of the system in terms of the variables i_f and θ.

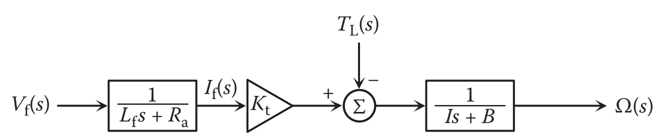


FIGURE 6.48 Block diagram of a field-controlled DC motor.

a. Assuming the angle θ to be the output, draw a block diagram to represent the dynamics of the field-controlled motor.
b. Derive the transfer functions $\Theta(s)/V_f(s)$ and $\Theta(s)/T_L(s)$. All of the initial conditions are assumed to be zero.
c. Determine the state-space form.

3. Consider the electromechanical system shown in Figure 6.49a. It consists of a cart of mass *m* moving without slipping on a ground track. The cart is equipped with an armature-controlled DC motor, which is coupled to a rack and pinion mechanism to convert the rotational motion to translation and to create the driving force *f* for the system. Figure 6.49b shows the equivalent electric circuit and the mechanical model of the DC motor, where *r* is the radius of the motor gear. The torque and the back emf constants of the motor are K_t and K_e, respectively.
   a. Derive the differential equation of the system relating the cart position *x* and the applied voltage v_a.
   b. Determine the transfer function $X(s)/V_a(s)$ using the differential equation obtained in Part (a). Assume that all initial conditions are zero.
4. Consider the single-link robot arm as shown in Figure 6.50a. It is driven by an armature-controlled DC motor through spur gears with a total gear ratio of *N*. The mass moments of inertia of the motor and the load are I_m and *I*, respectively. The coefficients of torsional viscous damping of the motor and the load are B_m and *B*, respectively. Figure 6.50b shows the equivalent electric circuit and the mechanical model of the DC motor. The torque and the back emf constants of the motor are K_t and K_e, respectively.
   a. Determine the transfer function $\Theta(s)/V_a(s)$. Assume that all initial conditions are zero.
   b. Determine the differential equation relating the applied voltage v_a and the link angular displacement θ.

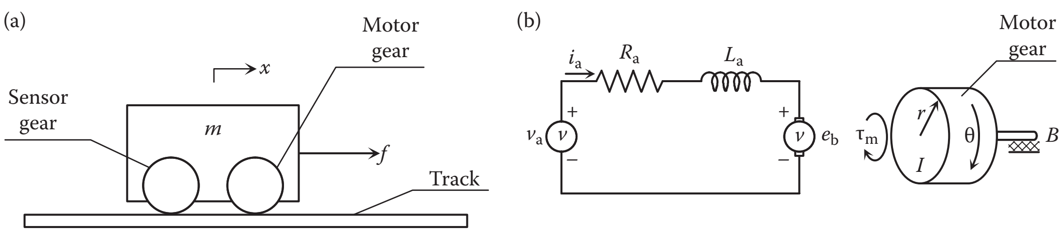


FIGURE 6.49 Problem 3. (a) a DC-motor driven cart, (b) the DC motor.

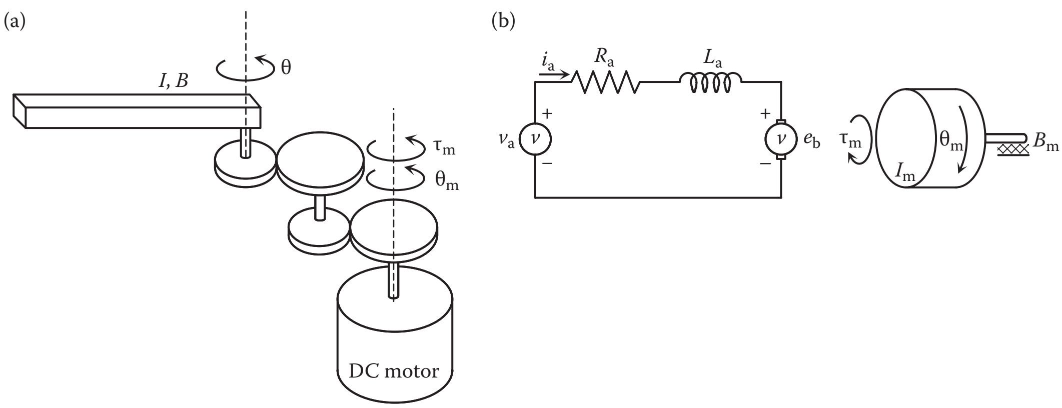


FIGURE 6.50 Problem 4. (a) a single-link robot arm, (b) the DC motor

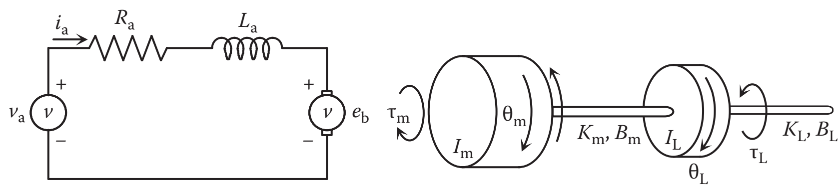


FIGURE 6.51 Problem 5.

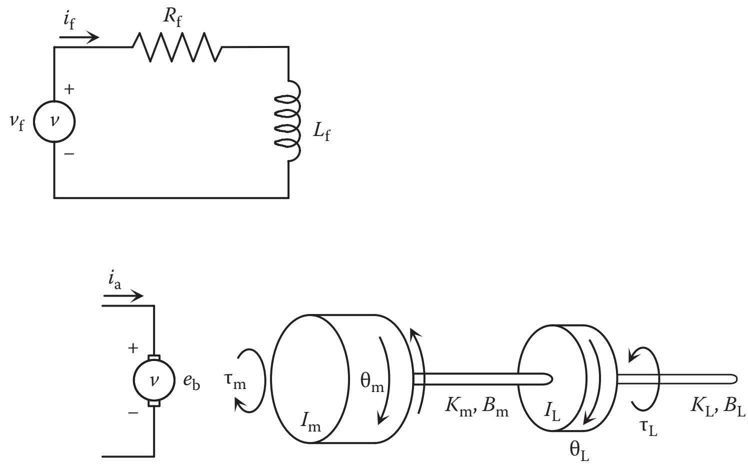


FIGURE 6.52 Problem 6.

5. A more complicated model of the armature-controlled motor is shown in Figure 6.51, in which the rotor is connected to an inertial load through a flexible and damped shaft. K_m and B_m represent the torsional stiffness and the torsional viscous damping of the shaft, respectively. The mass moments of inertia of the motor and the load are I_m and I_L, respectively. Let $\omega_m = \dot{\theta}_m$ and $\omega_L = \dot{\theta}_L$.
   a. Assuming zero initial conditions, derive the transfer functions $\Omega_L(s)/V_a(s)$ and $\Omega_L(s)/T_L(s)$.
   b. Assuming the angular velocity ω_L to be the output, draw a block diagram to represent the dynamics of the armature-controlled motor.
   c. Determine the state-space form.
6. A more complicated model of the field-controlled motor is shown in Figure 6.52, in which the rotor is connected to an inertial load through a flexible and damped shaft. K_m and B_m represent the torsional stiffness and the torsional viscous damping of the shaft, respectively. The mass moments of inertia of the motor and the load are I_m and I_L, respectively.
   a. Assuming zero initial conditions, derive the transfer functions $\Omega_L(s)/V_f(s)$ and $\Omega_L(s)/T_L(s)$.
   b. Assuming the angular velocity ω_L to be the output, draw a block diagram to represent the dynamics of the field-controlled motor.
   c. Determine the state-space form.

6.5 IMPEDANCE METHODS

The concept of impedance is very useful in electrical systems because it provides an alternative to transfer functions and differential equations for the derivation of system mathematical models.

6.5.1 Impedances of Electric Elements

Impedance is a generalization of the concept of resistance. Mathematically, electrical impedance is defined as the ratio of the voltage to the current in the s domain,

$$Z(s) = \frac{V(s)}{I(s)}. \tag{6.52}$$

For a resistor, we have $v = Ri$, hence the impedance is its resistance R,

$$Z(s) = R. \tag{6.53}$$

For an inductor, we have $v = L\mathrm{d}i/\mathrm{d}t$. Assuming zero initial conditions, this yields $V(s) = LsI(s)$. Thus, the impedance of an inductor is

$$Z(s) = Ls. \tag{6.54}$$

Similarly, for a capacitor, $i = C\mathrm{d}v/\mathrm{d}t$ yields $I(s) = CsV(s)$. Thus, the impedance of a capacitor is

$$Z(s) = \frac{1}{Cs}. \tag{6.55}$$

6.5.2 Series and Parallel Impedances

Because impedance can be viewed as a generalized resistance, it is easy to find the equivalent impedance for series-connected or parallel-connected electrical elements. Figure 6.53 shows n impedances in series. Note that the same current flows through n impedances and the total voltage drop across them is

$$V(s) = I(s)Z_1(s) + I(s)Z_2(s) + \cdots + I(s)Z_n(s). \tag{6.56}$$

In the equivalent diagram, the relation between the current and the voltage is

$$V(s) = I(s)Z_{\mathrm{eq}}(s). \tag{6.57}$$

Thus,

$$Z_{\mathrm{eq}}(s) = Z_1(s) + Z_2(s) + \cdots + Z_n(s). \tag{6.58}$$

That is, the equivalent impedance Z_{eq} is equal to the sum of all the individual impedances Z_i.

If there are n impedances in parallel as shown in Figure 6.54, then all the impedances have the same voltage drop. The total current through all the elements is

$$I(s) = \frac{V(s)}{Z_1(s)} + \frac{V(s)}{Z_2(s)} + \cdots + \frac{V(s)}{Z_n(s)}. \tag{6.59}$$

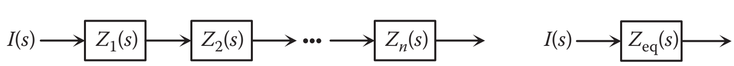


FIGURE 6.53 Equivalence for impedances in series.

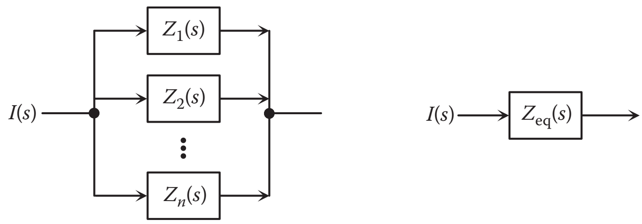


FIGURE 6.54 Equivalence for impedances in parallel.

For the equivalent impedance,

$$I(s) = \frac{V(s)}{Z_{eq}(s)}. \tag{6.60}$$

Therefore,

$$\frac{1}{Z_{eq}(s)} = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)} + \cdots + \frac{1}{Z_n(s)}. \tag{6.61}$$

That is, the reciprocal of the equivalent impedance Z_{eq} is equal to the sum of all the reciprocals of the individual impedances Z_i.

Note that impedance is essentially a transfer function, which has no integral or derivative signs. If we redraw an electrical system in the s domain by replacing passive elements with their corresponding impedances, we can determine the transfer function of the system using Kirchhoff's laws along with series and parallel laws. The differential equation of the system can then be obtained by converting the transfer function back from the s domain to the time-domain. Thus, the concept of impedance provides another way of modeling electrical systems without writing any time-domain equations.

Example 6.12: Impedance Method: An RLC Circuit

For the electric circuit in Example 6.4, use the impedance method to derive the differential equation relating the output voltage $v_o(t)$ to the input voltage $v_a(t)$. Assume zero initial conditions.

Solution

The original electric circuit is shown in Figure 6.55a. We can replace the passive elements with their impedance representations and redraw the circuit in the s domain as shown in Figure 6.55b. Note that the resistor R is in parallel connection with the capacitor C. The corresponding equivalent impedances are

$$Z_1(s) = Ls,$$

$$\frac{1}{Z_2(s)} = \frac{1}{R} + \frac{1}{1/Cs},$$

or

$$Z_2(s) = \frac{R}{RCs+1}.$$

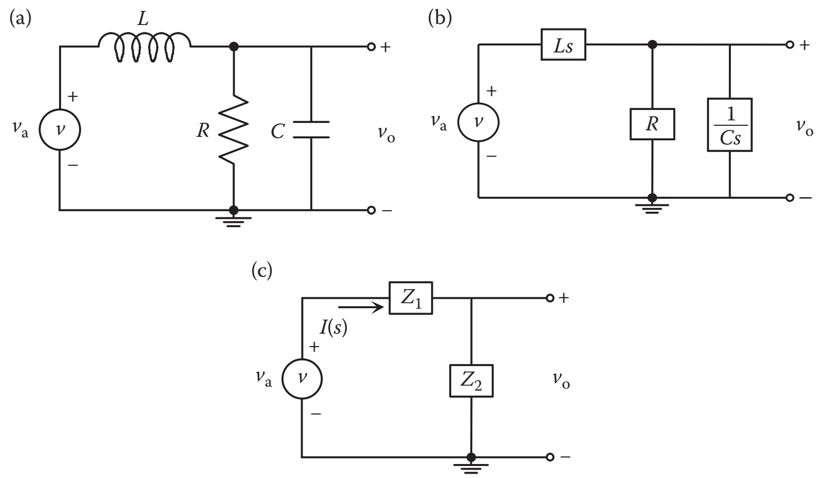


FIGURE 6.55 An RLC circuit drawn (a) in time-domain, (b) in s domain, and (c) using impedances.

For the equivalent impedance circuit in Figure 6.55c, we apply Kirchhoff's voltage law,

$$Z_1(s)I(s) + Z_2(s)I(s) - V_a(s) = 0,$$

where the current is related to the output voltage by

$$V_o(s) = Z_2(s)I(s).$$

Thus, we have

$$Z_1(s)\frac{V_o(s)}{Z_2(s)} + V_o(s) = V_a(s),$$

which gives the transfer function relating the input voltage v_a and the output voltage v_o,

$$\frac{V_o(s)}{V_a(s)} = \frac{Z_2(s)}{Z_1(s)+Z_2(s)} = \frac{R}{RLCs^2+Ls+R}.$$

By transforming $V_o(s)/V_a(s)$ from the s domain to the time-domain with the assumption of zero initial conditions, we obtain the differential equation of the system

$$RLC\ddot{v}_o + L\dot{v}_o + Rv_o = Rv_a,$$

which is the same as the one obtained in Example 6.4.

Example 6.13: Impedance Method: An Op-Amp Circuit

For the op-amp circuit in Example 6.10, use the impedance method to derive the differential equation relating the output voltage $v_o(t)$ to the input voltage $v_i(t)$. Assume zero initial conditions.

Solution

The original op-amp circuit is shown in Figure 6.56a. Replacing the passive elements with their impedance representations gives the equivalent op-amp circuit as shown in Figure 6.56b, where

$$\frac{1}{Z_1(s)} = \frac{1}{R_1} + \frac{1}{1/C_1 s},$$

$$\frac{1}{Z_2(s)} = \frac{1}{R_2} + \frac{1}{1/C_2 s},$$

or

$$Z_1(s) = \frac{R_1}{R_1 C_1 s + 1},$$

$$Z_2(s) = \frac{R_2}{R_2 C_2 s + 1}.$$

Because the current drawn by the op-amp is very small, applying Kirchhoff's current law to node 1 yields,

$$I_1(s) = I_2(s),$$

$$\frac{V_i(s) - V_1(s)}{Z_1(s)} = \frac{V_1(s) - V_o(s)}{Z_2(s)},$$

where the voltage at node 1 obeys

$$V_1(s) = V_-(s) = V_+(s) = 0.$$

Consequently, we have

$$\frac{V_i(s)}{Z_1(s)} = -\frac{V_o(s)}{Z_2(s)},$$

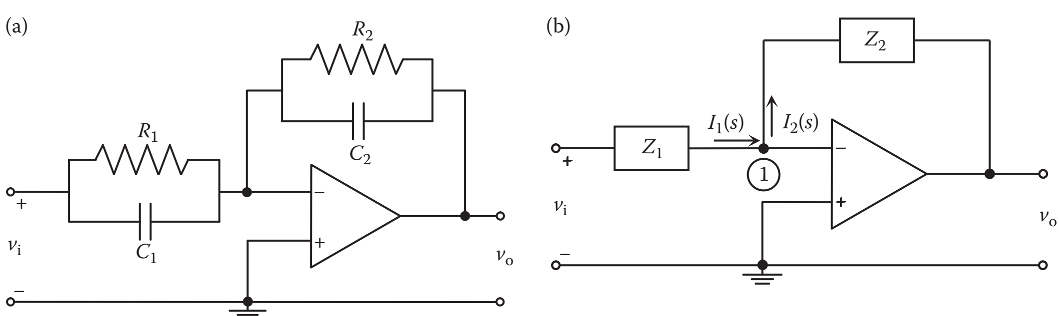


FIGURE 6.56 An op-amp circuit drawn (a) in time-domain and (b) using impedances.

which gives the transfer function relating the input voltage v_i and the output voltage v_o,

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_1R_2C_1s + R_2}{R_1R_2C_2s + R_1}.$$

By transforming $V_o(s)/V_i(s)$ from the s domain to the time-domain with the assumption of zero initial conditions, we obtain the differential equation of the system

$$R_1R_2C_2\dot{v}_o + R_1v_o = -R_1R_2C_1\dot{v}_i - R_2v_i,$$

which is the same as the one obtained in Example 6.10.

6.5.3 Mechanical Impedances

Analogous to electrical impedance, mechanical impedance is defined as

$$Z(s) = \frac{V(s)}{F(s)}, \tag{6.62}$$

where $V(s)$ and $F(s)$ are the Laplace transforms of velocity $v(t)$ and force $f(t)$, respectively. The impedance concept can also be used to obtain models of mechanical systems along with the linear graph, which is a topic beyond the scope of this text. Here, we only give the definitions of impedances for fundamental mechanical elements.

For a viscous damper, the damping force is related to the velocity by $f = bv$ or $F(s) = bV(s)$. Thus, the impedance of a damper is

$$Z(s) = \frac{1}{b}. \tag{6.63}$$

For a spring element, the spring force is proportional to the displacement, $f = kx = k\int v\,\mathrm{d}t$ or $F(s) = kV(s)/s$. Thus,

$$Z(s) = \frac{s}{k}. \tag{6.64}$$

For a mass element, by Newton's second law, $f = ma = m\dot{v}$ or $F(s) = msV(s)$. Thus, the impedance of a mass element is

$$Z(s) = \frac{1}{ms}. \tag{6.65}$$

When comparing the two sets of equations, Equations 6.53 through 6.55 and Equations 6.63 through 6.65, we note that the corresponding electrical and mechanical elements are not equivalent, although they have similar physical effects. For example, both the resistor and the damper dissipate energy. However, the mathematical expressions for their impedances are different.

PROBLEM SET 6.5

1. Reconsider the RC circuit shown in Figure 6.24. Use the impedance method to determine the transfer function $I(s)/V_a(s)$ and the input–output differential equation relating v_C and v_a. Assume that all the initial conditions are zero.

2. Reconsider the RL circuit shown in Figure 6.25. Use the impedance method to determine the transfer function $V_L(s)/V_a(s)$ and the input–output differential equation relating i_L and v_a. Assume zero initial conditions.
3. Reconsider the RLC circuit shown in Figure 6.26. Use the impedance method to determine the input–output differential equation relating v_o and v_a. Assume zero initial conditions.
4. Reconsider the RLC circuit shown in Figure 6.27. Use the impedance method to determine the input–output differential equation relating i and v_a. Assume that all the initial conditions are zero.
5. Reconsider the RLC circuit shown in Figure 6.28. Use the impedance method to determine the input–output differential equation relating v_o and v_a. Assume zero initial conditions.
6. Reconsider the RLC circuit shown in Figure 6.29. Use the impedance method to determine the input–output differential equation relating v_o and v_a. Assume zero initial conditions.
7. Reconsider the RLC circuit shown in Figure 6.30. Use the impedance method to determine the input–output differential equation relating v_o and v_a. Assume that all initial conditions are zero.
8. Reconsider the op-amp circuit shown in Figure 6.41. Use the impedance method to determine the differential equation relating the input voltage v_i and the output voltage v_o.

6.6 SYSTEM MODELING WITH SIMULINK AND SIMSCAPE

Electrical systems, or electrical circuits, can be modeled as systems with interconnected electrical elements, such as resistors, inductors, capacitors, op-amps, and others. These passive electrical elements are connected with active electrical elements, including current sources and voltage sources. The common output signals of electrical systems are currents and voltages. Similar to modeling of mechanical systems, the dynamics of electrical systems can be represented by ordinary differential equations, transfer functions, or the state-space form. Therefore, the Simulink modeling techniques discussed in Section 5.6 can also be applied to electrical systems.

This section focuses on physical modeling of electrical systems with Simscape. The blocks in the library of `Simscape\Foundation Library\Electrical` are categorized into three types, `Electrical Elements`, `Electrical Sources`, and `Electrical Sensors`. The Simscape library of `Electrical Elements` includes basic electrical building blocks, such as resistors, inductors, capacitors, op-amps, electromechanical convertors, switches, grounds, etc. The Simscape library of `Electrical sources` includes DC, AC, and controlled voltage and current sources. The Simscape library of `Electrical sensors` includes two types of sensors, current and voltage sensors. The examples in this section illustrate Simscape modeling of electric circuits, op-amp circuits, and DC motors.

6.6.1 ELECTRIC CIRCUITS

To correctly model an electric circuit using Simscape, it is very important to verify polarity and connection to the ground. First, the + and − signs seen on the ports of a block indicate how the current flows through that block. Second, each topologically distinct electric circuit must contain at least one `Electrical Reference` block, which represents a connection to the ground.

Example 6.14: A Series RLC Circuit

Consider the series RLC circuit shown in Figure 6.57, where $R = 1\ \Omega$, $L = 1$ H, and $C = 0.5$ F. When the switch is closed at 1 second, the circuit is driven by a 24 V DC voltage source. Assume that all initial conditions are zero.

a. Build a Simscape model of the physical system and find the loop current $i(t)$ and the voltage across the capacitor $v_C(t)$.

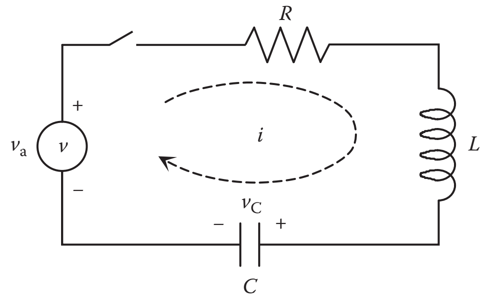


FIGURE 6.57 A series RLC circuit with a switch closed at 1 second.

b. Refer to the results obtained in Example 6.1. Build a Simulink model of the system based on the transfer function $I(s)/V_a(s)$ and find the loop current $i(t)$.
c. Refer to the results obtained in Example 6.1. Build a Simulink model of the system based on the transfer function $V_C(s)/V_a(s)$ and find the voltage across the capacitor $v_C(t)$.

Solution

a. The Simscape block diagram corresponding to the physical system is shown in Figure 6.58, which can be created by following these steps.
   1. Type `ssc_new` at the MATLAB Command window to open the main Simscape library and create a new model.
   2. Open the library of `Simscape/Foundation Library/Electrical/Electrical Elements` and drag the `Resistor`, `Inductor`, and `Capacitor` blocks into the model window. Double-click on these blocks to define the parameters `Resistance`, `Inductance`, and `Capacitance` as `1 Ohm`, `1 H`, and `0.5 F`. Also, drag the `Switch` and `Electrical Reference` blocks into the model window.
   3. To add a 24 V DC voltage source, open the library of `Simscape/Foundation Library/Electrical/Electrical Sources` and drag the `DC Voltage Source`

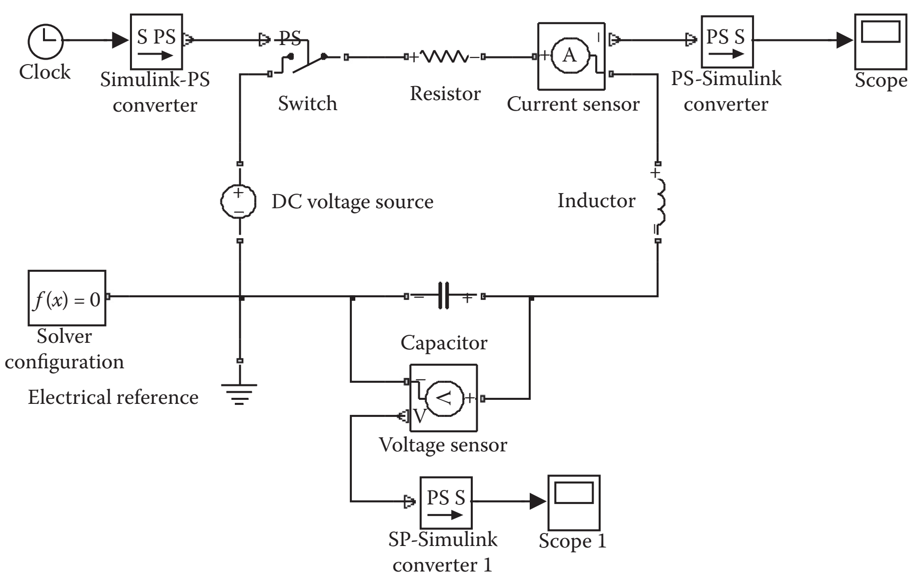


FIGURE 6.58 Simscape block diagram corresponding to Example 6.14.

block into the model window. Double-click on the block and define the parameter `Constant voltage` as `24 V`.

4. To measure the loop current and the voltage across the capacitor, open the library of `Simscape/Foundation Library/Electrical/Electrical Sensors` and drag both the `Current Sensor` and `Voltage Sensor` blocks into the model window. Each sensor has ports + and −, through which the sensor is connected to the circuit. The third port, `I` for the `Current Sensor` or `V` for the `Voltage Sensor`, is a physical signal port that outputs either current or voltage value.
5. Note that the Switch is closed at 1 second. To add the time source, open the library of `Simulink/Sources` and drag the `Clock` block, which is connected to the `Simulink-PS Converter` block and then the `Switch` block. Double-click on the `Switch` block and type `1` for the `Threshold`. This implies that the switch is closed if the time is greater than 1 second, otherwise the switch is open.
6. To display the loop current and the voltage across the capacitor, open the library of `Simulink/Sinks` and drag two `Scope` blocks, which are connected to the sensor blocks through the `PS-Simulink Converter` blocks. Note that the `Current Sensor` is connected in series with the circuit and the `Voltage Sensor` is connected in parallel with the `Capacitor` block.
7. Orient the blocks and connect them as shown in Figure 6.58.

   Set the simulation time to 15 seconds and run the model. The plots of the resulting loop current $i(t)$ and the voltage across the capacitor $v_C(t)$ are shown in Figures 6.59 and 6.60, respectively.

b. Refer to the results obtained in Example 6.1. The transfer function relating the input $v_a(t)$ to the output $i(t)$ is

$$\frac{I(s)}{V_a(s)} = \frac{Cs}{LCs^2 + RCs + 1}.$$

When the switch is closed at 1 second, the circuit is driven by a 24 V DC voltage source. Mathematically, this can be modeled using a `Step` block with the `Step time` set as `1` and the `Final value` set as `24`. The corresponding Simulink block diagram is shown in Figure 6.61, where a `Transfer Fcn` block is used to represent the series RLC

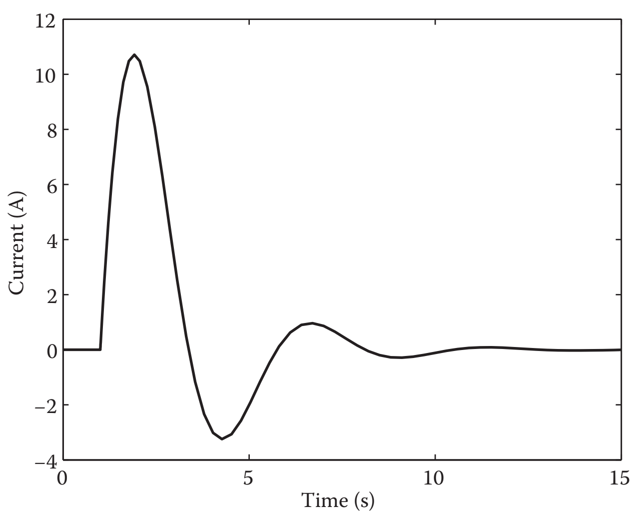


FIGURE 6.59 Loop current output $i(t)$ of the series RLC circuit in Example 6.14.

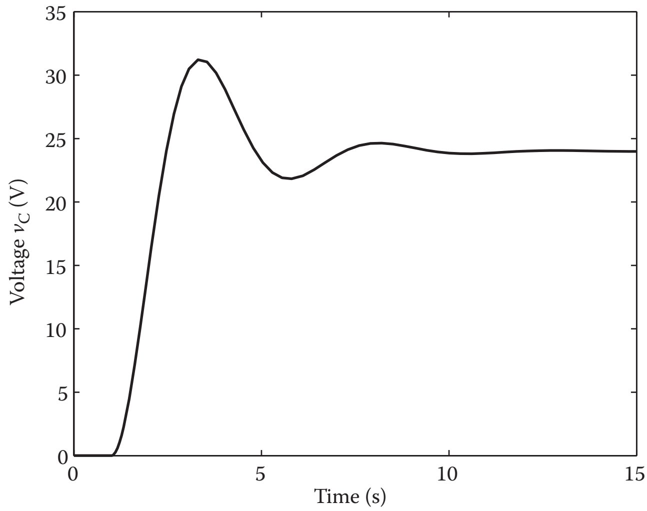


FIGURE 6.60 Voltage across the capacitor $v_C(t)$ in Example 6.14.

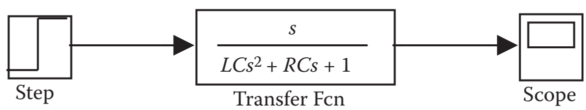


FIGURE 6.61 Simulink block diagram built based on the transfer function $I(s)/V_a(s)$.

circuit. Double-click on the block and type `[1 0]` for the `Numerator coefficient` and `[L*C R*C 1]` for the `Denominator coefficient` to define the transfer function $I(s)/V_a(s)$.

c. Similarly, the transfer function relating the input $v_a(t)$ to the output $v_C(t)$ is obtained as

$$\frac{V_C(s)}{V_a(s)} = \frac{1}{LCs^2 + RCs + 1}.$$

The corresponding Simulink block diagram is similar to the one shown in Figure 6.61, except that the `Numerator coefficient` of the `Transfer Fcn` block is `[1]`, instead of `[1 0]`. Running the Simulink models, we will obtain the same current output as shown in Figure 6.59 and the voltage output as shown in Figure 6.60.

Example 6.15: An RC High-Pass Filter

A passive, analog, first-order high-pass filter can be realized by an RC circuit (see Figure 6.62), which passes high-frequency signals but attenuates signals at low frequencies. Assume that the resistance is R = 100 Ω and the capacitance is C = 10 μF. The circuit is connected to an AC voltage source, which has an amplitude of 1 V and a frequency varying from 1 to 1000 Hz. Build a Simscape model of the physical system and find the output voltage $v_o(t)$ when the frequency of the input voltage is 1, 10, 100, and 1000 Hz.

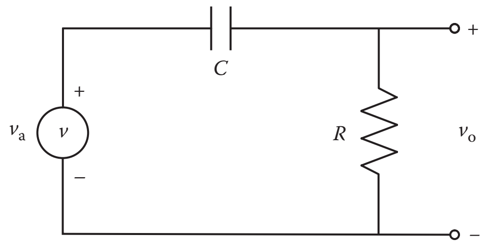


FIGURE 6.62 A passive, analog, first-order high-pass filter realized by an RC circuit.

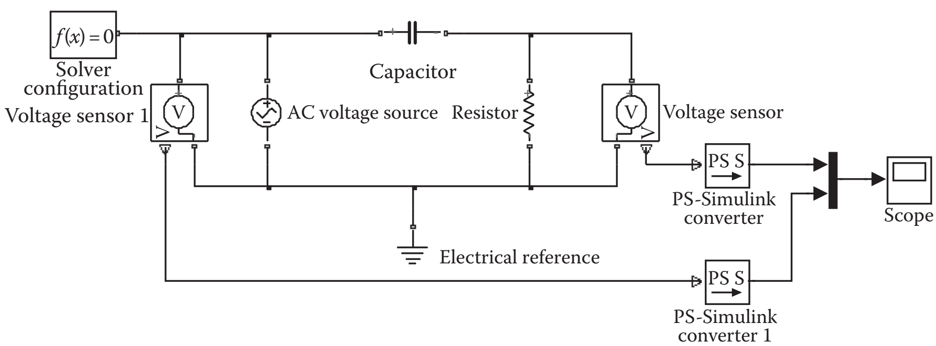


FIGURE 6.63 Simscape block diagram corresponding to Example 6.15.

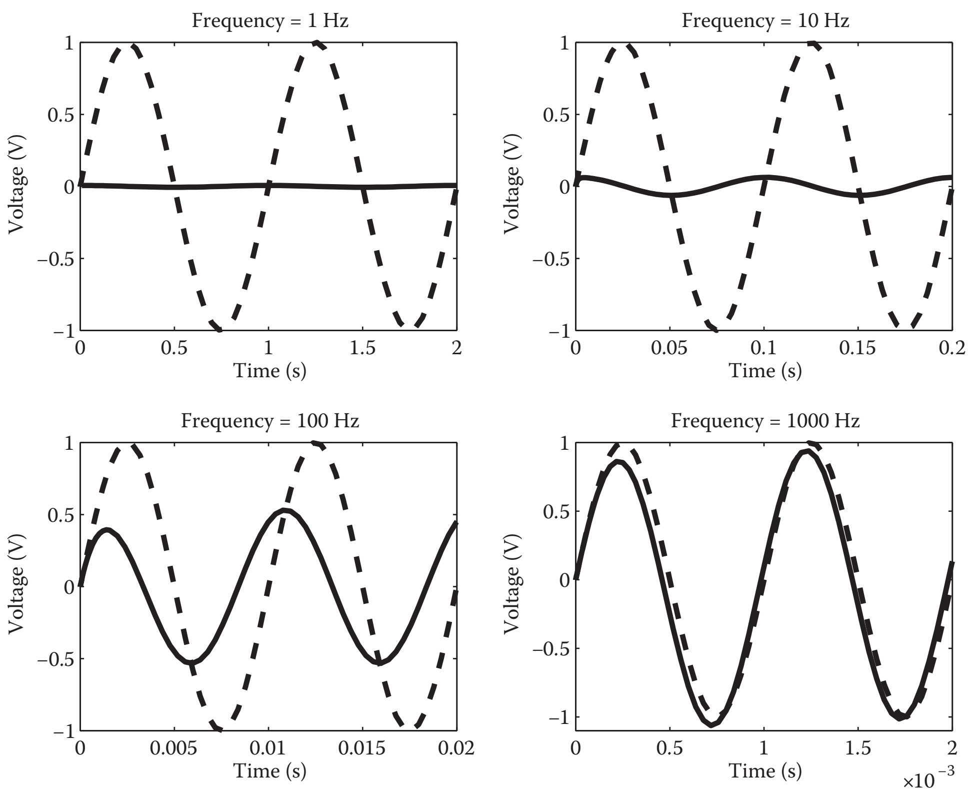


FIGURE 6.64 Comparison between the input and output voltages of the high-pass filter in Example 6.15.

Solution

The Simscape block diagram of the RC high-pass filter is shown in Figure 6.63, in which the `AC Voltage Source` block outputs a sinusoidal voltage. To better understand high-pass filtering, two `Voltage Sensor` blocks are included to measure the input and output voltages. They are displayed on the same scope through a `Mux` block, which can be found in the Simulink library of `Signal Routing`. Double-click on the `AC Voltage Source` block, type `1` for the `Peak amplitude`, and choose the unit as `V`. Vary the `Frequency` from `1`, `10`, `100`, to `1000 Hz` and run the simulations. The comparison between the input voltage and output voltage is given in Figure 6.64, in which solid lines are the output voltages and the dashed ones are the input voltages. It is obvious that the filter passes high-frequency signals but attenuates signals at low frequencies. The reader can derive the transfer function of the system and build a Simulink block diagram as an exercise.

6.6.2 Operational Amplifiers

The `Op-Amp` block in the Simscape library of `Electrical Elements` models the ideal behavior of an op-amp. The block has three electrical conserving ports. As discussed in Section 6.3, the voltage at the positive pin is equal to the voltage at the negative pin. In other words, the op-amp gain is assumed to be infinite. This implies that the current from the positive terminal to the negative terminal is zero. Building an op-amp circuit using the `Op-Amp` block combined with other electrical elements, such as resistors, capacitors, etc., is very straightforward.

Example 6.16: An Op-Amp Differentiator

Consider the op-amp differentiator in Example 6.9. Assume that the parameter values are $R = 1$ MΩ and $C = 1$ μF. Build a Simscape model of the op-amp circuit and find the output voltage $v_o(t)$ when the input voltage is $v_i = -0.1t$ V.

Solution

The Simscape block diagram of the op-amp differentiator is shown in Figure 6.65, in which a `Clock` block and a `Gain` block are used to generate the input voltage $v_i = -0.1t$ V. The Simulink

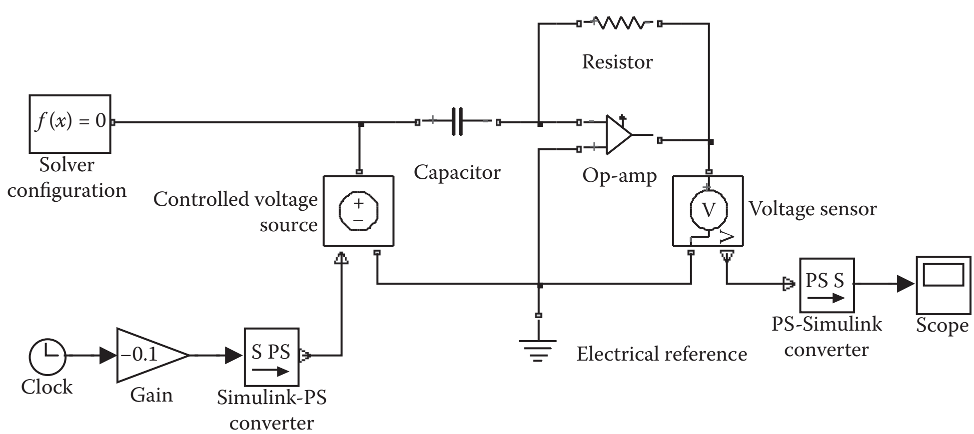


FIGURE 6.65 Simscape block diagram corresponding to Example 6.16.

input signal is converted into an equivalent voltage source using a `Controlled Voltage Source` block. According to the result in Example 6.9, the output voltage v_o is proportional to the time derivative of the input voltage v_i via $v_o = -RC\dot{v}_i$. Substitute the parameter values, and a constant value of 0.1 is expected to be the output voltage. The problem of building a Simulink block diagram based on the differential equation is left to the reader as an exercise.

6.6.3 DC Motors

As presented in Section 6.4, the mathematical model of a DC motor can be represented using a set of differential equations, a set of transfer functions, or in the state-space form. For each representation, a Simulink model can be built using the corresponding blocks, such as `Transfer Fcn` and `State-Space` blocks. To build a Simscape model of a DC motor, which is an electromechanical system, we need to use the blocks in both the `Simscape\Foundation Library\Electrical` and the `Simscape\Foundation Library\Mechanical` libraries. Two blocks, `Translational Electromechanical Converter` and `Rotational Electromechanical Converter`, in the Simscape library of `Electrical Elements` provide an interface between the electrical and mechanical translational or rotational domains. The following example illustrates how to model an armature-controlled DC motor with Simulink and Simscape.

Example 6.17: An Armature-Controlled DC Motor

Consider the dynamic system shown in Figure 6.66, which represents an armature-controlled DC motor. Assume that the armature inductance is negligibly small, that is, $L_a = 0$. The system dynamics can be expressed as

$$R_a i_a + K_e \dot{\theta}_m = v_a,$$

$$I_m \ddot{\theta}_m + B_m \dot{\theta}_m - K_t i_a = 0,$$

where the armature resistance is $R_a = 0.5\ \Omega$, the back emf constant is $K_e = 0.05$ V·s/rad, the torque constant is $K_t = 0.05$ N·m/A, the mass moment of inertia of the motor is $I_m = 0.00025$ kg·m^2, the coefficient of the torsional viscous damping of the motor is $B_m = 0.0001$ N·m·s/rad, and the applied voltage is $v_a = 10$ V.

a. Denote $\omega_m = \dot{\theta}_m$. Following Figure 6.45, build a Simulink block diagram using the given equations and find the armature current output $i_a(t)$ and the rotor speed $\omega_m(t)$.
b. Assume zero initial conditions, determine the transfer functions $I_a(s)/V_a(s)$ and $\Omega_m(s)/V_a(s)$. Build a Simulink block diagram using these two transfer functions and find the armature current output $i_a(t)$ and the rotor speed $\omega_m(t)$.

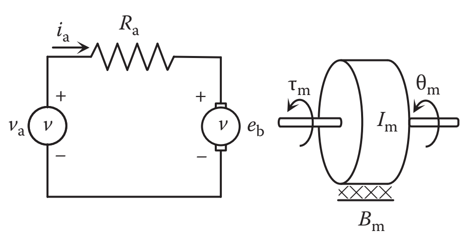


FIGURE 6.66 An armature-controlled DC motor with negligible inductance.

c. Choose θ_m and $\dot{\theta}_m$ as state variables and determine the state-space form of the system. Build a Simulink block diagram based on the state-space form and find the armature current output $i_a(t)$ and the rotor speed $\omega_m(t)$.
d. Build a Simscape model of the DC motor and find the armature current output $i_a(t)$ and the rotor speed $\omega_m(t)$.

Solution

a. This electromechanical system includes an armature circuit

$$R_a i_a = v_a - e_b,$$

a rotational system

$$I_m \dot{\omega}_m + B_m \omega_m = \tau_m,$$

and couplings between the electrical and mechanical subsystems

$$e_b = K_e \omega_m, \quad \tau_m = K_t i_a.$$

Following Figure 6.45, we can construct a Simulink block diagram (see Figure 6.67), which shows the major components of the DC motor and their interconnections. The dynamics of the mechanical rotational system is represented using a `Transfer Fcn` block. The armature resistance, torque constant, and back emf constant are represented using `Gain` blocks.

b. Taking Laplace transform of the two equations given and denoting $s\Theta_m(s) = \Omega_m(s)$ provides

$$R_a I_a(s) + K_e \Omega(s) = V_a(s),$$

$$I_m s \Omega_m(s) + B_m \Omega_m(s) - K_t I_a(s) = 0.$$

Using Cramer's rule to solve for the transfer functions $I_a(s)/V_a(s)$ and $\Omega_m(s)/V_a(s)$ yields

$$\frac{I_a(s)}{V_a(s)} = \frac{I_m s + B_m}{R_a I_m s + R_a B_m + K_t K_e},$$

$$\frac{\Omega_m(s)}{V_a(s)} = \frac{K_t}{R_a I_m s + R_a B_m + K_t K_e},$$

both of which can easily be represented using a `Transfer Fcn` block (see Figure 6.68).

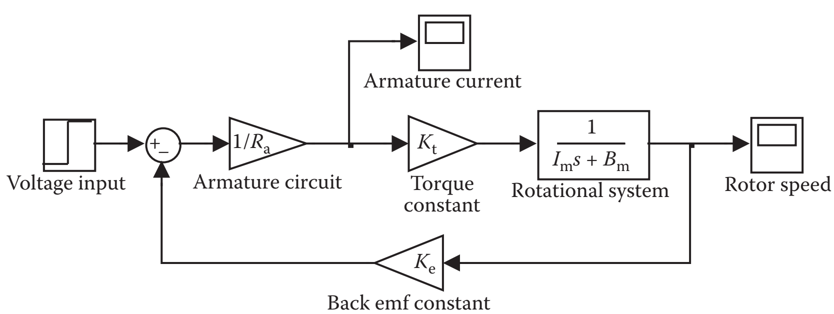


FIGURE 6.67 Simulink block diagram built based on the dynamics equations.

c. Let $x_1 = \theta_m$, $x_2 = \dot{\theta}_m$, and $u = v_a$, and the state-variable equations are

$$\dot{x}_1 = \dot{\theta}_m = x_2,$$

$$\dot{x}_2 = \ddot{\theta}_m = \frac{K_t}{I_m} i_a - \frac{B_m}{I_m}\dot{\theta}_m = \frac{K_t}{I_m}\left(\frac{v_a - K_e\dot{\theta}_m}{R_a}\right) - \frac{B_m}{I_m}\dot{\theta}_m$$
$$= -\left(\frac{K_t K_e}{I_m R_a} + \frac{B_m}{I_m}\right)x_2 + \frac{K_t}{I_m R_a}u,$$

or in matrix form

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\left(\dfrac{K_t K_e}{I_m R_a} + \dfrac{B_m}{I_m}\right) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ \dfrac{K_t}{I_m R_a} \end{bmatrix} u.$$

The output equations are

$$y_1 = i_a = \frac{v_a - K_e\dot{\theta}_m}{R_a} = -\frac{K_e}{R_a}x_2 + \frac{1}{R_a}u,$$

$$y_2 = \dot{\theta}_m = x_2,$$

or in matrix form

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{bmatrix} 0 & -\dfrac{K_e}{R_a} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} \dfrac{1}{R_a} \\ 0 \end{bmatrix} u.$$

The system can be represented using a `State-Space` block (see Figure 6.69) with A, B, C, D matrices defined above. A `Demux` block from the Simulink library of `Signal Routing` is used to split the output vector signal into two scalar signals $i_a(t)$ and $\omega_m(t)$.

d. The Simscape block diagram of the DC motor (see Figure 6.70) consists of elements from two domains, electrical and mechanical rotational. Note that each domain requires at least one reference block. As shown in Figure 6.70, both `Electrical Reference` and `Mechanical Rotational Reference` blocks are attached to the appropriate circuit.

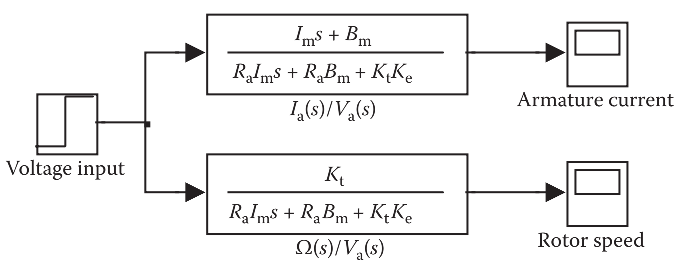


FIGURE 6.68 Simulink block diagram built based on the transfer functions.

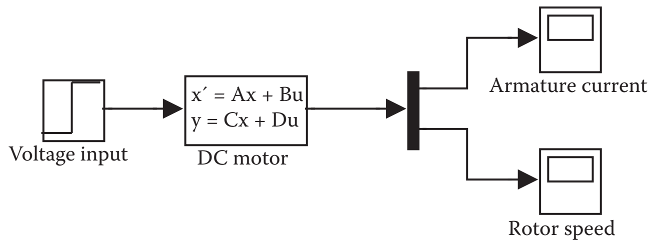


FIGURE 6.69 Simulink block diagram built based on the state-space form.

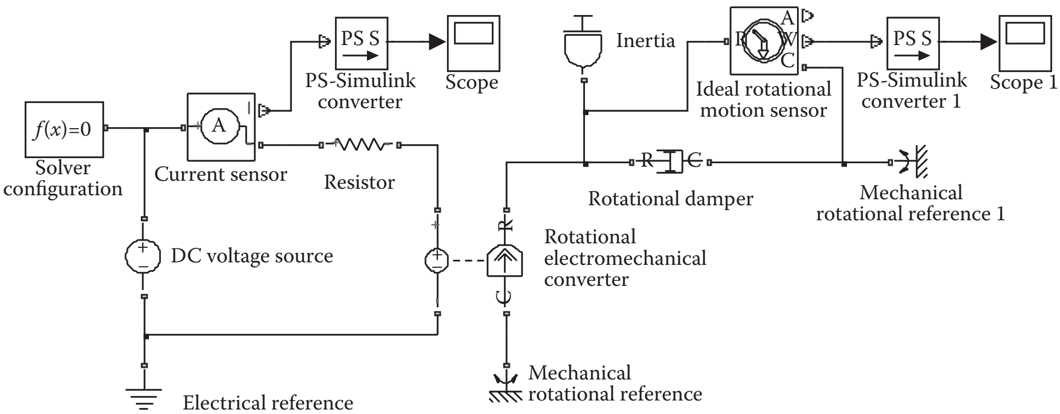


FIGURE 6.70 Simscape block diagram of the DC motor in Example 6.17.

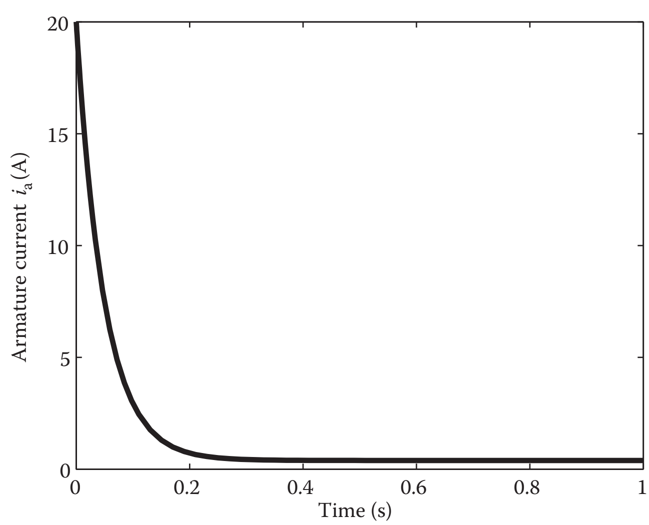


FIGURE 6.71 Armature current output $i_a(t)$ in Example 6.17.

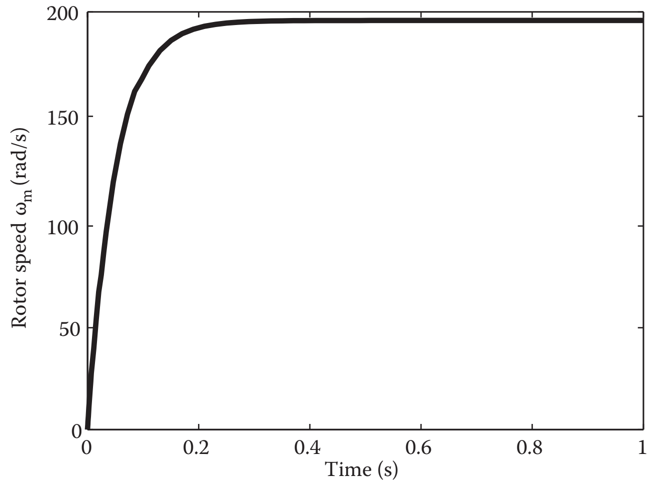


FIGURE 6.72 Rotor speed output $\omega_m(t)$ in Example 6.17.

Define all the parameters in MATLAB Workspace and run the Simulink or Simscape models. The result can be plotted as shown in Figures 6.71 and 6.72.

PROBLEM SET 6.6

1. Consider the RC circuit shown in Figure 6.24 (Problem Set 6.2, Problem 1), in which $R = 450\ \Omega$ and $C = 1000\ \mu F$. When the switch is closed at 1 second, the circuit is driven by a 5 V DC voltage source. Assume that all initial conditions are zero.
   a. Build a Simscape model of the physical system and find the loop current $i(t)$ and the voltage across the capacitor $v_C(t)$.
   b. Build a Simulink model of the system based on the differential equation relating v_C and v_a, and find the voltage across the capacitor $v_C(t)$.
   c. Build a Simulink model of the system based on the transfer function $I(s)/V_a(s)$, and find the loop current $i(t)$.
2. Consider the RL circuit shown in Figure 6.25 (Problem Set 6.2, Problem 2), in which $R = 35\ \Omega$ and $L = 10$ H. When the switch is closed at 0 seconds, the circuit is driven by a 6 V DC voltage source. Assume that all initial conditions are zero.
   a. Build a Simscape model of the physical system and find the loop current $i_L(t)$ and the voltage across the inductor $v_L(t)$.
   b. Build a Simulink model of the system based on the differential equation relating i_L and v_a, and find the loop current $i_L(t)$.
   c. Build a Simulink model of the system based on the transfer function $V_L(s)/V_a(s)$, and find the voltage across the inductor $v_L(t)$.
3. Consider the parallel RLC circuit shown in Example 6.2, in which $R = 2\ \Omega$, $L = 1$ H, and $C = 0.5$ F. The circuit is driven by a controlled current source $i_a(t) = 10u(t)$, where $u(t)$ is a unit-step function.
   a. Build a Simscape model of the physical system and find the voltage across the capacitor $v_C(t)$ and the current through the inductor $i_L(t)$.
   b. Refer to the results obtained in Example 6.2. Build a Simulink model of the system based on the transfer function $V_C(s)/I_a(s)$ and find the voltage across the capacitor $v_C(t)$.
   c. Refer to the results obtained in Example 6.2. Build a Simulink model of the system based on the transfer function $I_L(s)/I_a(s)$ and find the current through the inductor $i_L(t)$.

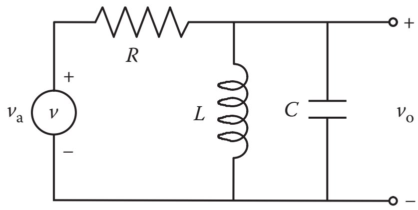


FIGURE 6.73 Problem 4.

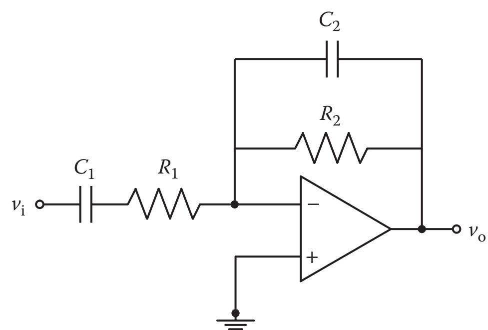


FIGURE 6.74 Problem 6.

4. A simple band-pass filter can be realized by an RLC circuit (see Figure 6.73), which passes frequencies within a certain range and attenuates frequencies outside that range. Assume that the parameter values are $R = 500\ \Omega$, $L = 100$ mH, and $C = 10\ \mu$F. The circuit is connected to an AC voltage source, which has an amplitude of 1 V and a varying frequency.
   a. Build a Simscape model of the physical system and find the output voltage $v_o(t)$ when the frequency of the input voltage is 1000, 800, and 1200 rad/s.
   b. Derive the transfer function $V_o(s)/V_a(s)$, build a Simulink model of the system based on this transfer function, and verify the results obtained in Part (a).
5. Consider the op-amp integrator in Figure 6.35. Assume that the parameter values are $R = 1$ MΩ and $C = 1\ \mu$F. Build a Simscape model of the op-amp circuit and find the output voltage $v_o(t)$ when the input voltage is $v_i = -0.1$ V.
6. Consider the op-amp circuit shown in Figure 6.74, in which the parameter values are $C_1 = 0.8\ \mu$F, $R_1 = 10$ kΩ, $C_2 = 80$ pF, and $R_2 = 100$ kΩ. The circuit is connected to an AC voltage source, which has an amplitude of 1 V and a frequency of 200 Hz.
   a. Build a Simscape model of the op-amp circuit and find the output voltage $v_o(t)$.
   b. Derive the transfer function $V_o(s)/V_i(s)$, build a Simulink model of the system based on this transfer function, and verify the results obtained in Part (a).

6.7 SUMMARY

This chapter was devoted to the modeling of electrical, electronic, and electromechanical systems. An electrical system or electrical circuit can be considered to be an interconnection of active and passive electrical elements. Active electrical elements include ideal current sources and ideal voltage sources, both of which can provide energy to the circuit and serve as the inputs. Passive electrical elements, including resistors, inductors, and capacitors, can either store or dissipate energy available in the circuit, but they cannot produce energy. The voltage–current relations for passive electrical elements are given as follows:

- Resistor: $v = Ri, \quad i = \dfrac{v}{R}$

- Inductor: $v = L\dfrac{\mathrm{d}i}{\mathrm{d}t}, \quad i = \dfrac{1}{L}\displaystyle\int v\,\mathrm{d}t$

- Capacitor: $v = \dfrac{1}{C}\displaystyle\int i\,\mathrm{d}t, \quad i = C\dfrac{\mathrm{d}v}{\mathrm{d}t}$

For the modeling of electrical, electronic, and electromechanical systems, Kirchhoff's voltage law and Kirchhoff's current law are the two main physical laws to derive the governing differential equations.

Kirchhoff's voltage law states that the algebraic sum of the voltages around a loop (closed path) must be zero,

$$\sum_j v_j = 0,$$

where v_j is the voltage across the jth element in the loop.

Kirchhoff's current law states that the sum of the currents entering a node must be equal to the sum of the currents leaving that node. If we assign a positive sign to the current entering the node and a negative sign to the current leaving the node, then the algebraic sum of the currents at the node must be zero,

$$\sum_j i_j = 0,$$

where i_j is the current of the jth element at the node.

It is usually not easy to obtain a set of differential equations for complicated circuits. For this purpose, two systematic methods, the node method that relies on Kirchhoff's current law and the loop method that relies on Kirchhoff's voltage law, were introduced in Section 6.2.

To represent a circuit model in state-space form, an appropriate set of state variables are normally chosen by identifying the energy storage elements. Both inductors and capacitors can store energy, and expressions for the stored electrical energy are given as follows:

- Inductor: $E(t) = \dfrac{1}{2}Li^2(t) - \dfrac{1}{2}Li^2(0)$

- Capacitor: $E(t) = \dfrac{1}{2}Cv^2(t) - \dfrac{1}{2}Cv^2(0)$

Generally, inductor currents and capacitor voltages are chosen as the state variables. To determine the state-space form of an electrical circuit, the expression of $\mathrm{d}i_\mathrm{L}/\mathrm{d}t$ or $\mathrm{d}v_\mathrm{C}/\mathrm{d}t$ for each inductor or capacitor is needed. Based on the voltage–current relations, we have $\mathrm{d}i_\mathrm{L}/\mathrm{d}t = 1/Lv_\mathrm{L}$ and $\mathrm{d}v_\mathrm{C}/\mathrm{d}t = 1/Ci_\mathrm{C}$. The problem is thus converted to expressing the inductor voltage v_L and the capacitor current i_C in terms of state variables and inputs using Kirchhoff's laws and voltage–current relations for electrical elements.

For an op-amp, which is an electronic element used to amplify electrical signals and drive physical devices, the differential equation relating the output voltage and the input voltage can be derived by applying Kirchhoff's laws and the op-amp equation

$$v_+ \approx v_-,$$

where v_+ and v_- are the voltages at the two input terminals of the op-amp.

For an electromechanical system, the dynamic model can be derived by applying electrical principles (e.g., Kirchhoff's laws) and mechanical principles (e.g., Newton's second law). The modeling of DC motors was discussed in Section 6.4. For an armature-controlled motor, the torque produced by the motor is

$$\tau_{\mathrm{m}} = K_{\mathrm{t}} i_{\mathrm{a}}$$

and the back emf generated in the armature due to the rotating motion is

$$e_{\mathrm{b}} = K_{\mathrm{e}}\omega = K_{\mathrm{e}}\dot{\theta}.$$

Armature-controlled motors are commonly used, in which a constant current source i_{f} is supplied to the field windings and the applied armature voltage v_{a} varies. Field-controlled motors are used in a different way, keeping the armature current i_{a} constant and letting the voltage v_{f} applied to the field windings vary. The torque generated by a field-controlled motor is proportional to the field current,

$$\tau_{\mathrm{m}} = K_{\mathrm{t}} i_{\mathrm{f}}.$$

The impedance concept provides an alternative to transfer functions and differential equations to obtain mathematical models of systems. The electrical impedance is defined as the ratio of the voltage to the current in the Laplace domain. The expressions of impedances for passive electrical elements are given as follows:

- Resistor: $Z(s) = R$
- Inductor: $Z(s) = Ls$
- Capacitor: $Z(s) = \dfrac{1}{Cs}$

Because the impedance can be viewed as a generalized resistance, it is easy to find the equivalent impedance for electrical elements connected in series or parallel and determine mathematical models of systems.

REVIEW PROBLEMS

1. Determine the equivalent resistance R_{eq} for the circuit shown in Figure 6.75.
2. Find R_{13} and R_{32} for the voltage divider shown in Figure 6.76 so that the current is limited to 0.5 A when $v_{\mathrm{i}} = 110$ V and $v_{\mathrm{o}} = 100$ V.
3. Consider the LC circuit shown in Figure 6.77. Derive the input–output differential equation relating v_{o} and v_{a} and find the order of this system.
4. Consider the second-order RC circuit shown in Figure 6.78. Assume that all the initial conditions are zero.
   a. Use the node or loop method to derive the input–output differential equation relating v_{o} and v_{a}, and find the transfer function $V_{\mathrm{o}}(s)/V_{\mathrm{a}}(s)$.
   b. Use the impedance method to determine the transfer function $V_{\mathrm{o}}(s)/V_{\mathrm{a}}(s)$, and compare with the result obtained in Part (a).

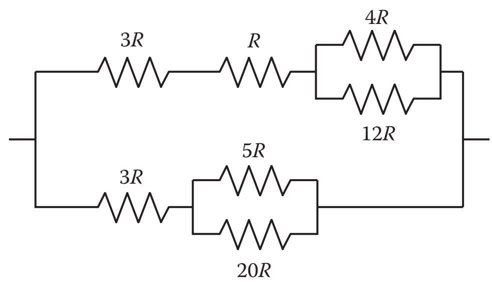


FIGURE 6.75 Problem 1.

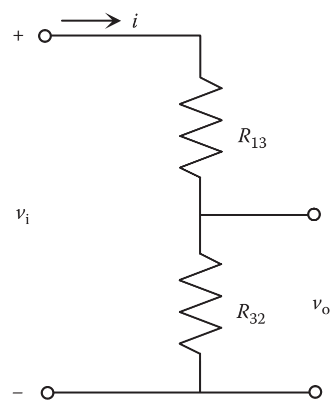


FIGURE 6.76 Problem 2.

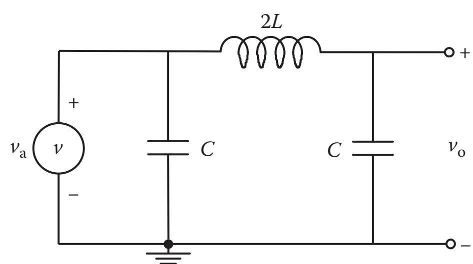


FIGURE 6.77 Problem 3.

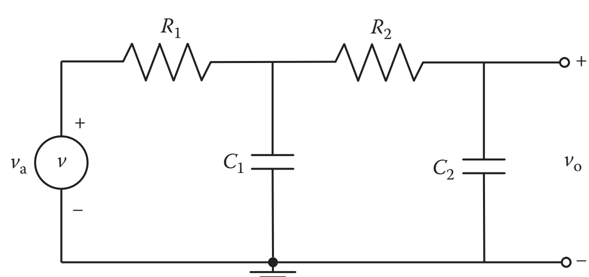


FIGURE 6.78 Problem 4.

5. Repeat Problem 4 for the RLC circuit shown in Figure 6.79. Assume zero initial conditions.
   a. Use the node or loop method to derive the input–output differential equation relating i and v_a, and find the transfer function $I(s)/V_a(s)$.
   b. Use the impedance method to determine the transfer function $I(s)/V_a(s)$, and compare with the result obtained in Part (a).
6. Consider the RLC circuit shown in Figure 6.80, and assume zero initial conditions.
   a. Use the node or loop method to derive the input–output differential equation relating v_o and v_a, and find the transfer function $V_o(s)/V_a(s)$.
   b. Use the impedance method to determine the transfer function $V_o(s)/V_a(s)$, and compare with the results obtained in Part (a).
7. Consider the circuit shown in Figure 6.77 (Review Problems, Problem 3).
   a. Determine a suitable set of state variables and obtain the state-space representation.
   b. Find the transfer function directly from the state-space form and compare with the result obtained in Problem 3.
8. Repeat Problem 7 for the circuit shown in Figure 6.78 (Review Problems, Problem 4).
9. Repeat Problem 7 for the circuit shown in Figure 6.79 (Review Problems, Problem 5).
10. Repeat Problem 7 for the circuit shown in Figure 6.80 (Review Problems, Problem 6).
11. The op-amp circuit shown in Figure 6.81 is an active low-pass filter. Derive the input–output differential equation relating the output voltage $v_o(t)$ and the input voltage $v_i(t)$. Assuming zero initial conditions, find the transfer function $V_o(s)/V_i(s)$ directly from the input–output equation.
12. Repeat Problem 11 for the op-amp circuit shown in Figure 6.82, which represents an active band-pass filter.

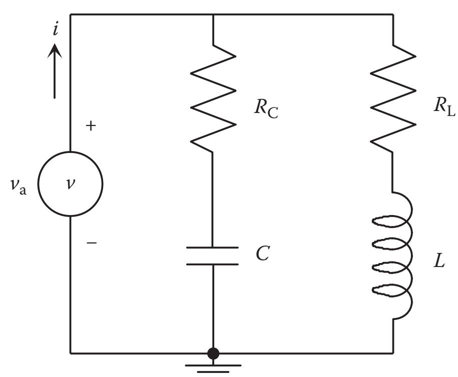


FIGURE 6.79 Problem 5.

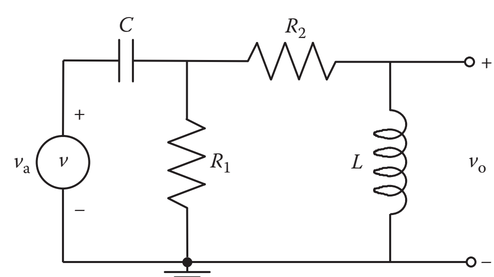


FIGURE 6.80 Problem 6.

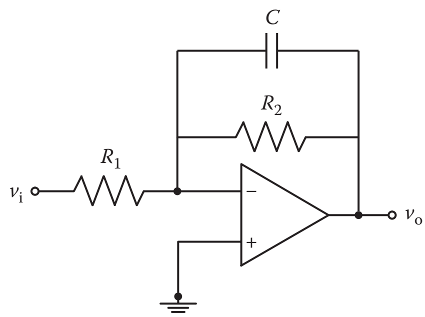


FIGURE 6.81 Problem 11.

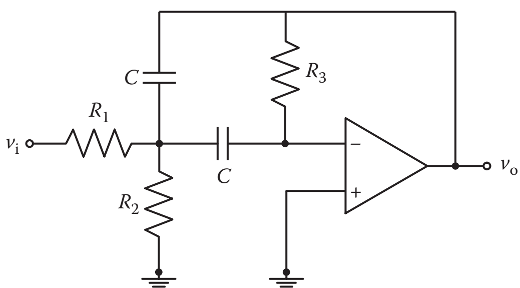


FIGURE 6.82 Problem 12.

13. Consider the RLC circuit shown in Figure 6.30 (Problem Set 6.2, Problem 15), where $R_1 = 100\ \Omega$, $L = 20$ H, $R_2 = 400\ \Omega$, and $C = 1/120$ F. The circuit is driven by a 100V DC voltage source.
    a. Build a Simscape model of the physical system and find the output voltage $v_o(t)$.
    b. Build a Simulink model of the system based on the state-space form and find the output voltage $v_o(t)$.
14. Consider the DC motor–driven wheeled mobile robot shown in Figure 6.83a, in which m is the mass of the wheeled mobile robot, r is the radius of the driving wheel, and τ is the

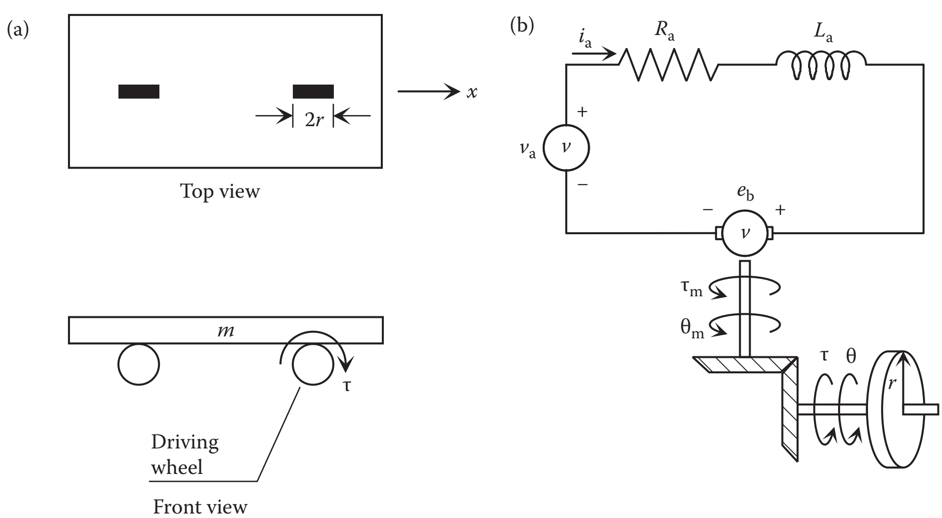


FIGURE 6.83 Problem 14. (a) DC-motor driven wheeled mobile robot, (b) simplified drive system.

torque delivered to the wheeled mobile robot by the DC motor. For simplicity, the motion is restricted to one spatial dimension. Figure 6.83b shows the simplified drive system, including the equivalent electrical circuit of the DC motor, the rotor of the DC motor, the gears, and the driving wheel. The motor parameter values are armature inductance $L_a = 0.001$ H, resistance $R_a = 2.6\ \Omega$, back emf constant $K_e = 0.008$ V·s/rad, and torque constant $K_t = 0.008$ N·m/A. The mass moment of inertia of the motor can be negligible. The gear ratio is $N = \theta/\theta_m = \tau_m/\tau = 1/3.7$. The wheel and axle mechanism converts the rotational motion to translation, and the wheel radius is $r = 0.00635$ m. The mass of the cart is $m = 0.455$ kg.

a. Derive the equations of motion of the system.
b. Choose the armature current i_a, the robot displacement x, and the robot velocity $\dot{x}$ as state variables and find the state-space form of the system.
c. Assuming zero initial conditions, find the transfer function $X(s)/V_a(s)$.
d. Following Figure 6.45, build a Simulink block diagram using the differential equations obtained in Part (a) and find the displacement output $x(t)$ when the voltage applied to the DC motor is a pulse function, $v_a(t) = 1$ V for $1 \le t \le 2$ s.
e. Build a Simscape model of the wheeled mobile robot and find the displacement output $x(t)$ when the voltage applied to the DC motor is a pulse function, $v_a(t) = 1$ V for $1 \le t \le 2$ s.

7 Fluid and Thermal Systems

Fluid is a general term used to represent a gas or a liquid. A fluid is said to be incompressible if its density does not change with pressure. All gases are considered compressible, whereas liquids can be considered incompressible. Although real liquids are actually compressible, the changes in their densities are insignificant when pressure is varied. Fluid systems can be divided into pneumatics and hydraulics. A pneumatic system is one in which the fluid is compressible. A hydraulic system is one in which the fluid is incompressible. A general type of incompressible liquid systems is liquid-level systems, which are operated by adjusting the heights or levels of liquids in storage tanks.

A thermal system is one in which thermal energy is stored or transferred. The mathematical model of a thermal system is often complicated because of the complex temperature distribution throughout the system. Partial, rather than ordinary, differential equations are required for precisely analyzing such a distributed-parameter system. This topic is beyond the scope of this text. To simplify analysis, a lumped-parameter model, rather than a distributed-parameter model governed by ordinary differential equations, may be used to approximate the dynamics of the system.

The modeling of fluid and thermal systems is presented in this chapter. The three major systems that are discussed include pneumatic, liquid-level, and heat-transfer systems. For each, we first introduce the concepts of capacitance and resistance. It is useful to think of fluid and thermal systems as electrical circuits. Along with the basic elements, two main laws, the conservation of mass and the conservation of energy, are then used to develop mathematical models of fluid and thermal systems, respectively. The chapter concludes with simulation of fluid and thermal systems using MATLAB®, Simulink®, and Simscape™ computer tools.

7.1 PNEUMATIC SYSTEMS

Pneumatic systems are often used in industry, particularly for pneumatic switches, pneumatic actuators, compressed-air engines, air brakes on buses and trucks, and so on. The working medium in a pneumatic system is compressible gas, typically air. To derive the mathematical model of a pneumatic system, it is important to understand the thermodynamic properties of gases.

7.1.1 Ideal Gases

The state of an amount of gas is determined by its pressure, volume, and temperature. In other words, pressure, volume, temperature, and mass are functionally related for gases. The ideal gas law is the model that is most often used to describe this relation. An ideal gas is a hypothetical gas whose quantity pV/T is constant, where p is the absolute pressure of the gas, V is the volume, and T is the absolute temperature (K or °R). All real gases behave as ideal gases if the pressure is low enough and the temperature is high enough. At low pressure and moderate temperature, real gases may be approximated as ideal gases to simplify calculations.

The ideal gas law states the relationship

$$pV = nR_{\mathrm{u}}T, \tag{7.1}$$

where n is the number of moles of the gas and R_{u} is the universal gas constant. The mole is the unit of the amount of substance. A mole of an element or a compound contains 6.023×10^{23} atoms

or molecules. The numerical value of the universal gas constant R_u is 8314.3 N·m/(kg·mol·K) or 1545.3 ft·lb/(lb·mol·°R).

The amount of substance can also be given in mass instead of moles. The number of moles n is equal to m/M, where m is the mass and M is the molar mass. Therefore, an alternative form of the ideal gas law is

$$pV = mR_g T, \tag{7.2}$$

where $R_g = R_u/M$ is the specific gas constant that depends on the particular type of gas. For dry air, R_g = 287.06 N·m/(kg·K) or 1716.6 ft·lb/(slug·°R). The ideal gas law can be used to solve for one of the four variables (p, V, T, or m) if the other three are known.

For a particular thermodynamic process from state 1 to state 2, the ideal gas equation can be simplified. Assume that the mass is constant. The following lists five possible processes, in which the state number is denoted by the subscript.

1. *Isobaric (or constant-pressure) process* ($p_1 = p_2$): The ideal gas law implies $T_1/V_1 = T_2/V_2$ or $T_1/T_2 = V_1/V_2$. When heat is added to the gas, some of it increases the temperature and some expands the volume to exert external work.
2. *Isochoric (or constant-volume) process* ($V_1 = V_2$): The ideal gas law implies $T_1/p_1 = T_2/p_2$ or $T_1/T_2 = p_1/p_2$. Because the volume is constant, there is no external work done. The heat added to the gas only increases the temperature.
3. *Isothermal (or constant-temperature) process* ($T_1 = T_2$): The ideal gas law implies $p_1V_1 = p_2V_2$ or $p_1/p_2 = V_2/V_1$. The heat added to the gas only does external work.
4. *Isentropic process (reversible adiabatic process)*: An adiabatic process is a process in which no heat is transferred to or from the gas. A reversible process is a process that, after it has taken place, can be reversed and causes no change in the thermodynamic conditions of either the system or its surroundings. Any reversible adiabatic process is an isentropic process. This process is described by $p_1V_1^\gamma = p_2V_2^\gamma$, where γ is defined as the heat capacity ratio.
5. *Polytropic process*: It is the most general thermodynamic process. The process is described by

$$p\left(\frac{V}{m}\right)^n = \frac{p}{\rho^n} = \text{constant}, \tag{7.3}$$

where ρ is the density of the gas. For an ideal gas with a constant mass, this process reduces to the previous four processes if n is chosen as 0, ∞, 1, or γ, respectively.

7.1.2 Pneumatic Capacitance

Fluid capacitance is the relation between the stored fluid mass and the resulting pressure caused by the stored mass. Specifically, the pneumatic capacitance C is defined as the ratio of the change in stored gas mass to the change in gas pressure:

$$C = \frac{\mathrm{d}m}{\mathrm{d}p}. \tag{7.4}$$

For a container of constant volume V with a gas of density ρ, Equation 7.4 can be rewritten as

$$C = \frac{\mathrm{d}(\rho V)}{\mathrm{d}p} = V\frac{\mathrm{d}\rho}{\mathrm{d}p}. \tag{7.5}$$

For a polytropic process, we have

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = n\rho^{n-1}\left(\frac{p}{\rho^n}\right) = \frac{np}{\rho}. \tag{7.6}$$

Introducing the ideal gas law presented in Equation 7.2 gives

$$\frac{p}{\rho} = \frac{pV}{m} = R_\mathrm{g}T. \tag{7.7}$$

Substituting this into Equation 7.6 yields

$$\frac{\mathrm{d}\rho}{\mathrm{d}p} = \frac{1}{nR_\mathrm{g}T}. \tag{7.8}$$

Thus, the capacitance of the container is

$$C = \frac{V}{nR_\mathrm{g}T}. \tag{7.9}$$

Example 7.1: Pneumatic Capacitance

Dry air passes through a valve into a rigid 27 m^3 container at a constant temperature of 25°C (298 K). The process is assumed to be isothermal. Determine the capacitance of the air container.

Solution

The filling of the container is modeled as an isothermal process. In Equation 7.9, let n = 1, and we have

$$C = \frac{V}{R_\mathrm{g}T} = \frac{27}{287.06\times 298} = 3.16\times 10^{-4}\,\mathrm{kg{\cdot}m^2/N}.$$

Note that the same container can have a different pneumatic capacitance. The value of C depends on the type of gas, the temperature of gas, and the type of thermodynamic process.

7.1.3 Modeling of Pneumatic Systems

It is rather difficult to precisely model pneumatic systems due to their highly nonlinear dynamics. To simplify the modeling, each mass storage element in a pneumatic system can be represented using a capacitance element, and the resistance due to a valve, an orifice, or pipe wall friction, can be represented using a resistance element. Such a simple model may be adequate to describe the dynamic behavior of the real system.

Consider a pneumatic system shown in Figure 7.1, where p_i is the inlet pressure, q_i is the volume flow rate (m^3/s), and p and ρ are the pressure and the density of the gas in a container of constant volume V. The gas passes through a valve and flows into the rigid container by the pressure difference, $\Delta p = p_\mathrm{i} - p$. Note that the gas meets resistance when flowing through the valve. The valve resistance depends on the pressure p and the mass flow rate q_m (kg/s). Generally, the p versus q_m curve is nonlinear. Thus, the value of the valve resistance R, which is defined as the slope of the curve, varies with operating conditions. The definition of the resistance R is valid for both pneumatic and hydraulic systems, and more details will be given in Section 7.2.

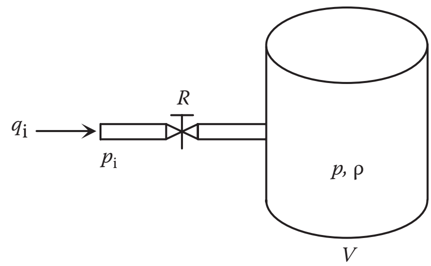


FIGURE 7.1 A pneumatic system with gas flowing into a container of constant volume.

Although the valve resistance R is nonlinear, it may be linearized about an operating point. For a constant pressure p_i at the inlet of the valve and a constant volume flow rate q_i through the valve, the resistance R is given as

$$R = \frac{\Delta p}{\Delta q_m} = \frac{p_i - p}{\rho_i q_i}. \tag{7.10}$$

Combining with the capacitance of the container given in Equation 7.9, the pneumatic system can be represented using a resistance–capacitance model.

The differential equation of the system can be derived by applying the law of conservation of mass:

$$\frac{dm}{dt} = q_{mi} - q_{mo}, \tag{7.11}$$

that is, the rate of mass increase in the container equals the mass flow rate into the container minus the mass flow rate out of the container. Note that

$$\frac{dm}{dt} = \frac{dm}{dp}\frac{dp}{dt} = C\frac{dp}{dt} \tag{7.12}$$

and

$$q_{mi} = \rho_i q_i = \frac{p_i - p}{R}. \tag{7.13}$$

For the pneumatic system in Figure 7.1, the mass flow rate out of the container is $q_{mo} = 0$. Thus, Equation 7.11 can be rewritten as

$$C\frac{dp}{dt} = \frac{p_i - p}{R} \tag{7.14}$$

or

$$RC\frac{dp}{dt} + p = p_i, \tag{7.15}$$

which is a first-order ordinary differential equation of the pressure p. Introducing the expression of the capacitance C given by Equation 7.9, we find

$$\frac{RV}{nR_gT}\frac{dp}{dt} + p = p_i. \tag{7.16}$$

Either one of Equations 7.15 or 7.16 is the mathematical model of the pneumatic system undergoing a polytropic process with an ideal gas. It is also valid for an isobaric, isochoric, isothermal, or isentropic process if the value of n in the capacitance C is chosen as 0, ∞, 1, or γ, respectively.

Example 7.2: A Pneumatic System

Dry air at a constant temperature of 20°C (293 K) passes through a valve into a rigid cubic container of 1 m on each side (see Figure 7.2). The pressure p_i at the inlet of the valve is constant, and it is greater than p. The valve resistance is approximately linear, and R = 1000 Pa·s/kg. Assume the filling process is isothermal. Develop a mathematical model of the pressure p in the container.

Solution

Applying the law of conservation of mass gives

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \rho_i q_i.$$

Note that

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}t} = C\frac{\mathrm{d}p}{\mathrm{d}t}.$$

Air at room temperature and low pressure can be approximated as an ideal gas. For an isothermal process, the pneumatic capacitance of the container is

$$C = \frac{V}{R_{air}T}.$$

The linear valve resistance R is defined as

$$R = \frac{p_i - p}{\rho_i q_i}.$$

Thus, the differential equation of the system is

$$\frac{V}{R_{air}T}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{p_i - p}{R}$$

or

$$\frac{RV}{R_{air}T}\frac{\mathrm{d}p}{\mathrm{d}t} + p = p_i$$

where $RV/(R_{air}T) = 1000 \times 1^3/(287.06 \times 293) = 1.19 \times 10^{-2}$ s.

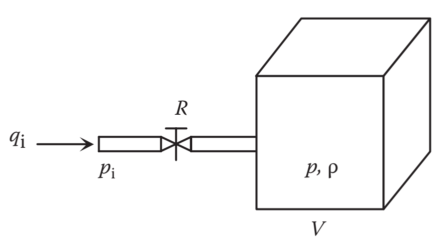


FIGURE 7.2 A pneumatic system with a rigid cubic container.

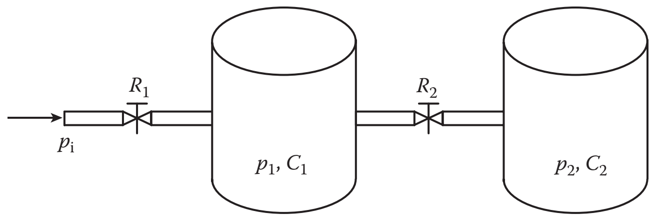


FIGURE 7.3 Problem 3.

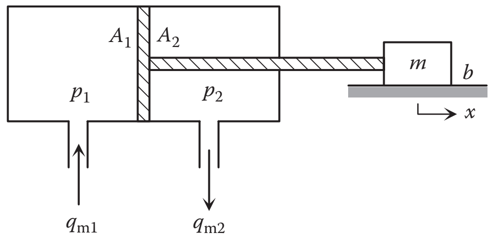


FIGURE 7.4 Problem 4.

PROBLEM SET 7.1

1. A car has an internal volume of 2.8 m^3. If the sun heats the car from a temperature of 20°C to a temperature of 40°C, what will the pressure inside the car be? Assume the pressure is initially 1 atm.
2. Find the pneumatic capacitance of dry air in a rigid container with volume 15 ft^3 for an isothermal process. Assume the air is initially at ambient temperature of 68°F.
3. Figure 7.3 shows a pneumatic system, in which the pneumatic capacitances of the two rigid containers are C_1 and C_2, respectively. Dry air at a constant temperature passes through a valve of linear resistance R_1 into container 1. The pressure p_i at the inlet of the valve is constant, and it is greater than p_1. The air flows from container 1 to container 2 through a valve of linear resistance R_2. Derive the differential equations in terms of the pressures p_1 and p_2. Write the equations in second-order matrix form.
4. Figure 7.4 shows a pneumatic piston, which can exert a force in one direction and serve as a translational actuator. The displacement of the piston is x, and m is the total mass of the piston and the load. Use subscript i to denote the chamber index and $i = 1, 2$. As shown in Figure 7.4, p_i is the absolute pressure in the chamber, q_{mi} is the mass flow rate at the port, and A_i is the effective area of the piston. The two chambers have the same initial volume of V_0. Assuming ideal gas and isothermal process, R is the specific gas constant and T is the absolute gas temperature. Derive the dynamic equations of the pressure change in the pneumatic chambers and the equation of motion of the mass block.

7.2 LIQUID-LEVEL SYSTEMS

Unlike gases, most liquids are generally considered incompressible, and this approximation greatly simplifies the modeling of hydraulic systems. A general category of hydraulic systems is liquid-level systems, which often appear in water treatment, water supply, and other chemical processing applications. Such a system usually consists of storage tanks interconnected to other systems through valves, pumps, or pistons.

The dynamic behavior of a liquid-level system can be described using volume flow rate q, pressure p, and liquid height h. Note that the hydrostatic pressure, rather than the dynamic pressure, will

be used in the modeling of liquid-level systems. The hydrostatic pressure is defined as the pressure that exists in a fluid at rest. It is caused by the weight of the fluid. For a liquid of density ρ, the absolute pressure p and the liquid height h are related by

$$p = p_a + \rho g h, \tag{7.17}$$

where p_a is the atmospheric pressure.

7.2.1 Hydraulic Capacitance

As introduced in Section 7.1, fluid capacitance is the ratio of the change in stored mass to the change in pressure. Because the density for an incompressible liquid is constant, the change in mass is equivalent to the change in volume. Some books define the capacitance for hydraulic systems in terms of volume instead of mass,

$$C_V = \frac{\mathrm{d}V}{\mathrm{d}p}, \tag{7.18}$$

which is related to Equation 7.4 via $C = \rho C_V$. The definition given in Equation 7.4 is used for both pneumatic and hydraulic systems in this text.

To find the expression of capacitance for a hydraulic system, let us consider a container, whose cross-sectional area varies with the liquid height. The mass stored in the container can be determined by integrating $\rho A(h)$ from the base of the container to the top of the liquid,

$$m = \int_0^h \rho A(y)\,\mathrm{d}y. \tag{7.19}$$

By Equation 7.4,

$$C = \frac{\mathrm{d}m}{\mathrm{d}p} = \frac{\mathrm{d}m}{\mathrm{d}h}\frac{\mathrm{d}h}{\mathrm{d}p}. \tag{7.20}$$

Note that Equation 7.19 implies that

$$\frac{\mathrm{d}m}{\mathrm{d}h} = \rho A(h). \tag{7.21}$$

Also, Equation 7.17 gives

$$\frac{\mathrm{d}p}{\mathrm{d}h} = \frac{\mathrm{d}}{\mathrm{d}h}(p_a + \rho g h) = \rho g. \tag{7.22}$$

Thus, the hydraulic capacitance of the container is

$$C = \rho A(h)\frac{1}{\rho g} = \frac{A(h)}{g}. \tag{7.23}$$

If the cross-sectional area of the container is constant, then $C = A/g$. Unlike the pneumatic capacitance (see Equation 7.9), which depends on the type of gas and its temperature, the hydraulic capacitance does not depend on any liquid properties.

Example 7.3: Hydraulic Capacitance of a Conical Tank

Derive the capacitance of the conical tank shown in Figure 7.5a using

a. $C = \mathrm{d}m/\mathrm{d}p$.
b. $C = A(h)/g$.

Solution

a. From Figure 7.5b, the radius r of the cross-section A at an arbitrary height is

$$r = h\tan\alpha = h\frac{R}{H}.$$

Thus, the volume of the liquid is

$$V(h) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\frac{\pi R^2}{H^2}h^3$$

and the stored mass is

$$m = \rho V(h) = \frac{1}{3}\frac{\rho\pi R^2}{H^2}h^3.$$

Note that the pressure caused by the height of the liquid is

$$p = p_{\mathrm{a}} + \rho g h,$$

which gives

$$\frac{\mathrm{d}p}{\mathrm{d}h} = \rho g.$$

Thus, the capacitance of the conical tank is

$$C = \frac{\mathrm{d}m}{\mathrm{d}p} = \frac{\mathrm{d}m}{\mathrm{d}h}\frac{\mathrm{d}h}{\mathrm{d}p} = \left(\frac{\rho\pi R^2}{H^2}h^2\right)\left(\frac{1}{\rho g}\right) = \frac{\pi R^2 h^2}{H^2 g}.$$

b. The hydraulic capacitance can also be derived directly using $C = A(h)/g$, which yields

$$C = \frac{\pi r^2}{g} = \left(\frac{\pi}{g}\right)\left(\frac{R^2h^2}{H^2}\right) = \frac{\pi R^2 h^2}{H^2 g}.$$

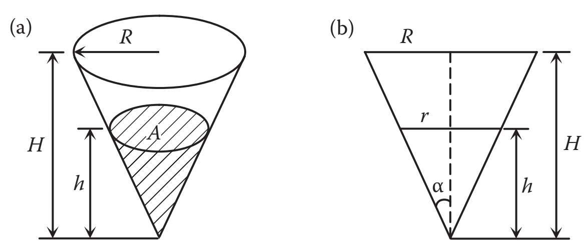


FIGURE 7.5 A conical tank. (a) Three-dimensional view and (b) cross-sectional view.

7.2.2 Hydraulic Resistance

When liquid flows through a pipe, a valve, or an orifice, the liquid meets resistance that creates a reduction in the pressure of the liquid. The pressure difference is associated with the mass flow rate q_m in a nonlinear relationship, $p = f(q_m)$, as illustrated in Figure 7.6. The slope of the curve is defined as the hydraulic resistance R, which depends on the reference mass flow rate q_{mr} and reference pressure p_r. The expression of the hydraulic resistance R is given by

$$R = \left.\frac{dp}{dq_m}\right|_{(q_{mr},p_r)}, \tag{7.24}$$

which is also used for pneumatic systems. Near a reference operating point (q_{mr},p_r), we can perform linearization and obtain the linearized resistance, which is

$$R = \frac{\Delta p}{\Delta q_m} = \frac{p - p_r}{q_m - q_{mr}}. \tag{7.25}$$

The resistance due to a valve, an orifice, or pipe wall friction can be represented by the valve-like symbol as in Figure 7.7. In many fluid systems, multiple valves, orifices, or pipes are used. They are arranged in different ways, such as in series or parallel connections. The equivalent linear hydraulic resistances can be obtained similar to electrical resistances.

Figure 7.8 shows resistances in series. Note that the mass flow rate remains the same through each resistance. The pressure decreases across the resistances R_1 and R_2 are $p_1 - p_2 = R_1q_m$ and $p_2 - p_3 = R_2q_m$, respectively. Consequently, the total pressure reduction across the two resistances in series is $p_1 - p_3 = (R_1 + R_2)q_m$. Comparing this result with the equivalent fluid system, $p_1 - p_3 = R_{eq}q_m$, we have

$$R_{eq} = R_1 + R_2 \tag{7.26}$$

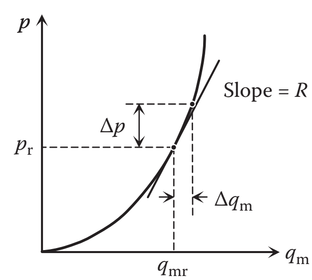


FIGURE 7.6 Linearized resistance near a reference point.

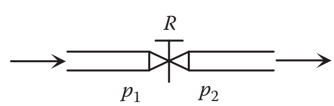


FIGURE 7.7 A symbol for hydraulic resistance.

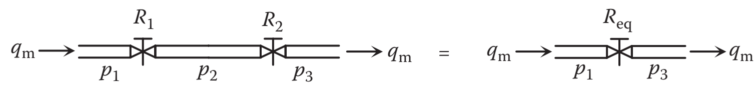


FIGURE 7.8 Equivalence for series hydraulic resistances.

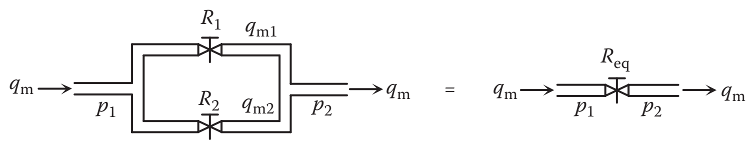


FIGURE 7.9 Equivalence for parallel hydraulic resistances.

A fluid system with resistances in parallel is shown in Figure 7.9. Note that the pressure decrease across each resistance must be the same. The mass flow rates through the resistances R_1 and R_2 are $q_{m1} = (p_1 - p_2)/R_1$ and $q_{m2} = (p_1 - p_2)/R_2$, respectively. Therefore, the total mass flow rate through the two resistances in parallel is $q_{m1} + q_{m2} = (p_1 - p_2)(1/R_1 + 1/R_2)$. Comparing this result with the equivalent fluid system, $q_m = (p_1 - p_2)/R_{eq}$, we have

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{7.27}$$

7.2.3 Modeling of Liquid-Level Systems

To obtain a simple model of a liquid-level system, we will use a capacitance element to represent each storage tank and a resistance element to represent each valve in the system. The resulting mathematical model may adequately describe the dynamics of the real system. Similar to the modeling of pneumatic systems, the basic law used to derive the differential equation of a liquid-level system is the law of conservation of mass presented in Equation 7.11: $\mathrm{d}m/\mathrm{d}t = q_{mi} - q_{mo}$. That is, the time rate of change of fluid mass in a tank equals the mass flow rate into the tank minus the mass flow rate out of the tank.

Consider a single tank with a valve as shown in Figure 7.10, where p_a is the atmospheric pressure, and q_i and q_o are the volume flow rates into and out of the tank, respectively. The cross-sectional area of the tank is A and the liquid height is h. The liquid leaves the tank through the valve, for which the hydraulic resistance is R. The density of the liquid is constant ρ.

Next, we will show how to derive the differential equation of the system by applying the law of conservation of mass presented in Equation 7.11. Note that the total fluid mass in the tank is $\rho A h$. For constant cross-sectional area and constant density, the left-hand side of Equation 7.11 can be rewritten as

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\rho A h) = \rho A \frac{\mathrm{d}h}{\mathrm{d}t}. \tag{7.28}$$

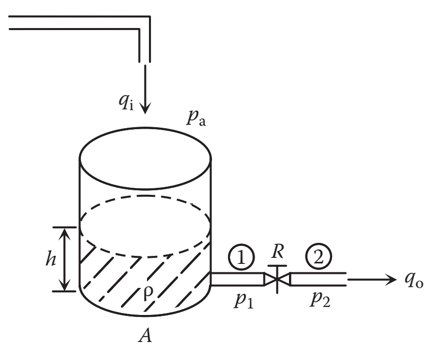


FIGURE 7.10 A single-tank liquid-level system with a valve.

The right-hand side of Equation 7.11 can be rewritten as

$$q_{mi} - q_{mo} = \rho q_i - \rho q_o. \tag{7.29}$$

Labeling point 1 at the upstream side of the valve and point 2 at the downstream side of the valve, the hydraulic resistance of the valve can be expressed as

$$R = \frac{\Delta p}{\Delta q_m} = \frac{p_1 - p_2}{\rho q_o}. \tag{7.30}$$

Assume that the pressure p_1 can be approximated as the hydrostatic pressure, $p_1 = p_a + \rho g h$, and the pressure p_2 can be approximated as the atmospheric pressure, $p_2 = p_a$. Substituting p_1 and p_2 into Equation 7.30 gives

$$R = \frac{\rho g h}{\rho q_o}. \tag{7.31}$$

Consequently,

$$\rho q_i - \rho q_o = \rho q_i - \frac{\rho g h}{R} \tag{7.32}$$

or

$$q_{mi} - q_{mo} = \rho q_i - \frac{\rho g h}{R}. \tag{7.33}$$

Substituting Equations 7.28 and 7.33 into Equation 7.11 results in

$$\rho A \frac{dh}{dt} = \rho q_i - \frac{\rho g h}{R}. \tag{7.34}$$

Rearranging the equation gives

$$\frac{RA}{g}\frac{dh}{dt} + h = \frac{R}{g} q_i, \tag{7.35}$$

which is a first-order ordinary differential equation relating the liquid height h and the inlet volume flow rate q_i. Introducing the expression of the hydraulic capacitance C given by Equation 7.23, we can rewrite Equation 7.35 as

$$RC\frac{dh}{dt} + h = \frac{R}{g} q_i. \tag{7.36}$$

Equation 7.36 describes the dynamic behavior of a single-tank liquid-level system with capacitance C and resistance R as shown schematically in Figure 7.10.

Hydraulic systems are usually connected with pumps, which can be considered as pressure sources. The following example shows how to derive the governing differential equation for a single-tank liquid-level system with a pump.

Example 7.4: A Single-Tank Liquid-Level System with a Pump

Consider the single-tank liquid-level system shown in Figure 7.11, where a pump is connected to the bottom of the tank through a valve of linear resistance R. The inlet to the pump is open to the

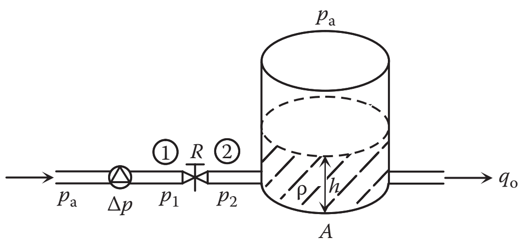


FIGURE 7.11 A single-tank liquid-level system with a pump.

atmosphere, and the pressure of the fluid increases by Δp when crossing the pump. Derive the differential equation relating the liquid height h and the volume flow rate q_o at the outlet. The tank's cross-sectional area A is constant. The density ρ of the liquid is constant.

Solution

We begin by applying the law of conservation of mass to the tank,

$$\frac{\mathrm{d}m}{\mathrm{d}t} = q_{mi} - q_{mo}.$$

The fluid mass inside the tank is ρAh. For constant fluid density and constant cross-sectional area,

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \rho A \frac{\mathrm{d}h}{\mathrm{d}t}.$$

The mass flow rate into the tank is

$$q_{mi} = \frac{p_1 - p_2}{R},$$

where $p_1 = p_a + \Delta p$ and $p_2 = p_a + \rho gh$, which is equal to the hydrostatic pressure at the bottom of the tank. Thus,

$$q_{mi} = \frac{p - \rho gh}{R}.$$

The mass flow rate out of the tank can be expressed in terms of the volume flow rate q_o as

$$q_{mo} = \rho q_o.$$

Substituting these expressions into the law of conservation of mass gives

$$\rho A \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{p - \rho gh}{R} - \rho q_o.$$

Rearranging the equation gives

$$\rho A \frac{\mathrm{d}h}{\mathrm{d}t} + \frac{\rho g}{R} h - \frac{p}{R} = -\rho q_o.$$

For a liquid-level system with two or more tanks, we apply the law of conservation of mass to each tank.

Example 7.5: A Two-Tank Liquid-Level System

Figure 7.12 shows a liquid-level system, in which two tanks have cross-sectional areas A_1 and A_2, respectively. A pump is connected to the bottom of tank 1 through a valve of linear resistance R_1. The liquid flows from tank 1 to tank 2 through a valve of linear resistance R_2, and leaves tank 2 through a valve of linear resistance R_3. The density ρ of the liquid is constant.

a. Derive the differential equations in terms of the liquid heights h_1 and h_2. Write the equations in second-order matrix form.
b. Assume the pump pressure Δp is the input and the liquid heights h_1 and h_2 are the outputs. Determine the state–space form of the system.

Solution

a. Applying the law of conservation of mass to tank 1 gives

$$\frac{\mathrm{d}m}{\mathrm{d}t} = q_{\mathrm{mi}} - q_{\mathrm{mo}}.$$

where

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \rho A_1 \frac{\mathrm{d}h_1}{\mathrm{d}t}.$$

The mass flow rates entering and leaving tank 1 can be written as

$$q_{\mathrm{mi}} = \frac{(p_{\mathrm{a}} + \quad p) - (p_{\mathrm{a}} + \rho g h_1)}{R_1} = \frac{\quad p - \rho g h_1}{R_1}$$

and

$$q_{\mathrm{mo}} = \frac{(p_{\mathrm{a}} + \rho g h_1) - p_{\mathrm{a}}}{R_2} = \frac{\rho g h_1}{R_2}.$$

Substituting these expressions results in

$$\rho A_1 \frac{\mathrm{d}h_1}{\mathrm{d}t} = \frac{\quad p - \rho g h_1}{R_1} - \frac{\rho g h_1}{R_2},$$

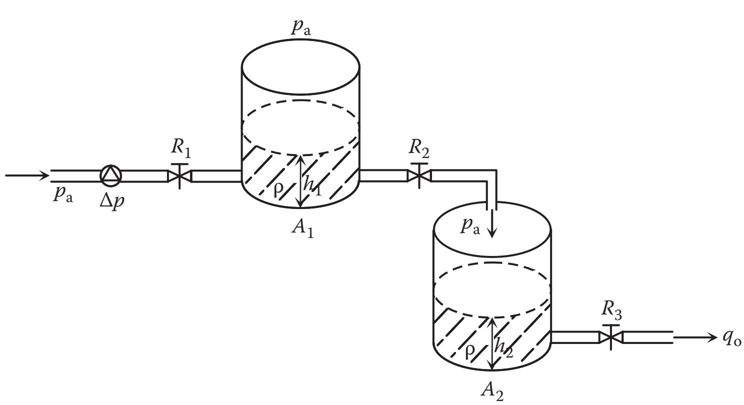


FIGURE 7.12 A two-tank liquid-level system.

which can be rearranged as

$$\rho A_1 \frac{\mathrm{d}h_1}{\mathrm{d}t} + \rho g h_1 \left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{p}{R_1}.$$

Applying the law of conservation of mass to tank 2 gives

$$\frac{\mathrm{d}m}{\mathrm{d}t} = q_{\mathrm{mi}} - q_{\mathrm{mo}},$$

where

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \rho A_2 \frac{\mathrm{d}h_2}{\mathrm{d}t}.$$

The mass flow rates entering and leaving tank 2 can be written as

$$q_{\mathrm{mi}} = \frac{(p_\mathrm{a} + \rho g h_1) - p_\mathrm{a}}{R_2} = \frac{\rho g h_1}{R_2}$$

and

$$q_{\mathrm{mo}} = \frac{(p_\mathrm{a} + \rho g h_2) - p_\mathrm{a}}{R_3} = \frac{\rho g h_2}{R_3}.$$

Substituting these expressions results in

$$\rho A_2 \frac{\mathrm{d}h_2}{\mathrm{d}t} = \frac{\rho g h_1}{R_2} - \frac{\rho g h_2}{R_3},$$

which can be rearranged as

$$\rho A_2 \frac{\mathrm{d}h_2}{\mathrm{d}t} - \frac{\rho g h_1}{R_2} + \frac{\rho g h_2}{R_3} = 0.$$

The system of differential equations in second-order matrix form is found to be

$$\begin{bmatrix} \rho A_1 & 0 \\ 0 & \rho A_2 \end{bmatrix} \begin{Bmatrix} \dfrac{\mathrm{d}h_1}{\mathrm{d}t} \\ \dfrac{\mathrm{d}h_2}{\mathrm{d}t} \end{Bmatrix} + \begin{bmatrix} \dfrac{\rho g}{R_1} + \dfrac{\rho g}{R_2} & 0 \\ -\dfrac{\rho g}{R_2} & \dfrac{\rho g}{R_3} \end{bmatrix} \begin{Bmatrix} h_1 \\ h_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{p}{R_1} \\ 0 \end{Bmatrix}.$$

b. As specified, the state, the input, and the output are

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} h_1 \\ h_2 \end{Bmatrix}, \quad u = p, \quad \mathbf{y} = \begin{Bmatrix} h_1 \\ h_2 \end{Bmatrix}.$$

The state-variable equations are

$$\dot{x}_1 = \frac{\mathrm{d}h_1}{\mathrm{d}t} = -\frac{g}{A_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) h_1 + \frac{1}{\rho A_1 R_1} p,$$

$$\dot{x}_2 = \frac{\mathrm{d}h_2}{\mathrm{d}t} = \frac{g}{A_2 R_2} h_1 - \frac{g}{A_2 R_3} h_2,$$

or in matrix form

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -\dfrac{g}{A_1}\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right) & 0 \\ \dfrac{g}{A_2R_2} & -\dfrac{g}{A_2R_3} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} \dfrac{1}{\rho A_1 R_1} \\ 0 \end{bmatrix} u.$$

The output equation is

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u.$$

PROBLEM SET 7.2

1. Derive the capacitance of the tank shown in Figure 7.13. There is an opening at the top of the tank at height H.
2. Derive the capacitance of the tank shown in Figure 7.14.
3. Consider the rectangular tank in Figure 7.15a and the pyramid tank in Figure 7.15b. The volume flow rate into each tank through a pipe is q_i. The liquid leaves each tank through a valve of linear resistance R. The density ρ of the liquid is constant.
   a. Derive the dynamic model of the liquid height h for each tank.
   b. For each tank, write the differential equation in terms of hydraulic capacitance and hydraulic resistance. Compare the models for the two single-tank systems.

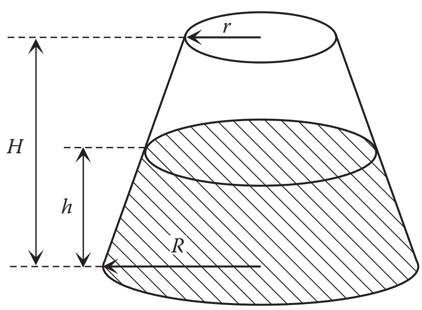


FIGURE 7.13 Problem 1.

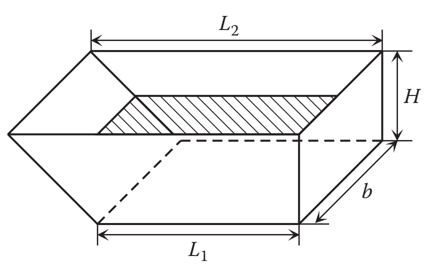


FIGURE 7.14 Problem 2.

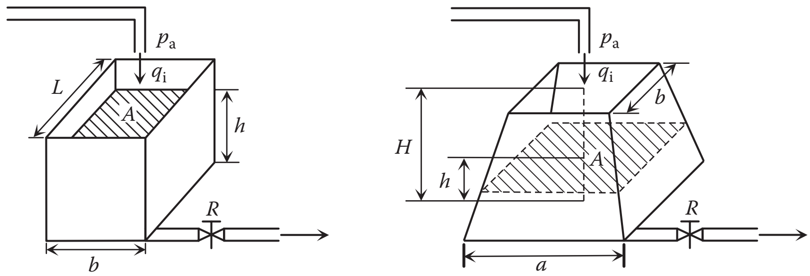


FIGURE 7.15 Problem 3.

4. Consider the single-tank liquid-level system shown in Figure 7.16, where the volume flow rate into the tank through a pipe is q_i. A pump is connected to the bottom of the tank through a valve of linear resistance R_1. The pressure of the fluid increases by Δp when crossing the pump. The liquid leaves the tank through a valve of linear resistance R_2. Derive the differential equation relating the liquid height h and the volume flow rate q_i at the inlet. The tank's cross-sectional area A is constant. The density ρ of the liquid is constant.
5. Consider the two-tank liquid-level system shown in Figure 7.17. The liquid is pumped into tanks 1 and 2 through valves of linear resistances R_1 and R_2, respectively. The pressure of the fluid increases by Δp when crossing the pump. The cross-sectional areas of the two tanks are A_1 and A_2, respectively. The liquid flows from tank 1 to tank 2 through a valve of linear resistance R_3 and leaves tank 2 through a valve of linear resistance R_4. The density ρ of the liquid is constant. Derive the differential equations in terms of the liquid heights h_1 and h_2. Write the equations in second-order matrix form.

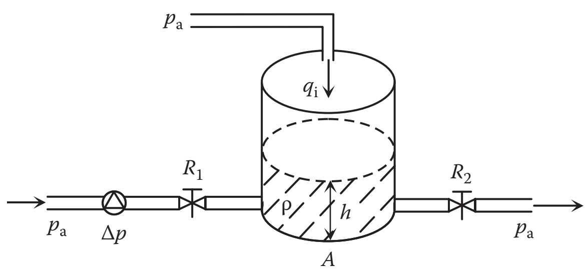


FIGURE 7.16 Problem 4.

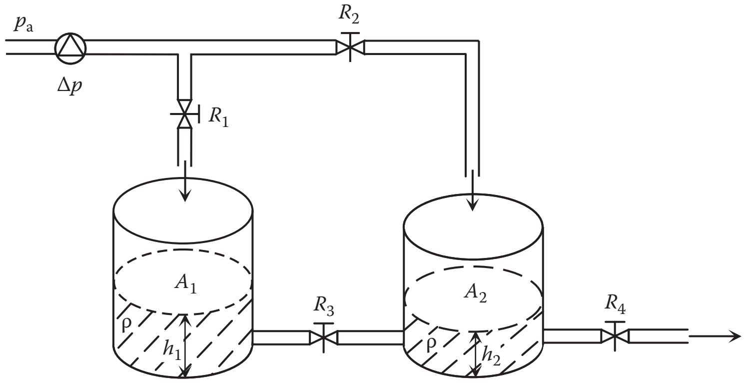


FIGURE 7.17 Problem 5.

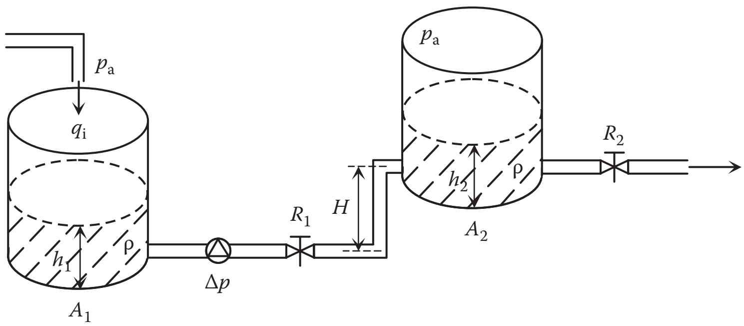


FIGURE 7.18 Problem 6.

6. Figure 7.18 shows a liquid-level system, in which two tanks have cross-sectional areas A_1 and A_2, respectively. The volume flow rate into tank 1 is q_i. A pump is connected to the bottom of tank 1 and the pressure of the fluid increases by Δp when crossing the pump. Tank 2 is located higher than tank 1 and the vertical distance between the two tanks is H. The liquid is pumped from tank 1 to tank 2 through a valve of linear resistance R_1 and leaves tank 2 through a valve of linear resistance R_2. The density ρ of the liquid is constant. Derive the differential equations in terms of the liquid heights h_1 and h_2. Write the equations in second-order matrix form.
7. Consider the single-tank liquid-level system shown in Figure 7.19, where the volume flow rate into the tank through a pipe is q_i. The liquid leaves the tank through an orifice of area A_o. Denote C_d as the discharge coefficient, which is the ratio of the actual mass flow rate to the theoretical one, and lies in the range of $0 < C_d < 1$ because of friction effects. Derive the differential equation relating the liquid height h and the volume flow rate q_i at the inlet. The tank's cross-sectional area is constant. The density ρ of the liquid is constant.
8. Figure 7.20 shows a hydraulic system of two interconnected tanks, which have the same cross-sectional area of A. A pump is connected to tank 1. Assume that the relationship between the voltage applied to the pump and the mass flow rate into tank 1 is linear, that is, $q_{mi} = k_p v_a$, where k_p is called the pump constant and can be obtained by experimental measurements. Tank 1 is connected to tank 2, which is connected to a reservoir. The liquid leaves each tank through an outlet of area A_o at the bottom. Derive the differential equations in terms of the liquid heights h_1 and h_2.

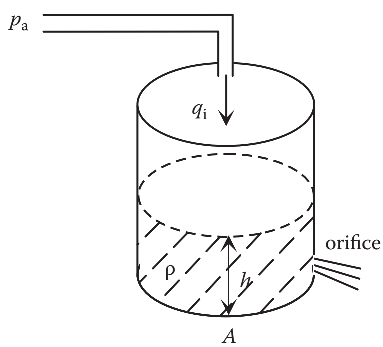


FIGURE 7.19 Problem 7.

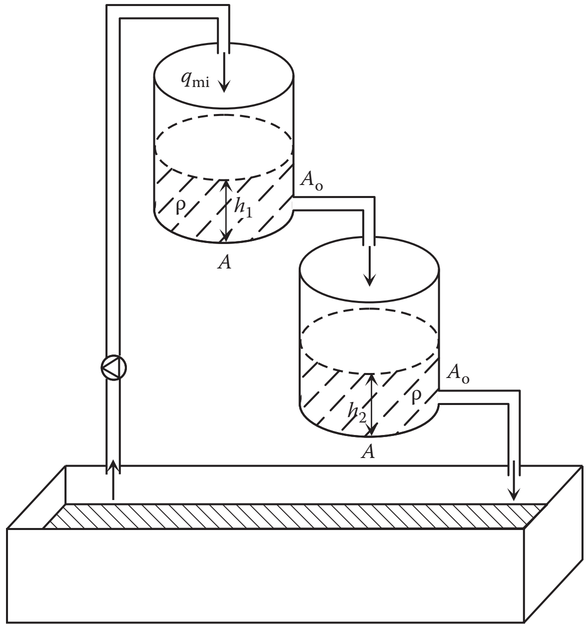


FIGURE 7.20 Problem 8.

7.3 THERMAL SYSTEMS

Thermal systems are those that involve the transfer of heat from one object to another. When an object is at a different temperature than its surroundings or another object, heat will transfer from the higher-temperature object to the lower-temperature one, obeying the law of conservation of energy. Examples of thermal systems include heaters, air conditioners, refrigerators, and so on. Just like with fluid systems (either pneumatic or hydraulic), for which fluid capacitance, fluid resistance, and the conservation of mass are the basis for system modeling, we will introduce thermal capacitance, thermal resistance, and the conservation of energy, which together form the basis of modeling for thermal systems.

7.3.1 First Law of Thermodynamics

The first law of thermodynamics is a version of the law of conservation of energy, adapted for thermodynamic systems. For a system with well-defined boundaries, the law of conservation of energy states

$$\Delta E = Q - W, \tag{7.37}$$

where ΔE is the change in energy of the system, Q is the heat flow into or out of the system, and W is the work done by or on the system. In Equation 7.37, Q is positive if heat is supplied to the system and negative if heat is given off by the system. W is positive if work is done by the system and negative if work is done to the system. Based on this sign convention, we have

$$\Delta E = (Q_{in} - Q_{out}) - (W_{out} - W_{in}), \tag{7.38}$$

where Q_{in}, Q_{out}, W_{in}, and W_{out} are all positive quantities.

In actuality, the net amount of energy added to the system is equal to the net increase in the energy stored internally in the system and any change in the mechanical energy of the system's center of mass,

$$\Delta E = \Delta U + \Delta ME_C. \tag{7.39}$$

U is the internal energy (or internal thermal energy), which is the energy stored at the molecular level. It includes the kinetic energy due to the motion of molecules and the potential energy that holds the atoms together. ME_C stands for the mechanical energy, which includes the kinetic energy and the potential energy associated with the system's mass center. For systems with negligible change in mechanical energy,

$$\Delta U = Q - W = (Q_{in} - Q_{out}) - (W_{out} - W_{in}), \tag{7.40}$$

which is the mathematical expression of the first law of thermodynamics. It basically states that the change in internal energy is equal to the amount of energy gained by heating minus the amount lost by doing work on the environment.

Heat Q is the energy transfer at the molecular level. Work W is the energy transfer that is capable of producing macroscopic mechanical motion of the system's mass center. For thermal systems with pure heat transfer and no work involved, that is, $W_{in} = W_{out} = 0$, the law of conservation of energy presented in Equation 7.40 can be rewritten as

$$\Delta U = Q = Q_{in} - Q_{out} \tag{7.41}$$

or

$$\frac{dU}{dt} = q_{hi} - q_{ho}, \tag{7.42}$$

where $q_h = dQ/dt$ is the heat flow rate having units of J/s, which is a watt or ft·lb/s.

7.3.2 Thermal Capacitance

For an object, the thermal capacitance C is defined as the ratio of the change in heat flow to the change in the object's temperature,

$$C = \frac{dQ}{dT}, \tag{7.43}$$

where C has units of J/K, J/°C, or ft·lb/°F. The thermal capacitance is a measure of the heat required to increase the temperature of an object by a certain temperature interval.

Strictly speaking, the value of the thermal capacitance of a substance depends on thermodynamic processes. For a constant-volume process, no work is involved and all the heat goes into the internal energy of the substance,

$$Q = \Delta U = mc_v \Delta T, \tag{7.44}$$

where m is the mass of the substance, c_v is the constant-volume specific heat capacity of the substance in units of J·K/kg, J·°C/kg, or ft·lb·°F/slug, and ΔT is the change in temperature of the substance. For a constant-pressure process,

$$Q = \Delta H = mc_p \Delta T, \tag{7.45}$$

where H is the enthalpy and c_p is the constant-pressure specific heat capacity. Combining Equation 7.43 with Equations 7.44 or 7.45 gives

$$C = mc_v \tag{7.46}$$

or

$$C = mc_p. \tag{7.47}$$

For incompressible liquids and solids, because the volume cannot expand, the heat flow in a constant-pressure process is equal to the internal energy: $Q = \Delta U$ and $c_p = c_v$. Assuming that the density and the volume of the mass are ρ and V, respectively, we have

$$C = mc = \rho Vc. \tag{7.48}$$

where c is the specific heat capacity and $c = c_p = c_v$. The subscripts p and v will be dropped in the rest of the chapter for simplicity.

Note that the value of the specific heat capacity c depends on the substance of the object, whereas the thermal capacitance C is an extensive property because its value is proportional to the mass of the object. For instance, the specific heat capacity of water at room temperature (25°C) is a constant value of 4186 J/(kg·°C). However, the thermal capacitance for a bathtub of water is greater than that for a cup of water.

7.3.3 Thermal Resistance

There are three mechanisms by which heat is transported: conduction, convection, and radiation. Conduction is the transfer of heat between substances that are in direct contact with each other. Convection is the transfer of heat due to a flowing fluid, which can be a gas or a liquid. Radiation is the transfer of heat through empty space. Here, we only consider conduction and convection.

The thermal resistance R for heat transfer is defined as the ratio of the change in temperature difference to the change in heat flow rate,

$$R = \frac{\mathrm{d}T}{\mathrm{d}q_h}, \tag{7.49}$$

The thermal resistance R has units of K·s/J, °C·s/J, or °F·s/(ft·lb).

For simple one-dimensional conduction as shown in Figure 7.21, Fourier's law, also known as the law of heat conduction, gives

$$q_h = kA\frac{\Delta T}{L} = kA\frac{T_1 - T_2}{L}, \tag{7.50}$$

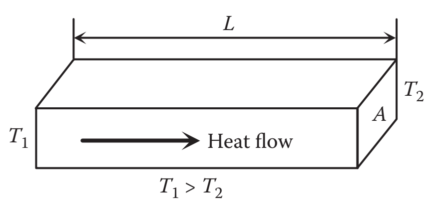


FIGURE 7.21 One-dimensional conduction: heat flow from higher to lower temperature.

where L is the length of the body in the direction of heat flow, A is the cross-sectional area normal to the heat flow direction, ΔT is the temperature difference along its length, and k is the thermal conductivity of the material in W/(m·K) or W/(m·°C). Note that the heat flow is in the direction of decreasing temperature. Fourier's equation is often used only for solids, although it is valid for both solids and fluids. Combining Equations 7.49 and 7.50 gives the thermal resistance for conduction

$$R = \frac{L}{kA}. \tag{7.51}$$

For convective heat transfer, Newton's law of cooling states that the rate of heat flow of a body is proportional to the difference in temperatures between the body and its surroundings or environment. The mathematical expression is

$$q_{\mathrm{h}} = hA\Delta T = hA(T_{\mathrm{s}} - T_{\mathrm{env}}), \tag{7.52}$$

where A is the surface area, from which the heat is being transferred, T_{s} is the temperature of the body's surface, T_{env} is the temperature of the environment, and h is the heat transfer coefficient in W/(m^2·K) or W/(m^2·°C). Combining Equations 7.49 and 7.52 gives the thermal resistance for convection

$$R = \frac{1}{hA}. \tag{7.53}$$

It is very useful to utilize the concept of thermal resistance and represent heat transfer by thermal circuits. The heat flow rate q_{h} is analogous to the current, the temperature difference ΔT is analogous to the voltage, and the thermal resistance is analogous to the electric resistance.

Figure 7.22 shows the heat transfer across a composite slab, which can be represented using a thermal resistance network with series interconnection. The heat flow rate remains the same through each component. Assume that the thermal resistances are R_1 and R_2. As a result, the temperature differences across the resistances R_1 and R_2 are $T_1 - T_2 = R_1 q_{\mathrm{h}}$ and $T_2 - T_3 = R_2 q_{\mathrm{h}}$, respectively. The total temperature difference across the composite slab is $T_1 - T_3 = (R_1 + R_2)q_{\mathrm{h}}$. Thus, the equivalent thermal resistance for a series interconnection is

$$R_{\mathrm{eq}} = R_1 + R_2. \tag{7.54}$$

Figure 7.23 shows heat transfer across a wall with different materials and how it can be represented by a thermal resistance network with parallel interconnection. Note that the temperature decrease across each material must be the same. Consequently, the heat flow rates through the resistances R_1 and R_2 are $q_{\mathrm{h1}} = (T_1 - T_2)/R_1$ and $q_{\mathrm{h2}} = (T_1 - T_2)/R_2$, respectively. The total heat flow

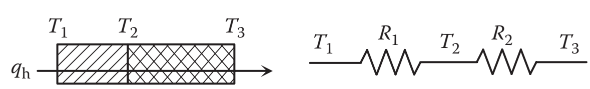


FIGURE 7.22 Heat transfer across series thermal resistance.

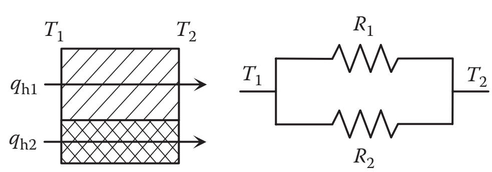


FIGURE 7.23 Heat transfer across parallel thermal resistance.

rate through the wall is $q_{h1} + q_{h2} = (T_1 - T_2)(1/R_1 + 1/R_2)$. Thus, the equivalent thermal resistance for a parallel interconnection is

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{7.55}$$

If there are different heat transfer modes in a system, a thermal circuit with thermal resistances representing the different modes of heat transfer can be used to analyze the system.

Example 7.6: Thermal Resistance

Consider heat transfer through an insulated wall as shown in Figure 7.24. The wall is made of a layer of brick with thermal conductivity k_1 and two layers of foam with thermal conductivity k_2 for insulation. The left surface of the wall is at temperature T_1 and exposed to air with heat transfer coefficient h_1. The right surface of the wall is at temperature T_2 and exposed to air with heat transfer coefficient h_2. Assume that k_1 = 0.5 W/(m·°C), k_2 = 0.17 W/(m·°C), $h_1 = h_2$ = 10 W/(m²·°C), T_1 = 38°C, and T_2 = 20°C. The thickness of the brick layer is 0.1 m, the thickness of each foam layer is 0.03 m, and the cross-sectional area of the wall is 16 m².

a. Determine the heat flow rate through the wall.
b. Determine the temperature distribution through the wall.

Solution

a. The heat transfer through the insulated wall can be represented using a thermal circuit with five thermal resistances connected in series as shown in Figure 7.25. Two modes of heat transfer, conduction and convection, are involved. The corresponding thermal resistances are

$$R_1 = \frac{1}{h_1 A} = \frac{1}{10 \times 16} = 6.25 \times 10^{-3}\,^\circ\text{C·s/J},$$

$$R_2 = R_4 = \frac{L_2}{k_2 A} = \frac{0.03}{0.17 \times 16} = 1.10 \times 10^{-2}\,^\circ\text{C·s/J},$$

$$R_3 = \frac{L_1}{k_1 A} = \frac{0.1}{0.5 \times 16} = 1.25 \times 10^{-2}\,^\circ\text{C·s/J},$$

$$R_5 = \frac{1}{h_2 A} = \frac{1}{10 \times 16} = 6.25 \times 10^{-3}\,^\circ\text{C·s/J}.$$

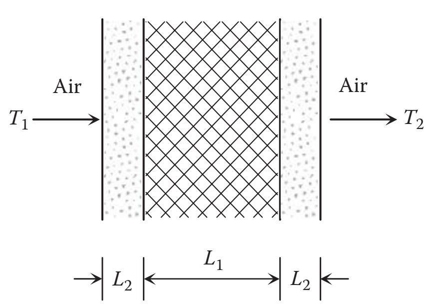


FIGURE 7.24 Heat transfer through an insulated wall.

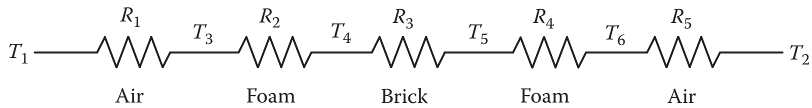


FIGURE 7.25 The equivalent thermal circuit for the heat transfer system in Figure 7.24.

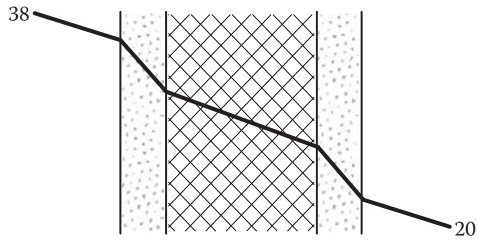


FIGURE 7.26 Temperature distribution through the insulated wall.

The total thermal resistance is

$$R_{\mathrm{eq}} = \sum_{i=1}^{5} R_i = 4.70 \times 10^{-2} \,°\mathrm{C{\cdot}s/J}.$$

Thus, the heat flow rate through the insulated wall is

$$q_{\mathrm{h}} = \frac{\Delta T}{R_{\mathrm{eq}}} = \frac{38 - 20}{4.70 \times 10^{-2}} = 382.98\ \mathrm{W}.$$

b. Note that the heat flow rate stays the same through the insulated wall. Thus, from left to right, the heat flow rate through each layer is

$$\begin{aligned}
&\text{Air:} && q_{\mathrm{h}} = R_1(T_1 - T_3) \\
&\text{Foam:} && q_{\mathrm{h}} = R_2(T_3 - T_4) \\
&\text{Brick:} && q_{\mathrm{h}} = R_3(T_4 - T_5) \\
&\text{Foam:} && q_{\mathrm{h}} = R_4(T_5 - T_6) \\
&\text{Air:} && q_{\mathrm{h}} = R_5(T_6 - T_2)
\end{aligned}$$

With the given temperatures T_1 and T_2, we have $T_3 = 35.61°\mathrm{C}$, $T_4 = 31.40°\mathrm{C}$, $T_5 = 26.61°\mathrm{C}$, and $T_6 = 22.40°\mathrm{C}$. Figure 7.26 shows the temperature distribution through the wall. Note that the temperatures shown in Figure 7.26 are the values when the heat transfer process reaches steady state.

7.3.4 Modeling of Heat Transfer Systems

The mathematical model of a thermal system is often complicated because of the complex temperature distribution throughout the system. To simplify analysis, in this section, we discuss how to obtain a lumped-parameter model, which may approximate the gross system dynamics. The validity of this lumped-parameter assumption can be checked using the so-called Biot number.

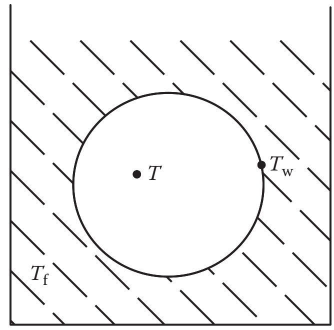


FIGURE 7.27 A solid object submerged in a fluid.

Consider a solid body submerged in hot fluid as shown in Figure 7.27, in which T_f is the fluid temperature, T_w is the temperature at the wall of the object, and T is the temperature of an arbitrary point inside the object. There are two modes of heat transfer involved, conduction within the body and convection between the fluid and the body. The heat flow rates for the two different modes can be approximated to have the same magnitude,

$$\frac{kA}{L}(T_w - T) \approx hA(T_f - T_w), \tag{7.56}$$

where h is the heat transfer coefficient, k is the thermal conductivity for the material of the object, and L is the relevant length between the wall and the point. The ratio of the temperature differences caused by the different modes of heat transfer is

$$\frac{T_w - T}{T_f - T_w} \approx \frac{hL}{k}. \tag{7.57}$$

Note that the temperature difference within the object can be negligible if the object is thin or small enough. The criterion for determining a solid being thin or small is based on the Biot number, which is defined as

$$Bi = \frac{hL_c}{k}, \tag{7.58}$$

where L_c is the characteristic length of the solid object, defined as the volume of the body divided by the surface area of the body,

$$L_c = \frac{V_{\text{body}}}{A_{\text{surface}}}. \tag{7.59}$$

For a body whose Biot number is much less than one, typically, $Bi < 0.1$, the interior of the body may be assumed to have the same temperature.

Example 7.7: Temperature Dynamics of a Heated Object

Consider a steel sphere with a radius of $r = 0.01$ m submerged in a hot water bath with a heat transfer coefficient of $h = 350$ W/(m^2·°C). For steel, the density is $\rho = 7850$ kg/m^3, the specific heat capacity is $c = 440$ J/(kg·°C), and the thermal conductivity is $k = 43$ W/(m·°C). The temperature of the water T_f is 100°C and the initial temperature of the sphere T_0 is 25°C.

a. Determine whether the sphere's temperature can be considered uniform.
b. Derive the differential equation relating the sphere's temperature $T(t)$ and the water's temperature T_f.
c. Using the differential equation obtained in Part (b), construct a Simulink block diagram to find the sphere's temperature $T(t)$.

Solution

a. The characteristic length of the sphere is

$$L_c = \frac{V_{\text{body}}}{A_{\text{surface}}} = \frac{4/3\pi r^3}{4\pi r^2} = \frac{1}{3}r = \frac{0.01}{3}.$$

The Biot number of the steel sphere is

$$Bi = \frac{hL_c}{k} = \frac{350(0.01)}{43(3)} = 2.71\times10^{-2} < 0.1.$$

Thus, the sphere can be treated as a lump-temperature system, and its temperature can be considered uniform within the body.

b. The dynamic model of the sphere's temperature can be derived using the law of conservation of energy,

$$\frac{\mathrm{d}U}{\mathrm{d}t} = q_{hi} - q_{ho}.$$

Note that $U = mcT = \rho VcT$, and we have

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\rho VcT) = \rho Vc\frac{\mathrm{d}T}{\mathrm{d}t}.$$

The heat flow rate into the body is

$$q_{hi} = \frac{T_f - T}{R}$$

and the heat flow rate out of the body is $q_{ho} = 0$. Thus, the differential equation of the system is

$$\rho Vc(\mathrm{d}T/\mathrm{d}t) = (T_f - T)/R.$$

Introducing the expression for the thermal capacitance $\rho Vc = C$, we find

$$RC\frac{\mathrm{d}T}{\mathrm{d}t} + T = T_f.$$

The thermal capacitance of the sphere is

$$C = \rho Vc = 7850\left(\frac{4}{3}\right)(\pi)(0.01)^3(440) = 14.47\ \text{J/°C},$$

and the thermal resistance due to convection is

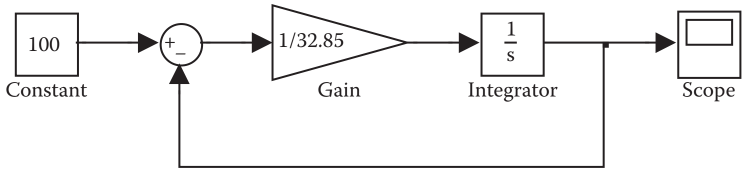


FIGURE 7.28 A Simulink block diagram representing the thermal system in Example 7.7.

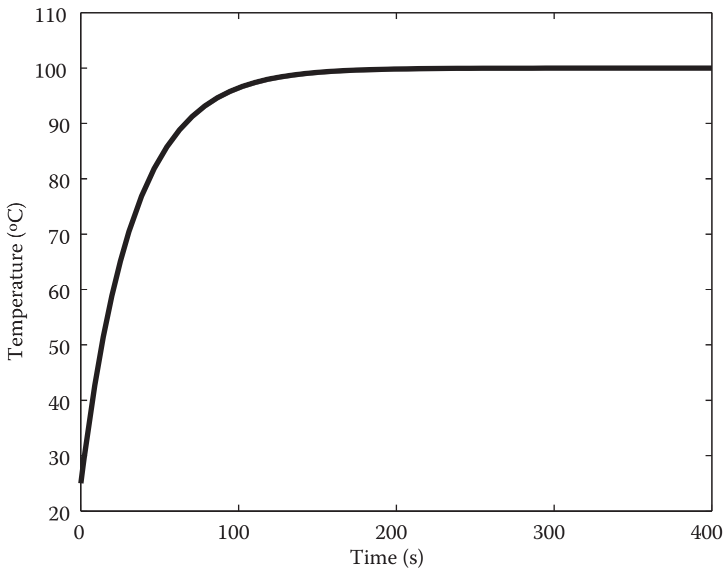


FIGURE 7.29 Temperature output $T(t)$ of the thermal system in Example 7.7.

$$R = \frac{1}{hA} = \frac{1}{350(4)(\pi)(0.01)^2} = 2.27°\mathrm{Cs/J}.$$

Thus, the dynamic model of the sphere's temperature is

$$32.85\frac{dT}{dt} + T = T_f.$$

c. Given $T_f = 100°$C, solving for the highest derivative of the output T gives

$$\frac{dT}{dt} = \frac{1}{32.85}(100 - T),$$

which can be represented using the Simulink block diagram shown in Figure 7.28. Double-click on the `Integrator` block and define the initial temperature of the sphere to be 25°C. Run the simulation. The results can be plotted as shown in Figure 7.29.

Example 7.7 shows that the temperature dynamics of a thermal system can be expressed in terms of thermal capacitance and thermal resistance. For multiple thermal capacitances, we apply the law of conservation of energy to each of them.

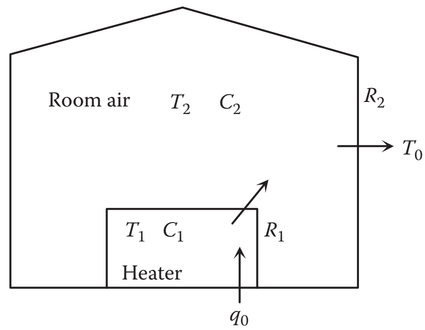


FIGURE 7.30 A room with a heater.

Example 7.8: Temperature Dynamics of a House with a Heater

The room shown in Figure 7.30 has a heater with heat flow rate input of q_0. The thermal capacitances of the heater and the room air are C_1 and C_2, respectively. The thermal resistances of the heater–air interface and the room wall–ambient air interface are R_1 and R_2, respectively. The temperatures of the heater and the room air are T_1 and T_2, respectively. The temperature outside the room is T_0, which is assumed to be constant.

a. Derive the differential equations relating the temperatures T_1, T_2, the input q_0, and the outside temperature T_0.
b. Using the differential equations obtained in Part (a), determine the state–space form of the system. Assume the temperatures T_1 and T_2 as the outputs.

Solution

a. Applying the law of conservation of energy to the heater, we have

$$\frac{\mathrm{d}U}{\mathrm{d}t} = q_{\mathrm{hi}} - q_{\mathrm{ho}},$$

where

$$\frac{\mathrm{d}U}{\mathrm{d}t} = C_1 \frac{\mathrm{d}T_1}{\mathrm{d}t},$$

$$q_{\mathrm{hi}} = q_0,$$

$$q_{\mathrm{ho}} = \frac{T_1 - T_2}{R_1}.$$

Substituting these expressions gives

$$C_1 \frac{\mathrm{d}T_1}{\mathrm{d}t} = q_0 - \frac{T_1 - T_2}{R_1},$$

which can be rearranged into

$$C_1 \frac{\mathrm{d}T_1}{\mathrm{d}t} + \frac{1}{R_1} T_1 - \frac{1}{R_1} T_2 = q_0.$$

Applying the law of conservation of energy to the room air, we have

$$\frac{\mathrm{d}U}{\mathrm{d}t} = q_{\text{hi}} - q_{\text{ho}},$$

where

$$\frac{\mathrm{d}U}{\mathrm{d}t} = C_2 \frac{\mathrm{d}T_2}{\mathrm{d}t},$$

$$q_{\text{hi}} = \frac{T_1 - T_2}{R_1},$$

$$q_{\text{ho}} = \frac{T_2 - T_0}{R_2}.$$

Substituting these expressions gives

$$C_2 \frac{\mathrm{d}T_2}{\mathrm{d}t} = \frac{T_1 - T_2}{R_1} - \frac{T_2 - T_0}{R_2},$$

which can be rearranged into

$$C_2 \frac{\mathrm{d}T_2}{\mathrm{d}t} - \frac{1}{R_1} T_1 + \left(\frac{1}{R_1} + \frac{1}{R_2} \right) T_2 = \frac{1}{R_2} T_0.$$

The system of differential equations can be written in second-order matrix form as

$$\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{Bmatrix} \dfrac{\mathrm{d}T_1}{\mathrm{d}t} \\ \dfrac{\mathrm{d}T_2}{\mathrm{d}t} \end{Bmatrix} + \begin{bmatrix} \dfrac{1}{R_1} & -\dfrac{1}{R_1} \\ -\dfrac{1}{R_1} & \left(\dfrac{1}{R_1} + \dfrac{1}{R_2} \right) \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} q_0 \\ \dfrac{1}{R_2} T_0 \end{Bmatrix}.$$

b. To represent a thermal system in the state–space form, the temperature of each thermal capacitance is often chosen as a state variable. As specified, the state, the input, and the output are

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}, \quad \mathbf{u} = \begin{Bmatrix} q_0 \\ T_0 \end{Bmatrix}, \quad \mathbf{y} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}.$$

The state-variable equations are

$$\dot{x}_1 = \frac{\mathrm{d}T_1}{\mathrm{d}t} = -\frac{1}{R_1 C_1} T_1 + \frac{1}{R_1 C_1} T_2 + \frac{1}{C_1} q_0,$$

$$\dot{x}_2 = \frac{\mathrm{d}T_2}{\mathrm{d}t} = \frac{1}{R_1 C_2} T_1 - \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} \right) T_2 + \frac{1}{R_2 C_2} T_0,$$

or in matrix form

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -\dfrac{1}{R_1 C_1} & \dfrac{1}{R_1 C_1} \\ \dfrac{1}{R_1 C_2} & -\left(\dfrac{1}{R_1 C_2} + \dfrac{1}{R_2 C_2} \right) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} \dfrac{1}{C_1} & 0 \\ 0 & \dfrac{1}{R_2 C_2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}.$$

The output equation is

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}.$$

PROBLEM SET 7.3

1. Consider heat transfer through an insulated frame wall of a house. The thermal conductivity of the wall is 0.055 W/(m·°C). The wall is 0.15 m thick and has an area of 15 m^2. The inside air temperature is 20°C and the heat transfer coefficient for convection between the wall and the inside air is 2.6 W/(m·°C). On the outside of the wall, the heat transfer coefficient for convection between the wall and the outside air is 10.4 W/(m·°C) and the outside air temperature is –20°C. Determine the heat flow rate through the wall.
2. Consider heat transfer through a double-pane window as shown in Figure 7.31a. Two layers of glass with thermal conductivity k_1 are separated by a layer of stagnant air with thermal conductivity k_2. The inner surface of the window is at temperature T_1 and exposed to room air with heat transfer coefficient h_1. The outer surface of the wall is at temperature T_2 and exposed to air with heat transfer coefficient h_2. Assume that k_1 = 0.95 W/(m·°C), k_2 = 0.0285 W/(m·°C), $h_1 = h_2$ = 10 W/(m^2·°C), T_1 = 20°C, and T_2 = 35°C. The thickness of each glass layer is 4 mm, the thickness of the air layer is 8 mm, and the cross-sectional area of the window is 1.5 m^2.
   a. Determine the heat flow rate through the double-pane window.
   b. Determine the temperature distribution through the double-pane window.
   c. Repeat Parts (a) and (b) for the single-pane glass window shown in Figure 7.31b.
3. The junction of a thermocouple can be approximated as a sphere with a diameter of 1 mm. As shown in Figure 7.32, the thermocouple is used to measure the temperature of a gas stream. For the junction, the density is ρ = 8500 kg/m^3, the specific heat capacity is c = 320 J/(kg·°C), and the thermal conductivity is k = 40 W/(m·°C). The temperature of the gas T_f is 100°C and the initial temperature of the sphere T_0 is 20°C. The heat transfer coefficient between the gas and the junction is h = 70 W/(m^2·°C).
   a. Determine if the junction's temperature can be considered uniform.
   b. Derive the differential equation relating the junction's temperature $T(t)$ and the gas's temperature T_f.
   c. Using the differential equation obtained in Part (b), construct a Simulink block diagram to find out how long it will take the thermocouple to read 99% of the initial temperature difference.

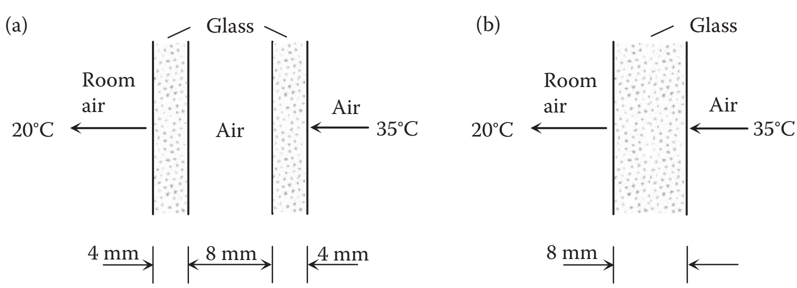


FIGURE 7.31 Problem 2 (a) double-pane window, (b) single-pane window.

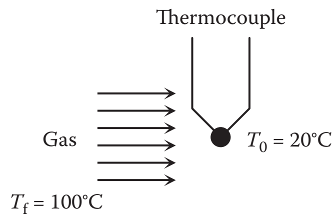


FIGURE 7.32 Problem 3.

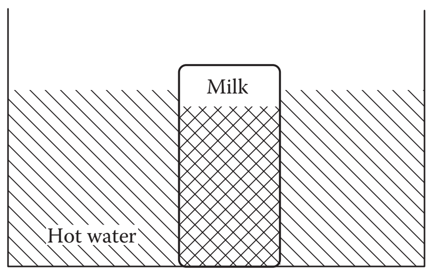


FIGURE 7.33 Problem 4.

4. Figure 7.33 shows a thin-walled glass of milk, which is taken out of the refrigerator at a uniform temperature of 3°C and is placed in a large pan filled with hot water at 60°C. Assume that the assumption of the lumped system analysis is applicable because the milk is stirred constantly, so that its temperature is uniform at all times. The glass container is cylindrical in shape with a radius of 3 cm and a height of 6 cm. The estimated parameters of the milk are density $\rho = 1035$ kg/m^3, specific heat capacity $c = 3980$ J/(kg·°C), and thermal conductivity $k = 0.56$ W/(m·°C). The heat transfer coefficient between the water and the glass is $h = 250$ W/(m^2·°C).
   a. Derive the differential equation relating the milk's temperature $T(t)$ and the water temperature.
   b. Using the differential equation obtained in Part (a), construct a Simulink block diagram. How long will it take for the milk to warm up from 3°C to 58°C?
5. As shown in Figure 7.34, the wall of a room consists of two layers, for which the thermal capacitances are C_1 and C_2. Assume that the temperatures in both layers are uniform and they are T_1 and T_2, respectively. The temperatures inside and outside the room are T_i and T_o, respectively. Both layers exchange heat by convection with air and the thermal resistances are R_1 and R_3, respectively. The thermal resistance of the interface between the layers is R_2.
   a. Derive the differential equations for this system.

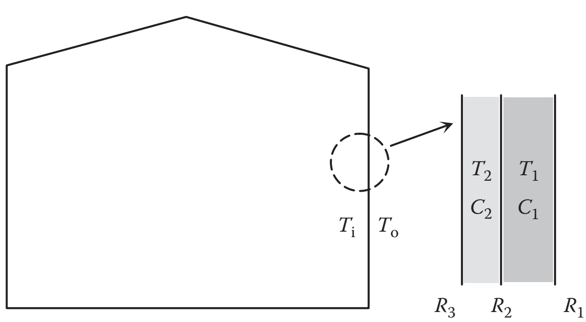


FIGURE 7.34 Problem 5.

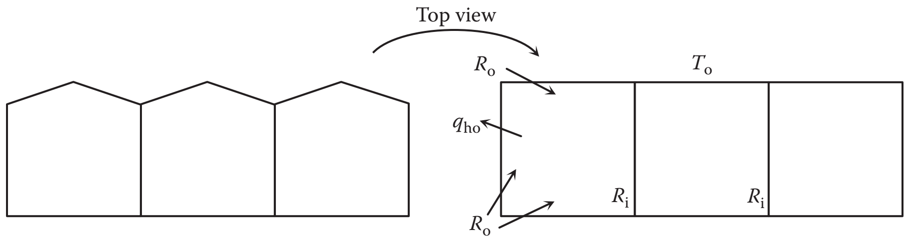


FIGURE 7.35 Problem 6.

b. Using the differential equations obtained in Part (a), determine the state–space form of the system. Assume the temperatures T_1 and T_2 are the outputs.

6. For the three-room house shown in Figure 7.35, all rooms are perfectly square and have the same dimensions. An air conditioner produces an equal amount of heat flow q_{ho} out of each room. The temperature outside the house is T_o. Assume that there is no heat flow through the floors or ceilings. The thermal resistances through the inner walls and the outer walls are R_i and R_o, respectively. The thermal capacitance of each room is C. Derive the differential equations for this system.

7.4 SYSTEM MODELING WITH SIMULINK AND SIMSCAPE

Similar to the modeling of mechanical and electrical systems, the dynamics of fluid and thermal systems can be represented by ordinary differential equations, transfer functions, or the state–space form. Therefore, the Simulink modeling techniques discussed in Sections 5.6 and 6.6 can also be applied to fluid and thermal systems.

Example 7.9: Modeling of a Pneumatic System with Simulink

Consider the pneumatic system in Example 7.2. Construct a Simulink block diagram to find the pressure inside the container, $p(t)$, which is assumed to be 0 Pa initially. The pressure at the inlet is assumed to be 101.325 kPa.

Solution

The dynamics equation obtained in Example 7.2 is

$$\frac{RV}{R_{air}T}\frac{\mathrm{d}p}{\mathrm{d}t}+p=p_i,$$

where $RV/(R_{air}T) = 1.19 \times 10^{-2}$ s. Solving for the highest derivative of the output p gives

$$\dot{p}=84.03(101,325-p),$$

which can be represented by the block diagram shown in Figure 7.36. One `Integrator` block is used to form the container pressure p, which is fed back to form the variation rate $\dot{p}$. Note that

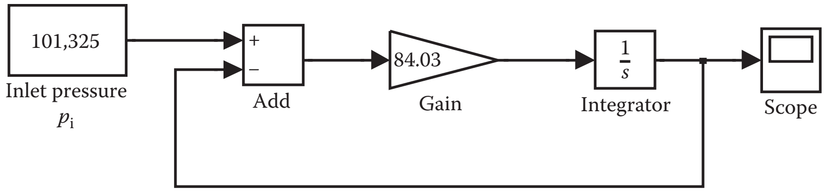


FIGURE 7.36 Simulink block diagram representing the pneumatic system in Example 7.2.

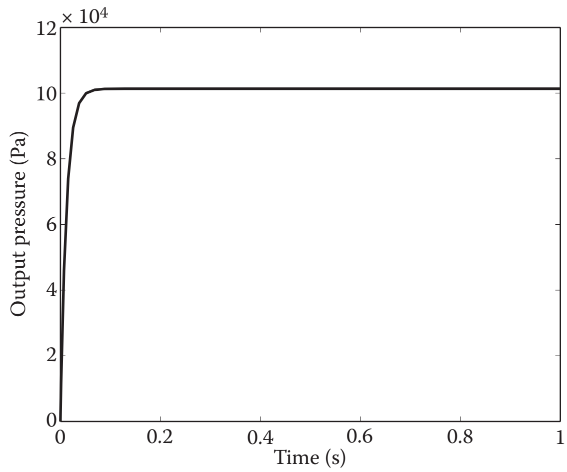


FIGURE 7.37 Pressure output $p(t)$ for constant inlet pressure p_i = 101.325 kPa.

the system input is the inlet pressure p_i, which is constant and is represented using a `Constant` block. Run the simulation. Double-click the `Scope` block and the resulting output of the pneumatic system $p(t)$ is shown in Figure 7.37.

Example 7.10: Modeling a Two-Tank Liquid-Level System with Simulink

Consider the two-tank liquid-level system in Example 7.5. Construct a Simulink block diagram to find the liquid levels $h_1(t)$ and $h_2(t)$. Assume ρ = 1000 kg/m^3, g = 9.81 m/s^2, A_1 = 2 m^2, A_2 = 3 m^2, $R_1 = R_2 = R_3$ = 400 N·s/(kg·m^2), and initial liquid heights $h_1(0)$ = 1 m and $h_2(0)$ = 0 m. The pump pressure Δp is a step function with a magnitude of 0 before t = 0 s and a magnitude of 130 kPa after t = 0 s.

Solution

The Simulink block diagram can be constructed based on either the differential equations obtained in Part (a) or the state–space form obtained in Part (b) in Example 7.5. Substituting the values of the parameters into the differential equations gives

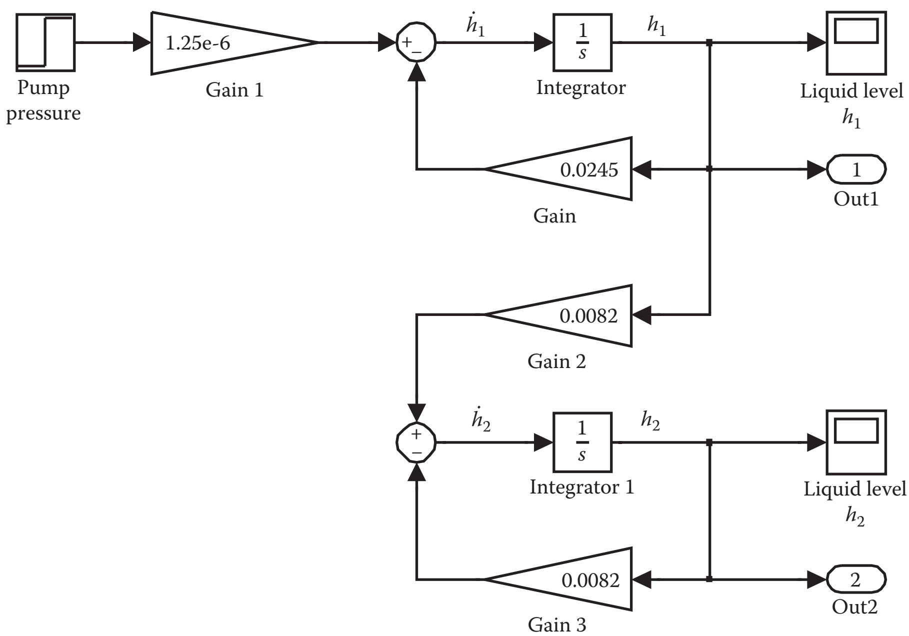


FIGURE 7.38 Simulink block diagram constructed based on differential equations.

$$\frac{dh_1}{dt} = -0.0245h_1 + 1.25\times10^{-6}\ \ p,$$

$$\frac{dh_2}{dt} = 0.0082h_1 - 0.0082h_2.$$

Figure 7.38 shows the resulting Simulink block diagram, in which two `Integrator` blocks are used to form h_1 and h_2. Double-clicking on each `Integrator` block, we can enter the initial liquid level for each tank.

Substituting the values of the parameters into the state–space equations gives

$$\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\end{Bmatrix} = \begin{bmatrix}-0.0245 & 0\\ 0.0082 & -0.0082\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} + \begin{bmatrix}1.25\times10^{-6}\\ 0\end{bmatrix}u,$$

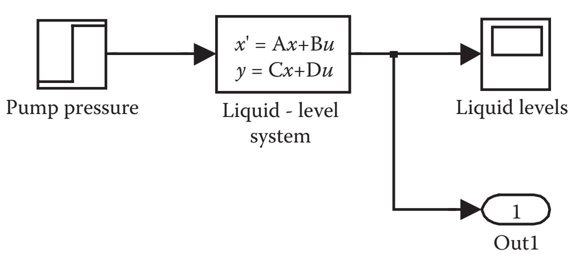


FIGURE 7.39 Simulink block diagram constructed based on state–space equations.

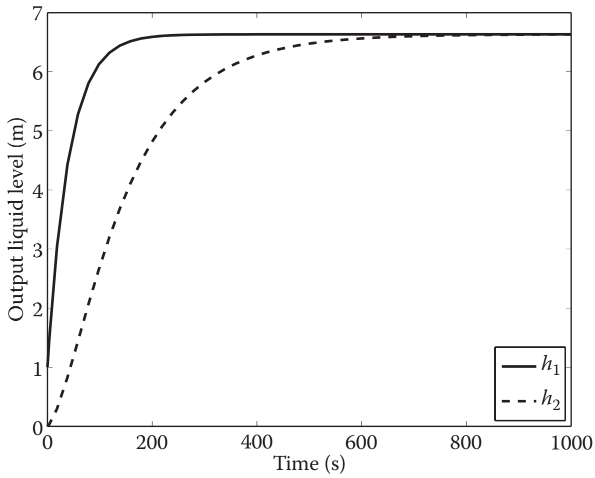


FIGURE 7.40 Liquid level outputs $h_1(t)$ and $h_2(t)$.

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u.$$

The Simulink block diagram based on the state–space form is shown in Figure 7.39, in which a `State-Space` block is used to represent the liquid-level system. Double-clicking on the `State-Space` block with the name `Liquid-level system`, we can define the matrices **A**, **B**, **C**, and **D**. The initial liquid level is the vector `[1; 0]`. Running either of the two simulations, we can obtain the same results as plotted in Figure 7.40.

To model a fluid or thermal system with Simscape, the `Simscape\Foundation Library` can be used, which contains basic pneumatic, hydraulic, and thermal blocks. However, connecting pneumatic or hydraulic components requires a good knowledge of the physical domains and the equations involved. Because this is beyond the scope of this text, this section focuses on Simscape modeling of thermal systems only.

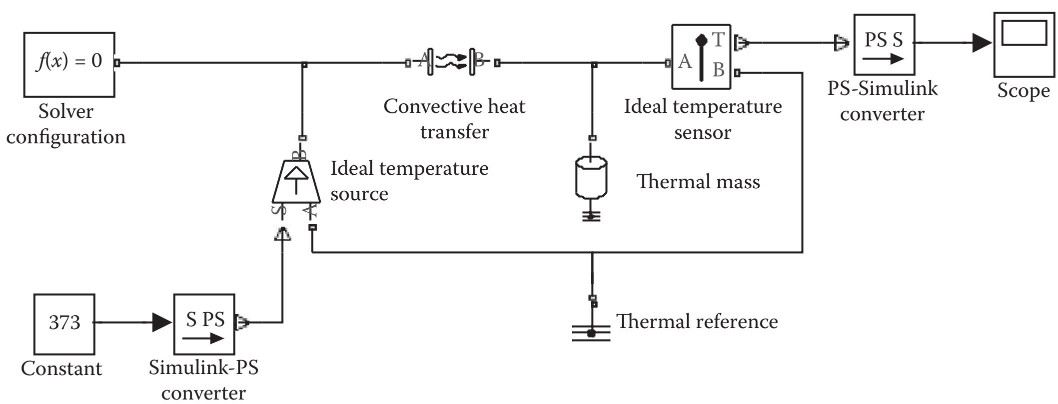


FIGURE 7.41 Simscape block diagram of the thermal system in Example 7.7.

Example 7.11: Temperature Dynamics of a Heated Object

Consider the heat transfer system in Example 7.7, in which a steel sphere is submerged in hot water and the temperature of the sphere is assumed to be uniform. Build a Simscape model of the physical system and find the sphere's temperature output $T(t)$. Compare the result with that obtained in Example 7.7.

Solution

The Simscape block diagram corresponding to the physical system is shown in Figure 7.41. The temperature of the hot water is known as 100°C or 373 K. The `Ideal Temperature Source` block in the library of `Simscape/Foundation Library/Thermal/Thermal Sources` is used to represent the temperature input. A `Simulink-PS Converter` block converts the constant value of 373 K into a physical signal. Double-click on the block and define the `Input signal unit` as `K`.

Because the temperature of the sphere is assumed to be uniform, only the convective heat transfer between the water and the sphere is considered in modeling. The corresponding block can be found in the library of `Simscape/Foundation Library/Thermal/Thermal Elements`. Double-click on the block and define `Area` as `0.0013` m^2, and `Heat transfer coefficient` as `350` W/(m^2·K). The block `Thermal Mass` in the same library is used to represent the steel sphere. The associated parameters are `Mass`, `Specific heat`, and `Initial temperature`, and their values are `0.0329` kg, `440` J/(kg·K), and `25°C` or `298 K`. All parameter values can be determined using the information given in Example 7.7.

To measure the sphere's temperature, drag the `Ideal Temperature Sensor` block in the library of `Simscape/Foundation Library/Thermal/Thermal Sensors`. The `PS-Simulink Converter` block converts the physical signal to a Simulink signal. Double-click on the block and define the `Output signal unit` as `K`. Because the simulation result in Example 7.7 is given in units of °C, we can also define the unit as `C` and check the box of `Apply affine conversion`.

Run the simulation and the same curve as shown in Figure 7.29 can be obtained, which is the resulting temperature output $T(t)$ of the heated sphere.

Example 7.12: Temperature Dynamics of Two Adjacent Objects

Figure 7.42 represents the temperature dynamics of two adjacent objects, in which the thermal capacitances of the objects are C_1 and C_2, respectively. Assume that the temperatures of both objects are uniform, and they are T_1 and T_2, respectively. The heat flow rate into object 1 is q_0, and the temperature surrounding object 2 is T_0. There are two modes of heat transfer involved, conduction between the objects and convection between object 2 and the air. The corresponding thermal resistances are R_1 and R_2, respectively.

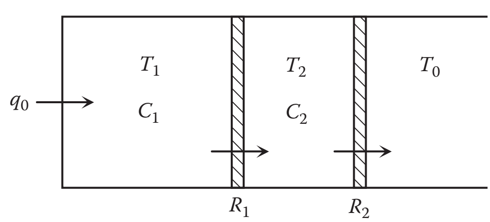


FIGURE 7.42 Thermal dynamics of two adjacent objects.

a. Derive the differential equations relating the temperatures T_1, T_2, the input q_0, and the outside temperature T_0.
b. Build a Simscape model of the physical system and find the temperature outputs $T_1(t)$ and $T_2(t)$. Use default values for the blocks of `Thermal Mass` (mass = 1 kg, specific heat = 447 J·K/kg, and initial temperature = 300 K), `Conductive Heat Transfer` (area = 1×10^{-4} m^2, thickness = 0.1 m, and thermal conductivity = 401 W/(m·K)), and `Convective Heat Transfer` (area = 1×10^{-4} m^2 and heat transfer coefficient = 20 W/(m^2·K)). Assume that the heat flow rate is q_0 = 400 J/s and the surrounding temperature is T_0 = 298 K.
c. Build a Simulink block diagram based on the differential equations obtained in Part (a) and find the temperature outputs $T_1(t)$ and $T_2(t)$.

Solution

a. Similar to Example 7.8, the differential equations relating the temperatures T_1, T_2, the input q_0, and the outside temperature T_0 can easily be obtained as

$$C_1 \frac{dT_1}{dt} = q_0 - \frac{T_1 - T_2}{R_1},$$

$$C_2 \frac{dT_2}{dt} = \frac{T_1 - T_2}{R_1} - \frac{T_2 - T_0}{R_2}.$$

The equations can be rearranged as

$$C_1 \frac{dT_1}{dt} + \frac{1}{R_1} T_1 - \frac{1}{R_1} T_2 = q_0,$$

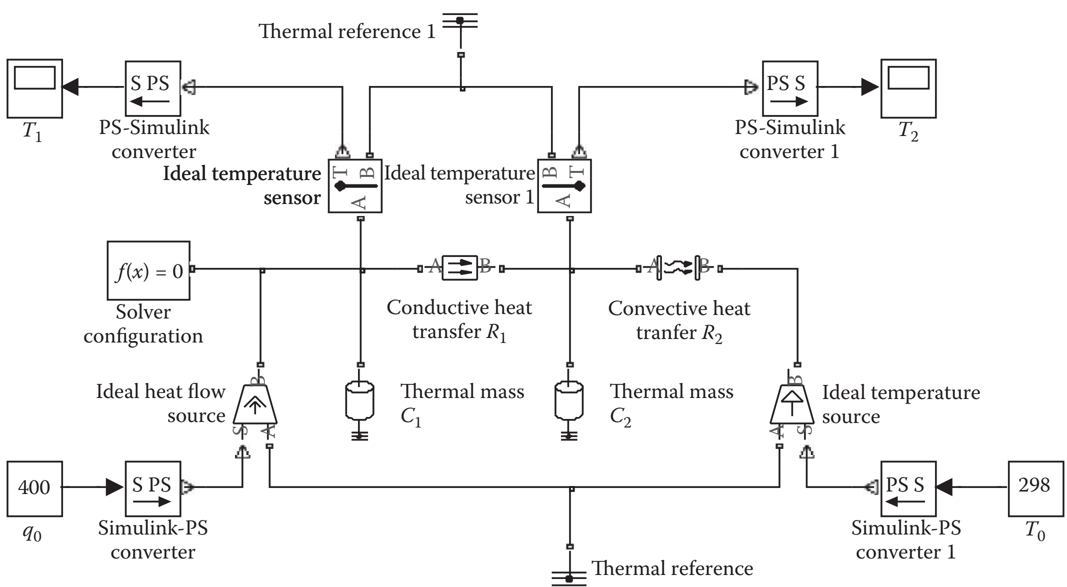


FIGURE 7.43 Simscape block diagram of the thermal system in Figure 7.42.

$$C_2\frac{dT_2}{dt}-\frac{1}{R_1}T_1+\left(\frac{1}{R_1}+\frac{1}{R_2}\right)T_2=\frac{1}{R_2}T_0.$$

b. The Simscape block diagram corresponding to the physical system is shown in Figure 7.43. Two `Thermal Mass` blocks are used to represent objects 1 and 2. A `Conductive Heat Transfer` block and a `Convective Heat Transfer` block are used to represent the two modes of heat transfer involved, conduction between the objects and convection between object 2 and the air. An `Ideal Heat Flow Source` block is included to represent the heat flow rate input q_0, and an `Ideal Temperature Source` block is used to represent the surrounding temperature T_0.

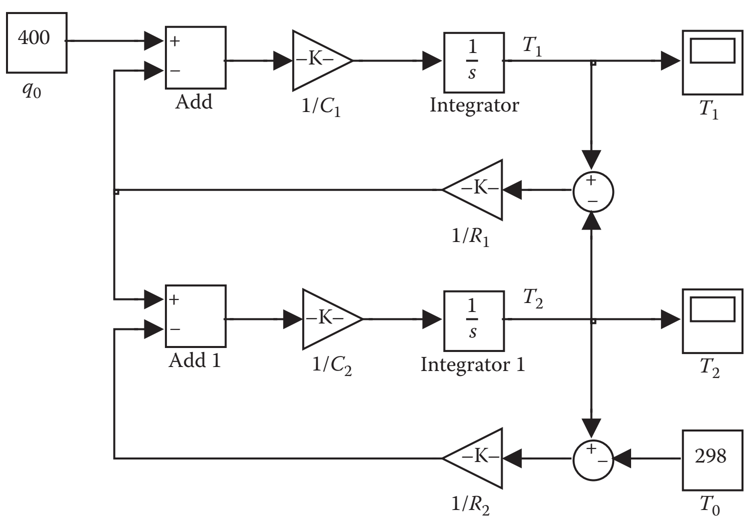


FIGURE 7.44 Simulink block diagram of the thermal system in Figure 7.42.

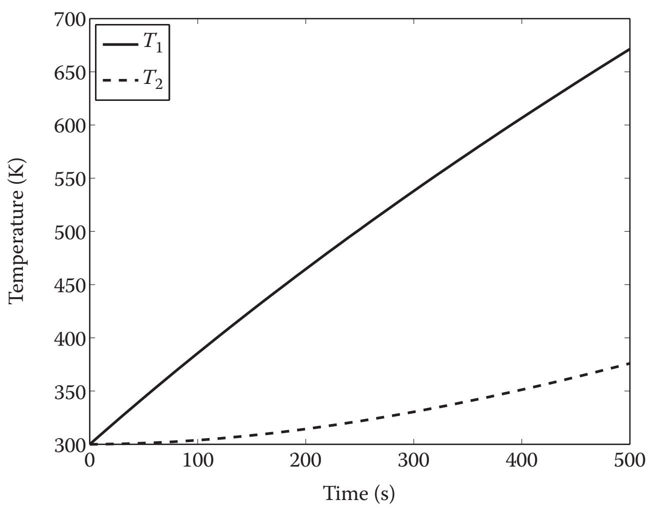


FIGURE 7.45 Temperature outputs $T_1(t)$ and $T_2(t)$.

c. The Simulink block diagram built based on the differential equations obtained in Part (a) is shown in Figure 7.44. Based on the default values used in Simscape modeling, the following system parameters can be determined,

$$C_1 = C_2 = mc = 1 \times 447 = 447 \text{ J/K},$$

$$R_1 = \frac{L}{kA} = \frac{0.1}{401 \times 1 \times 10^{-4}} = 2.49 \text{ K·s/J},$$

$$R_2 = \frac{1}{hA} = \frac{1}{20 \times 1 \times 10^{-4}} = 500 \text{ K·s/J}.$$

Run either simulation to generate Figure 7.45, showing the resulting temperature outputs $T_1(t)$ and $T_2(t)$ of the two adjacent objects.

PROBLEM SET 7.4

1. Dry air at a constant temperature of 20°C passes through a valve out of a rigid cubic container of 1 m on each side (see Figure 7.46). The pressure p_o at the outlet of the valve is constant, and it is less than p. The valve resistance is approximately linear, and R = 1000 Pa·s/kg. Assume the process is isothermal.
   a. Develop a mathematical model of the pressure p in the container.
   b. Construct a Simulink block diagram to find the output $p(t)$ of the pneumatic system if the pressure inside the container initially is 2 atm and the pressure at the outlet is 1 atm.
2. Figure 7.47 shows a liquid-level system in which two tanks have hydraulic capacitances C_1 and C_2, respectively. The volume flow rate into tank 1 is q_i. The liquid flows from tank 1 to tank 2 through a valve of linear resistance R_1 and leaves tank 2 through a valve of linear resistance R_2. The density ρ of the liquid is constant.
   a. Derive the differential equations in terms of the liquid heights h_1 and h_2. Write the equations in second-order matrix form.

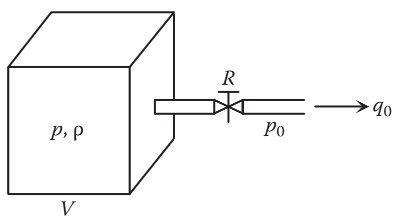


FIGURE 7.46 Problem 1.

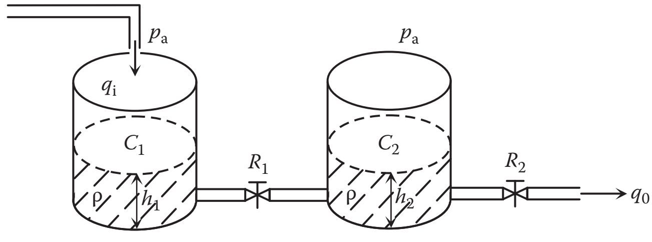


FIGURE 7.47 Problem 2.

b. Assume the volume flow rate q_i is the input and the liquid heights h_1 and h_2 are the outputs. Determine the state–space form of the system.
c. Construct a Simulink block diagram to find the outputs $h_1(t)$ and $h_2(t)$ of the liquid-level system. Assume $\rho = 1000$ kg/m^3, $g = 9.81$ m/s^2, $C_1 = 0.2$ kg·m^2/N, $C_2 = 0.3$ kg·m^2/N, $R_1 = R_2 = 400$ N·s/(kg·m^2), and initial liquid heights $h_1(0) = 1$ m and $h_2(0) = 0$ m. The volume flow rate q_i is a step function with a magnitude of 0 before $t = 0$ s and a magnitude of 0.5 m^3/s after $t = 0$ s.

3. A chicken is taken out of the oven at a uniform temperature of 150°C and is left out in the open air at the room temperature of 25°C. Assume that the chicken can be approximated as a lumped model. The estimated parameters are mass $m = 2$ kg, heat transfer surface area $A = 0.32$ m^2, specific heat capacity $c = 3220$ J/(kg·°C), and heat transfer coefficient $h = 15$ W/(m^2·°C).
   a. Derive the differential equation relating the chicken's temperature $T(t)$ and the room temperature.
   b. Using the differential equation obtained in Part (a), construct a Simulink block diagram and find the temperature of the chicken.
   c. Build a Simscape model of the system and find the temperature of the chicken.
   d. Assume that the chicken can be served only if its temperature is higher than 80°C. Based on the simulation results obtained in Parts (b) and (c), can the chicken be left at the room temperature of 25°C for 1 hour?

7.5 SUMMARY

This chapter was devoted to the modeling of fluid and thermal systems. For each of them, we first introduced the concepts of capacitance and resistance. It is useful to think of fluid and thermal systems as electrical circuits. Together with the basic elements, the conservation of mass and the conservation of energy are the main laws used to develop mathematical models of fluid and thermal systems, respectively.

Fluid systems can be divided into pneumatics and hydraulics. A pneumatic system is one in which the fluid is compressible. At low pressure and moderate or high temperature, real gases may be approximated as ideal gases to simplify calculations for pneumatic systems. A hydraulic system is one in which the fluid is incompressible. Most liquids are generally considered incompressible, and this approximation greatly simplifies the modeling of hydraulic systems. A general category of hydraulic systems is liquid-level systems.

Fluid capacitance is the relation between the stored fluid mass and the resulting pressure caused by the stored mass,

$$C = \frac{\mathrm{d}m}{\mathrm{d}p}.$$

The pneumatic capacitance of a container of constant volume V is defined as

$$C = \frac{V}{nR_g T},$$

where the value of n depends on the type of thermodynamic process. For a tank of cross-sectional area A with a liquid of height h, the hydraulic capacitance of the tank is defined as

$$C = \frac{A(h)}{g}.$$

When a fluid flows through a valve, a pipe, or an orifice, the fluid meets resistance and there is a decrease in the pressure of the fluid. The pressure difference is associated with the mass flow rate q_m in a nonlinear relationship. Near a reference operating point, linearization can be performed to obtain the linearized resistance,

$$R = \frac{\Delta p}{\Delta q_m}.$$

To obtain a simple model of a fluid system, each mass storage element can be represented by a capacitance element and each valve can be represented by a resistance element. The differential equation of the system can be derived by applying the law of conservation of mass:

$$\frac{dm}{dt} = q_{mi} - q_{mo},$$

where the mass flow rate into or out of the system can be related to the resistance at the inlet or the outlet of the system, respectively. The resulting mathematical model may adequately describe the dynamics of the real system.

A thermal system is one that involves the transfer of heat from one object to another. For an object, the thermal capacitance is defined as the ratio of the change in heat flow to the change in the object's temperature,

$$C = \frac{dQ}{dT}.$$

For incompressible liquids and solids,

$$C = mc = \rho Vc,$$

where c is the specific heat capacity.

The thermal resistance for heat transfer is defined as the ratio of the change in temperature difference to the change in heat flow rate,

$$R = \frac{dT}{dq_h}.$$

For simple one-dimensional conduction, Fourier's law gives

$$q_h = kA\frac{\Delta T}{L} = kA\frac{T_1 - T_2}{L}$$

and the thermal resistance for conduction is

$$R = \frac{L}{kA}.$$

For convective heat transfer, Newton's law of cooling gives

$$q_h = hA\Delta T = hA(T_s - T_{env})$$

and the thermal resistance for convection is

$$R = \frac{1}{hA}.$$

The mathematical model of a thermal system is often complicated because of the complex temperature distribution throughout the system. To simplify analysis, a lumped-parameter model may be used to approximate the gross system dynamics. The validity of this lumped-parameter assumption can be checked using the so-called Biot number,

$$Bi = \frac{hL_c}{k},$$

where L_c is the characteristic length of the solid object,

$$L_c = \frac{V_{body}}{A_{surface}}.$$

For a body whose Biot number is much less than one, that is, $Bi < 0.1$, the interior of the body may be assumed to have a uniform temperature. Then, the dynamic model of a heat transfer system can be derived using the law of conservation of energy,

$$\frac{dU}{dt} = q_{hi} - q_{ho}.$$

REVIEW PROBLEMS

1. Dry air at a constant temperature of T passes through a valve into a rigid spherical container (see Figure 7.48). The pressure at the inlet is p_i. The linear resistances of the two valves at the inlet and the outlet are R_1 and R_2, respectively. Assume that the process is isothermal.
   a. Develop a mathematical model of the pressure p in the container.
   b. Denote the volume flow rate at the outlet as q_o. Determine the transfer function relating p_i and p for this pneumatic system if $q_o = 0$.
2. A single-tank liquid-level system is shown in Figure 7.49, in which water flows into the tank at a volume flow rate q_i and out of the tank through two valves at points 1 and 2. The linear resistances of the two valves are R_1 and R_2, respectively. Assuming $h > h_1$, derive the differential equation relating the liquid height h and the volume flow rate q_i at the inlet. The cross-sectional area of the tank A is constant. The density ρ of the liquid is constant.
3. A watermelon is taken out of the refrigerator at a uniform temperature of 5°C and is exposed to 27°C air. Assume that the watermelon can be approximated as a sphere and the temperature of the watermelon is uniform. The estimated parameters are the density

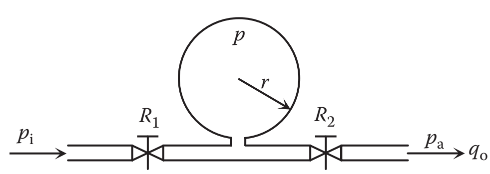


FIGURE 7.48 Problem 1.

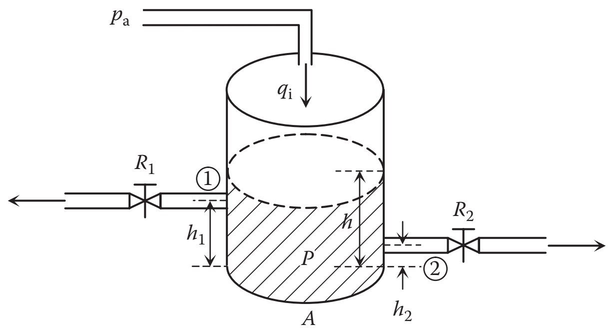


FIGURE 7.49 Problem 2.

ρ = 120 kg/m^3, diameter D = 40 cm, specific heat capacity c = 4200 J/(kg·°C), and heat transfer coefficient of the watermelon h = 15 W/(m^2·°C).

a. Derive the differential equation relating the watermelon's temperature $T(t)$ and the air temperature.
b. Using the differential equation obtained in Part (a), construct a Simulink block diagram and find the temperature of the watermelon.
c. Build a Simscape model of the system.
d. Based on the simulation results obtained in Parts (b) and (c), how long will it take before the watermelon is warmed up to 20°C?

8 System Response

Derivations of the mathematical models of dynamic systems were presented in Chapters 5 through 7. And the different forms of system model representations were thoroughly discussed in Chapter 4. This chapter is mainly concerned with the transient response and frequency-response analysis of dynamic systems. Systematic methods to solve the state equation will also be presented. For the most part, linear dynamic systems are considered in this chapter. The response of nonlinear systems using MATLAB® and Simulink® is presented in the final section of the chapter.

8.1 TYPES OF RESPONSE

Consider an nth-order dynamic system governed by the differential equation

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1}\dot{x} + a_n x = f(t) \tag{8.1}$$

where the coefficients $a_1, a_2, \ldots, a_n$ are constants, $x(t)$ is the dependent variable, t denotes time, and $f(t)$ is the input known as the forcing function. As seen in Chapter 2, the solution $x(t)$, known as the total response, comprises the complementary (homogeneous) solution $x_c(t)$ and the particular solution $x_p(t)$, that is, $x(t) = x_c(t) + x_p(t)$. Note that $x_c(t)$ is the solution of Equation 8.1 when $f(t) = 0$, and $x_p(t)$ depends on the nature of the forcing function $f(t)$. The complementary solution $x_c(t)$ is called the natural response (free response) because it represents the natural behavior of the system in the absence of the input. The particular solution $x_p(t)$ is known as the forced response of the system.

8.2 TRANSIENT RESPONSE AND STEADY-STATE RESPONSE

The total response $x(t)$ can also be decomposed into transient response $x_{tr}(t)$ and steady-state response $x_{ss}(t)$, that is, $x(t) = x_{tr}(t) + x_{ss}(t)$. The transient response consists of those terms in $x(t)$ that decay to zero as $t \to \infty$. The portion of the response $x(t)$ that remains after the transient terms have vanished is called the steady-state response.

Example 8.1: Transient and Steady-State Responses

The mathematical model of a dynamic system is defined by

$$\ddot{x} + 4x = 17e^{-t/2}, \quad x(0) = 4, \quad \dot{x}(0) = 0$$

Find the system's total response, and identify the transient and steady-state responses.

Solution

The complementary solution is readily found as $x_c(t) = c_1\cos 2t + c_2\sin 2t$. The particular solution is found via the method of undetermined coefficients (Chapter 2) as $x_p(t) = 4e^{-t/2}$. Therefore,

$$x(t) = c_1\cos 2t + c_2\sin 2t + 4e^{-t/2}$$

Initial conditions yield $c_1 = 0$, $c_2 = 1$. The total response is then formed as

$$x(t) = \sin 2t + 4e^{-t/2}$$

Consequently, $x_{tr}(t) = 4e^{-t/2}$ and $x_{ss}(t) = \sin 2t$.

8.2.1 Transient Response of First-Order Systems

Linear, first-order dynamic systems are described by

$$\tau\dot{x} + x = f(t), \quad \tau = \text{const} > 0, \quad x(0) = x_0 \tag{8.2}$$

where τ is called the time constant. Note that upon division by τ, Equation 8.2 agrees with the general form of Equation 8.1. To perform the transient-response analysis, we first take the Laplace transform of Equation 8.2, as

$$\tau[sX(s) - x_0] + X(s) = F(s) \quad \xrightarrow[\text{and rearrange}]{\text{Simplify}} \quad X(s) = \frac{\tau}{\tau s + 1}x_0 + \frac{1}{\tau s + 1}F(s)$$

Subsequently, inverse Laplace transformation yields

$$x(t) = \mathcal{L}^{-1}\left\{\frac{\tau}{\tau s + 1}\right\}x_0 + \mathcal{L}^{-1}\left\{\frac{1}{\tau s + 1}F(s)\right\}$$

Noting the inverse Laplace transform in the first term is $e^{-t/\tau}$, the total response is expressed as

$$x(t) = \underbrace{\boxed{e^{-t/\tau}x_0}}_{\text{Zero-input response}} + \underbrace{\boxed{\mathcal{L}^{-1}\left\{\frac{1}{\tau s + 1}F(s)\right\}}}_{\text{Zero-state response}} \tag{8.3}$$

The first term on the right side of Equation 8.3 represents the system response to initial condition (excitation) only, and is called the zero-input response. The second term describes the response to the input, and is known as the zero-state response. In this section, we will study the response of first-order systems subjected to specific types of input such as step and ramp functions. The system's free response is discussed first.

8.2.1.1 Free Response of First-Order Systems

Free response is defined as the response to the initial condition only, hence given by the first term in Equation 8.3,

$$x(t) = e^{-t/\tau}x_0 \tag{8.4}$$

It is readily seen that the smaller the time constant, the faster the response reaches equilibrium.

Example 8.2: Free Response

Suppose a first-order system is described by

$$3\dot{x} + 2x = 0, \quad x(0) = \tfrac{1}{2}$$

The response $x(t)$ is clearly a free (natural) response as there is no forcing function present. Rewriting the ODE as $\frac{3}{2}\dot{x} + x = 0$, the time constant is found as $\tau = \frac{3}{2}$. With this, and noting $x(0) = \frac{1}{2}$, Equation 8.4 yields

$$x(t) = \tfrac{1}{2}e^{-2t/3}$$

8.2.1.2 Impulse Response of First-Order Systems

Impulse response refers to $x(t)$ in Equation 8.2 when $f(t) = A\delta(t)$, where $\delta(t)$ is the unit-impulse and A is a constant magnitude (Section 2.3). Inserting $F(s) = A$ in Equation 8.3, we find

$$x(t) = e^{-t/\tau}x_0 + \mathcal{L}^{-1}\left\{\frac{A}{\tau s + 1}\right\}$$

which simplifies to

$$x(t) = e^{-t/\tau}x_0 + \frac{A}{\tau}e^{-t/\tau} \tag{8.5}$$

Because $\tau > 0$, we have $x(t) \to 0$ as $t \to \infty$. That is, the impulse response has a steady-state value of 0.

8.2.1.3 Step Response of First-Order Systems

Step response refers to $x(t)$ in Equation 8.2 when $f(t) = Au(t)$, where $u(t)$ is the unit-step and A is a constant magnitude (Section 2.3). Inserting $F(s) = A/s$ in Equation 8.3, we find

$$x(t) = e^{-t/\tau}x_0 + \mathcal{L}^{-1}\left\{\frac{A}{s(\tau s + 1)}\right\}$$

Using partial-fraction expansion, it can be shown that

$$\mathcal{L}^{-1}\left\{\frac{A}{s(\tau s + 1)}\right\} = A(1 - e^{-t/\tau})$$

Therefore, the step response is described by

$$x(t) = e^{-t/\tau}x_0 + A(1 - e^{-t/\tau}) \tag{8.6}$$

Because $\tau > 0$, we have $x(t) \to A$ as $t \to \infty$. In other words, the step response has a steady-state value of A.

Example 8.3: Step Response of an RL Circuit

Consider the RL circuit shown in Figure 8.1a, which consists of a resistance R and inductance L, and assume that the initial current is $i(0) = i_0$. If the applied voltage is modeled as a step function with magnitude V, find the zero-input, zero-state, and steady-state responses.

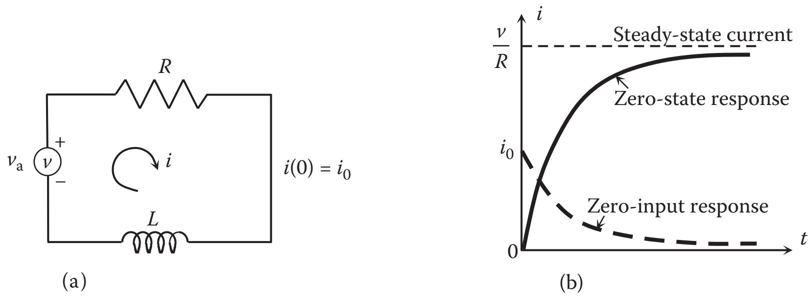


FIGURE 8.1 (a) RL circuit, and (b) response to a step input.

Solution

Using Kirchhoff's voltage law (KVL), the mathematical model of the circuit is obtained as

$$L\frac{\mathrm{d}i}{\mathrm{d}t}+Ri=v_{\mathrm{a}}(t),\quad i(0)=i_0$$

Division of the differential equation by R yields

$$\frac{L}{R}\frac{\mathrm{d}i}{\mathrm{d}t}+i=\frac{1}{R}v_{\mathrm{a}}(t) \qquad \underbrace{\tau=\frac{L}{R}}_{\text{Time constant}}$$

Comparing this equation with Equation 8.2, we find $f(t) = (1/R)v_a(t)$. But because $v_a(t) = Vu(t)$, the forcing function can be written as $f(t) = (V/R)u(t)$. Therefore, we can interpret $f(t)$ as a step function with magnitude V/R. As a result, using $A = V/R$ and $x_0 = i_0$ in Equation 8.6, we find

$$i(t)=e^{-Rt/L}i_0+\frac{V}{R}(1-e^{-Rt/L})$$

The first term represents the zero-input response, whereas the second term describes the zero-state response. It is clear that $i(t)$ reaches a steady-state value of V/R after a sufficiently long time (Figure 8.1b), hence the steady-state current is $i_{ss} = V/R$.

8.2.1.3.1 The Role of Time Constant

The role of time constant τ is examined as follows. Consider the RL circuit in Example 8.3, and assume zero initial current, $i(0) = 0$. As a result, the current is given by

$$i(t)=\frac{V}{R}(1-e^{-t/\tau})$$

where we have replaced L/R with the generic notation τ. After one time constant ($t = \tau$), the current is

$$i(\tau)=\frac{V}{R}(1-e^{-1})=0.632\left(\frac{V}{R}\right)=0.632i_{ss}$$

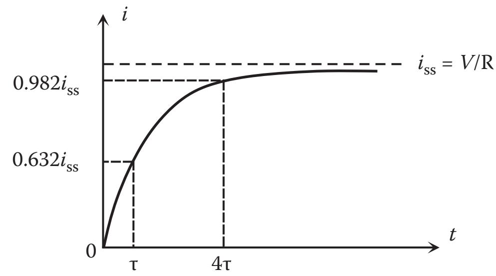


FIGURE 8.2 Role of time constant.

This means 63.2% of the steady-state current is recovered after one time constant. Similarly, the percentage of recovery at $t = 2\tau$ and $t = 3\tau$ may be calculated. At $t = 4\tau$, we have

$$i(4\tau) = \frac{V}{R}(1 - e^{-4}) = 0.982\left(\frac{V}{R}\right) = 0.982 i_{ss}$$

Therefore, after four time constants, the response is within 2% of the steady-state value (see Figure 8.2). This essentially serves as the settling time for the step response curve. Settling time is one of the four transient-response specifications of second-order systems and plays a central role in the control of such systems (see Chapter 10).

8.2.1.4 Ramp Response of First-Order Systems

Ramp response refers to response $x(t)$ in Equation 8.2 when the forcing function is $f(t) = Au_r(t)$, where $u_r(t)$ is the unit-ramp and A is a constant slope (Section 2.3). Inserting $F(s) = A/s^2$ in Equation 8.3, we find

$$x(t) = e^{-t/\tau}x_0 + \mathcal{L}^{-1}\left\{\frac{A}{s^2(\tau s + 1)}\right\}$$

But

$$\mathcal{L}^{-1}\left\{\frac{A}{s^2(\tau s + 1)}\right\} = A[t - \tau(1 - e^{-t/\tau})]$$

Therefore, the ramp response is given by

$$x(t) = e^{-t/\tau}x_0 + A[t - \tau(1 - e^{-t/\tau})] \tag{8.7}$$

8.2.1.4.1 Steady-State Error

For simplicity, assume the initial condition is zero so that Equation 8.7 reduces to

$$x(t) = A[t - \tau(1 - e^{-t/\tau})]$$

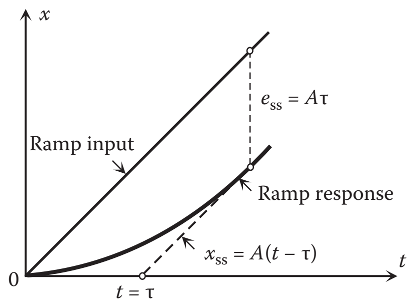


FIGURE 8.3 Ramp response of a first-order system.

The error between the ramp input and the ramp response is

$$e(t) = At - A[t - \tau(1 - e^{-t/\tau})] = A\tau(1 - e^{-t/\tau})$$

As $t \to \infty$, this error approaches $A\tau$ so that the steady-state error is

$$e_{ss} = A\tau$$

The ramp input, ramp response, and the associated steady-state error are shown in Figure 8.3.

PROBLEM SET 8.1

1. Consider a first-order system with time constant τ and zero initial condition. Find the system's unit-impulse response for $\tau = \frac{1}{4}$ and $\tau = \frac{1}{2}$, plot the two curves versus $0 \le t \le 2$ in the same graph, and comment.
2. Consider a first-order system with time constant τ and zero initial condition. Find the system's unit-step response for $\tau = \frac{1}{3}$ and $\tau = \frac{2}{3}$, plot the two curves versus $0 \le t \le 2$ in the same graph, and comment.
3. A thermostat, initially at ambient temperature $T(0) = T_a$, is placed inside a water tank whose temperature is fixed at T_b. The thermostat temperature is the response of $\tau\dot{T} + T = T_b$ where τ = const depends on thermal resistance and capacitance.
   a. Find the zero-state and zero-input responses.
   b. Find the steady-state thermostat temperature.
4. Repeat Problem 3 for the case when the temperature of the water tank increases linearly with time at a rate of r.
5. A single-tank liquid-level system with inflow rate q_i as its input and liquid level h as its output is modeled as $RA\dot{h} + gh = Rq_i(t)$, $h(0) = 0$, where R,A,g = const. If the inflow rate is a unit-step, find the system response in terms of the physical parameters. Also find the steady-state response.
6. The equation of motion of the mechanical system in Figure 8.4 is $b\dot{y} + k(y - x) = 0$, where x and y are the input and the output, respectively, and b,k = const. Assuming zero initial condition, find the response when x is a
   a. unit step
   b. unit ramp

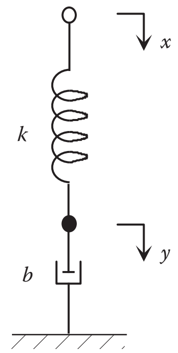


FIGURE 8.4 Problem 6.

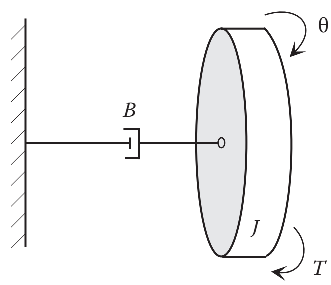


FIGURE 8.5 Problem 7.

7. The torsional mechanical system in Figure 8.5 is modeled as $J\ddot{\theta} + B\dot{\theta} = T(t)$, where J,B = const, θ is the angular displacement, and T is a constant applied torque. Rewrite the model as first-order in angular velocity $\omega = \dot{\theta}$. Assuming $\omega(0) = \omega_0$, determine $\omega(t)$. Also identify the transient and steady-state responses.
8. Find the unit-ramp response of the RL circuit in Example 8.3.
9. A first-order dynamic system is modeled as

$$\tfrac{1}{2}\dot{v} + 5v = F(t), \quad v(0) = \tfrac{2}{3}$$

Find the response $v(t)$ if the input $F(t)$ is a ramp function with a slope of $\frac{3}{2}$. Also find the steady-state response and the steady-state error.

10. A first-order dynamic system is modeled as

$$\dot{w} + 3w = g(t), \quad w(0) = 1$$

Find $w(t)$ if the input $g(t)$ is a step function with magnitude 10. Also find w_{ss}. How many time units will it take for the response curve to reach within 2% of w_{ss}?

8.3 TRANSIENT RESPONSE OF SECOND-ORDER SYSTEMS

Linear, second-order dynamic systems are mathematically modeled as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 \tag{8.8}$$

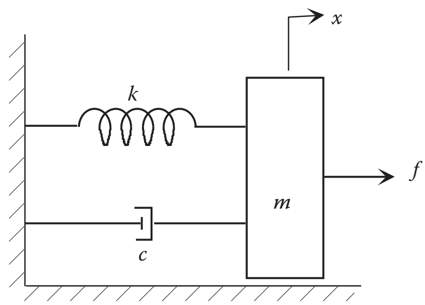


FIGURE 8.6 A mass–spring–damper system.

where ζ is the damping ratio and ω_n is the (undamped) natural frequency, in radians per second. Although Equation 8.8 represents the model for any second-order dynamic system, it is best understood when it is viewed in relation to a mechanical system. To that end, consider the mass–spring–damper system in Figure 8.6.

The system's (undamped) natural frequency is defined as

$$\omega_n = \sqrt{\frac{k}{m}} \text{ rad/s}$$

The damping ratio is defined as the ratio of the actual damping c and the critical damping* $c_{cr} = 2\sqrt{mk}$, that is,

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{mk}}$$

The system's equation of motion is derived as (Chapter 5)

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad \xrightarrow{\text{Divide by } m} \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}f(t)$$

Noting that

$$\frac{k}{m} = \omega_n^2, \quad \frac{c}{m} = 2\frac{c}{2\sqrt{mk}}\sqrt{\frac{k}{m}} = 2\zeta\omega_n$$

the equation of motion can be expressed in the form of Equation 8.8, with the force $(1/m)f(t)$ renamed as $f(t)$. The transient-response analysis of second-order systems is performed as follows. Taking the Laplace transform of Equation 8.8, taking into account the initial conditions, we find

$$[s^2X(s) - sx_0 - \dot{x}_0] + 2\zeta\omega_n[sX(s) - x_0] + \omega_n^2X(s) = F(s)$$

* This is defined as the value of c that satisfies $c^2 - 4mk = 0$.

Collecting like terms and solving for $X(s)$, yields

$$X(s) = \frac{(s + 2\zeta\omega_n)x_0 + \dot{x}_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} F(s)$$

Finally,

$$x(t) = \underbrace{\boxed{\mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x_0 + \dot{x}_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\}}}_{\text{Zero-input response}} + \underbrace{\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} F(s)\right\}}}_{\text{Zero-state response}} \tag{8.9}$$

The first term on the right side represents the zero-input response, which is the response to initial conditions only. The second term describes the zero-state response, which is the response to the input only.

8.3.1 Free Response of Second-Order Systems

Free response is the response to initial conditions only, defined by the first term in Equation 8.9,

$$x(t) = \mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x_0 + \dot{x}_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} \tag{8.10}$$

This inverse Laplace transform depends on the nature of the poles (Section 2.3), that is, the roots of the characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Solution of this equation yields the poles

$$s = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{8.11}$$

The system is undamped if $\zeta = 0$, underdamped if $0 < \zeta < 1$, critically damped if $\zeta = 1$, and overdamped if $\zeta > 1$. The free response analysis for all these damping cases is conducted as follows.

Case (1) Undamped ($\zeta = 0$)

Inserting $\zeta = 0$ in Equation 8.10, we find

$$x(t) = \mathcal{L}^{-1}\left\{\frac{sx_0 + \dot{x}_0}{s^2 + \omega_n^2}\right\} = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t \tag{8.12}$$

Case (2) Underdamped $(0 < \zeta < 1)$
In this case, rewrite

$$\begin{aligned} s^2 + 2\zeta\omega_n s + \omega_n^2 &= (s + \zeta\omega_n)^2 - (\zeta\omega_n)^2 + \omega_n^2 \\ &= (s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2) \\ &= (s + \zeta\omega_n)^2 + \omega_d^2 \end{aligned}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the (damped) natural frequency. With this, Equation 8.10 yields

$$x(t) = \mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x_0 + \dot{x}_0}{(s + \zeta\omega_n)^2 + \omega_d^2}\right\} = e^{-\zeta\omega_n t}\left[x_0\cos\omega_d t + \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\sin\omega_d t\right] \tag{8.13}$$

Case (3) Critically Damped $(\zeta = 1)$
Using $\zeta = 1$ in Equation 8.10 leads to

$$x(t) = \mathcal{L}^{-1}\left\{\frac{(s + 2\omega_n)x_0 + \dot{x}_0}{(s + \omega_n)^2}\right\} = e^{-\omega_n t}\left[x_0 + (\omega_n x_0 + \dot{x}_0)t\right] \tag{8.14}$$

Case (4) Overdamped $(\zeta > 1)$
In this case, the two poles are real and distinct as in Equation 8.11. Let

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \quad s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Then

$$x(t) = \mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x_0 + \dot{x}_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} = \frac{-s_2 x_0 + \dot{x}_0}{s_1 - s_2}e^{s_1 t} - \frac{-s_1 x_0 + \dot{x}_0}{s_1 - s_2}e^{s_2 t} \tag{8.15}$$

Note that besides the response for the undamped case, which is oscillatory, all other responses stabilize at zero. The response for the underdamped case exhibits decaying oscillations.

8.3.1.1 Initial Response in MATLAB

The `initial` command calculates the free response of a state–space model (Section 4.2) with an initial condition on the states:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \ \ \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}$$

Then, `initial(sys,x0)` plots the response of system `sys` to an initial condition vector `x0`. The duration of simulation is determined automatically to adequately reflect the response transients.

Example 8.4: Free Response

Consider

$$\ddot{x} + 2\dot{x} + 2x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 1$$

a. Determine $x(t)$.
b. Find and plot the free response in MATLAB.

Solution

a. Comparing with the standard form, Equation 8.8, we have

$$\begin{cases} \omega_n^2 = 2 \\ 2\zeta\omega_n = 2 \end{cases} \qquad \begin{cases} \omega_n = \sqrt{2} \text{ rad/s} \\ \zeta = \dfrac{1}{\sqrt{2}} < 1 \text{ (underdamped)} \end{cases}$$

The characteristic equation $s^2 + 2s + 2 = 0$ has roots $s_{1,2} = -1 \pm j$. Because the system is underdamped, we also calculate $\omega_d = \omega_n\sqrt{1-\zeta^2} = 1$ rad/s. Using these and the given initial conditions in Equation 8.13, we find $x(t) = e^{-t}\sin t$.

b. Choosing the state variables as $x_1 = x$ and $x_2 = \dot{x}$, the state-variable equations are

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 - 2x_2 \end{cases} \quad \text{so that} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

Because we are seeking x, which happens to be the first state x_1, the output equation is $y = x_1$ so that $\mathbf{C} = [1 \quad 0]$. The following script will generate the plot of x versus t.

```
>> A = [0 1;-2 -2]; C = [1 0]; x0 = [0;1];
>> sys = ss(A,[],C,[]);
>> initial(sys,x0)    % Figure 8.7
```

Note that `t = 8(sec)` has been automatically determined by the `initial` command. As expected, the result agrees with that in Part (a).

8.3.2 Impulse Response of Second-Order Systems

Impulse response refers to $x(t)$ in Equation 8.8 when the input is $f(t) = A\delta(t)$, where $\delta(t)$ is the unit-impulse and A is a constant magnitude. Using $F(s) = A$ in Equation 8.9, the response is

$$x(t) = \mathcal{L}^{-1}\left\{\frac{(s + 2\zeta\omega_n)x_0 + \dot{x}_0 + A}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} \tag{8.16}$$

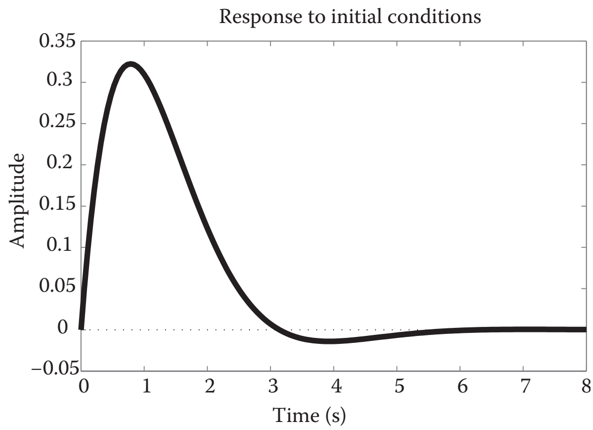


FIGURE 8.7 Initial response (Example 8.4).

As in free response analysis, different damping cases are studied separately as follows.
Case (1) Undamped ($\zeta = 0$)

$$x(t) = x_0 \cos\omega_n t + \frac{\dot{x}_0 + A}{\omega_n} \sin\omega_n t \tag{8.17}$$

Case (2) Underdamped ($0 < \zeta < 1$)

$$x(t) = e^{-\zeta\omega_n t}\left[x_0 \cos\omega_d t + \frac{\zeta\omega_n x_0 + \dot{x}_0 + A}{\omega_d} \sin\omega_d t\right] \tag{8.18}$$

Case (3) Critically Damped ($\zeta = 1$)

$$x(t) = e^{-\omega_n t}[x_0 + (\omega_n x_0 + \dot{x}_0 + A)t] \tag{8.19}$$

Case (4) Overdamped ($\zeta > 1$)

$$x(t) = \frac{-s_2 x_0 + \dot{x}_0 + A}{s_1 - s_2} e^{s_1 t} - \frac{-s_1 x_0 + \dot{x}_0 + A}{s_1 - s_2} e^{s_2 t} \tag{8.20}$$

where

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \quad s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Once again, as it was the case with the free response of second-order systems, the impulse response for the undamped case is oscillatory. In each of the other cases, the response will stabilize at zero, whereas the underdamped impulse response exhibits decaying oscillations.

8.3.2.1 Impulse Response in MATLAB

The command `impulse(sys)` returns the unit-impulse response of the linear, time-invariant (LTI) model `sys` created with either transfer function (`TF`) or state–space (`ss`), with the assumption of zero initial conditions. For multi-input models, independent impulse commands are applied to each input channel. The time range and number of points are chosen automatically in MATLAB. For nonzero initial conditions, it is best to use Equations 8.17 through 8.20.

Example 8.5: Impulse Response

Consider

$$\ddot{x} + 2\dot{x} + kx = 3\delta(t), \quad x(0) = 0, \quad \dot{x}(0) = 0$$

a. Determine the response $x(t)$ corresponding to $k = 2$ and $k = 3$.
b. For the two parameter values $k = 2,3$, find and plot the responses in the same graph using the `impulse` command.

Solution

a. Comparing with Equation 8.8, we have

$$\begin{cases} \omega_n^2 = k \\ 2\zeta\omega_n = 2 \end{cases} \qquad \begin{cases} \omega_n = \sqrt{k} \text{ rad/s} \\ \zeta = \dfrac{1}{\sqrt{k}} < 1 \text{ (underdamped, } k = 2,3) \end{cases}$$

Because the system is underdamped in both cases, we calculate $\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{k-1}$ for each. By Equation 8.18, the two responses are then found as

$$\begin{matrix} k = 2, \omega_d = 1 & \quad x_1(t) = 3e^{-t}\sin t \\ k = 3, \omega_d = \sqrt{2} & \quad x_2(t) = \frac{3}{\sqrt{2}}e^{-t}\sin\sqrt{2}t \end{matrix}$$

b. Because the initial conditions are zero, it is appropriate to use the `impulse` command. We will create our system using the transfer function. Because the `impulse` command returns the unit-impulse response, we must define the transfer function as

$$\frac{3}{s^2 + 2s + k} \quad (k = 2,3)$$

The following script will generate and plot the impulse responses:

```
>> n = 3; d1 = [1 2 2]; d2 = [1 2 3];
>> sys1 = tf(n,d1); sys2 = tf(n,d2);
>> impulse(sys1,sys2) % Figure 8.8
```

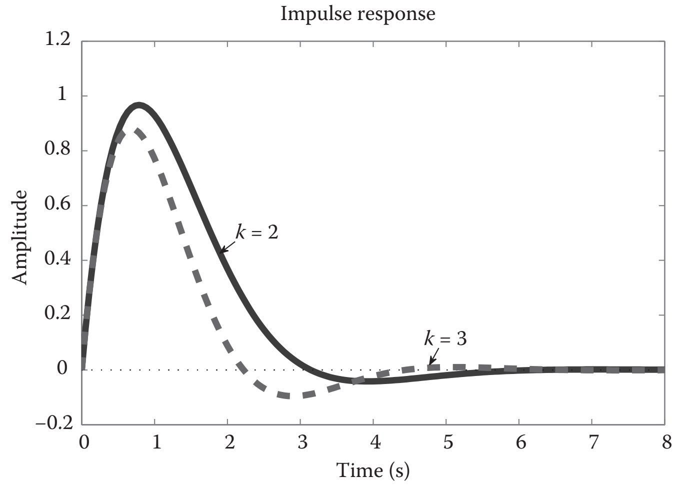


FIGURE 8.8 Impulse responses (Example 8.5).

The damping ratio and the value of parameter k are related via $\zeta = \dfrac{1}{\sqrt{k}}$. Therefore, the damping ratio associated with $k = 2$ is larger than that for $k = 3$. The response curves in Figure 8.8 are in line with this assertion.

8.3.3 Step Response of Second-Order Systems

Step response refers to $x(t)$ in Equation 8.8 when the input is $f(t) = Au(t)$, where $u(t)$ is the unit-step and A is a constant magnitude. Using $F(s) = A/s$ in Equation 8.9, the response is obtained as

$$x(t) = \mathcal{L}^{-1}\left\{\frac{(s+2\zeta\omega_n)x_0 + \dot{x}_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} + \mathcal{L}^{-1}\left\{\frac{A}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}\right\} \tag{8.21}$$

As always, different cases of damping are considered separately as follows.

Case (1) Undamped ($\zeta = 0$)

$$x(t) = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{A}{\omega_n^2}(1 - \cos\omega_n t) \tag{8.22}$$

Case (2) Underdamped ($0 < \zeta < 1$)

$$\begin{aligned} x(t) &= e^{-\zeta\omega_n t}\left[x_0 \cos\omega_d t + \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\sin\omega_d t\right] \\ &\quad + \frac{A}{\omega_n^2}\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\right] \end{aligned} \tag{8.23}$$

Case (3) Critically Damped ($\zeta = 1$)

$$x(t) = e^{-\omega_n t}[x_0 + (\omega_n x_0 + \dot{x}_0)t] + \frac{A}{\omega_n^2}[1 - e^{-\omega_n t}(1 + \omega_n t)] \tag{8.24}$$

Case (4) Overdamped ($\zeta > 1$)

$$\begin{aligned} x(t) &= \frac{-s_2 x_0 + \dot{x}_0}{s_1 - s_2} e^{s_1 t} - \frac{-s_1 x_0 + \dot{x}_0}{s_1 - s_2} e^{s_2 t} \\ &\quad + \frac{A}{\omega_n^2}\left[1 + \frac{\omega_n^2}{s_1 - s_2}\left(\frac{1}{s_1} e^{s_1 t} - \frac{1}{s_2} e^{s_2 t}\right)\right] \end{aligned} \tag{8.25}$$

where

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \quad s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Besides the undamped case in which the response oscillates, it is readily seen that for each of the other cases of damping, the step response has a steady-state value of A/ω_n^2.

8.3.3.1 Step Response in MATLAB

The command `step(sys)` returns the unit-step response of the LTI model `sys` created with either transfer function (`TF`) or state–space (`ss`), with the assumption of zero initial conditions. For multi-input models, independent step commands are applied to each input channel. The time range and number of points are chosen automatically in MATLAB. For nonzero initial conditions, it is best to use Equations 8.22 through 8.25.

Example 8.6: Step Response

For the mechanical system in Figure 8.9, assuming zero initial displacement and velocity,

a. Determine the system response $x(t)$.
b. Find and plot the response using the `step` command.

Solution

a. Using the assumed physical parameter values, the equation of motion is written as

$$2\ddot{x} + 2\dot{x} + 3x = 20u(t), \quad x(0) = 0, \quad \dot{x}(0) = 0$$

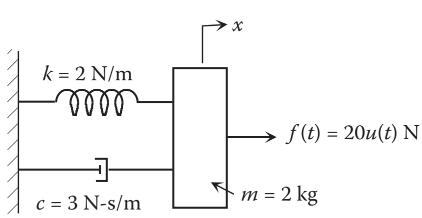


FIGURE 8.9 Mechanical system (Example 8.6).

Division of the equation of motion by 2 yields $\ddot{x}+\dot{x}+\frac{3}{2}x=10u(t)$. Comparing with the standard form of Equation 8.8, we find

$$\begin{cases}\omega_n^2=\frac{3}{2}\\ 2\zeta\omega_n=1\end{cases} \qquad \begin{cases}\omega_n=\sqrt{\frac{3}{2}}\ \text{rad/s}\\ \zeta=\dfrac{1}{\sqrt{6}}<1\ \text{(underdamped)}\end{cases}$$

This implies that the system is underdamped and $\omega_d=\omega_n\sqrt{1-\zeta^2}=\frac{\sqrt{5}}{2\sqrt{2}}$ rad/s. Using Equation 8.23, with zero initial conditions and A = 10, we find the response as (in meters)

$$x(t)=\frac{A}{\omega_n^2}\left[1-e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\right]=\frac{10}{\frac{3}{2}}\left[1-e^{-t/2}\left(\cos\frac{\sqrt{5}}{2\sqrt{2}}t+\frac{1}{\sqrt{5}}\sin\frac{\sqrt{5}}{2\sqrt{2}}t\right)\right]$$

Note that the steady-state value of response is $\dfrac{A}{\omega_n^2}=\dfrac{20}{3}$.

b. Because the `step` command returns the unit-step response, we will define the transfer function as

$$\frac{20}{2s^2+2s+3}$$

Subsequently, the following script will generate the desired plot.

```
>> n = 20; d = [2 2 3]; sys = tf(n,d);
>> step(sys)     % Figure 8.10
```

Note that `t = 12(sec)` has been automatically determined by the `step` command, and that the step response has a steady-state value of $\frac{20}{3}$, as asserted in Part (a).

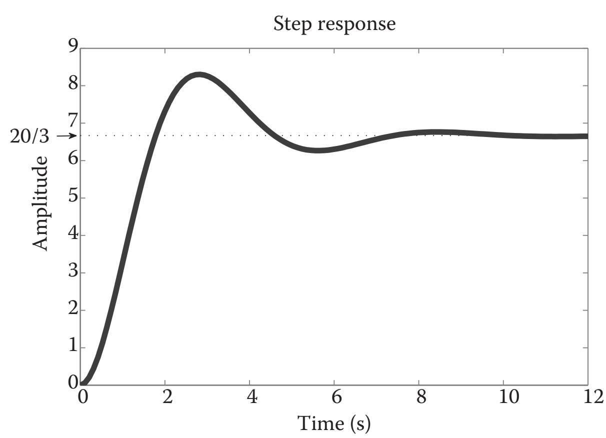


FIGURE 8.10 Step response of the mechanical system in Example 8.6.

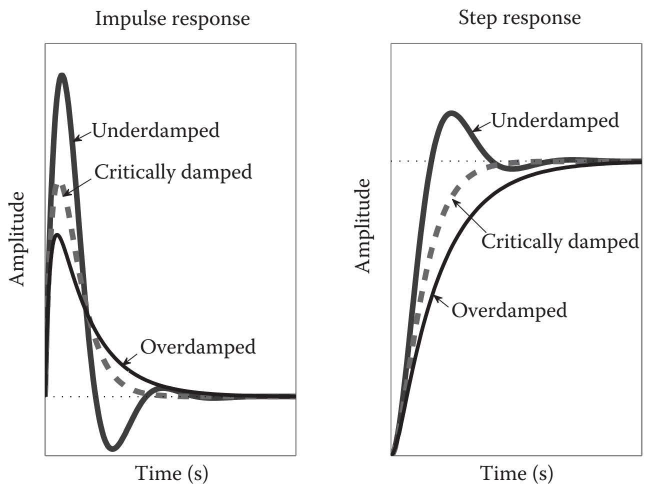


FIGURE 8.11 Unit-impulse and unit-step responses of second-order systems.

Figure 8.11 presents a summary of the unit-impulse and unit-step responses of second-order systems subjected to zero initial conditions for all three cases of damping. Note that the oscillatory response curves associated with the undamped cases have been omitted.

8.3.3.2 Response Analysis Using MATLAB Simulink

We learned that the MATLAB built-in function `initial` returns the system response to initial excitations only, whereas `impulse` and `step` return responses to impulse and step functions with the assumption of zero initial conditions. These tasks may also be achieved by simulating the Simulink model of the system at hand. This approach proves particularly valuable in applications in which the input is not one of the aforementioned special functions, and the initial conditions are nonzero. These general cases can also be handled by using the built-in command `lsim`.

8.3.3.2.1 The `lsim` Command

The command `lsim`, with function call `lsim(sys,u,t,x0)`, plots the time response of the LTI model `sys` to the arbitrary input signal described by `u` and `t`. The time vector `t` consists of regularly spaced time samples and `u` is a matrix with as many columns as inputs and whose *i*th row specifies the input value at time `t(i)`. Vector `x0` is the initial state vector at `t(1)` (for state–space models only). It is set to zero when omitted.

Example 8.7: Response Using MATLAB Simulink

A dynamic system is modeled as

$$\ddot{x}+4\dot{x}+5x=\tfrac{1}{3}\sin t, \quad x(0)=0, \quad \dot{x}(0)=-1$$

Plot the response $x(t)$ for $0 \le t \le 10$ by

a. Using the `lsim` command.
b. Simulating the Simulink model.

Solution

a. With state variables $x_1 = x$, $x_2 = \dot{x}$, we first obtain the state–space model as

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}u, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad u = \sin t \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} \end{cases}$$

Note that the output matrix is chosen as $\mathbf{C} = [1 \quad 0]$ because $x = x_1$ is to be plotted. The following script will generate the desired plot.

```
>> A = [0 1;-5 -4]; B = [0;1/3]; C = [1 0];
>> sys = ss(A,B,C,[]); x0 = [0;-1];
>> t = 0:0.01:10; u = sin(t);
>> lsim(sys,u,t,x0)   % Figure 8.12
```

The `lsim` command returns the input, as well as the response curve. Note that the limits along the vertical axis have been slightly modified for more clarity.

b. Based on the state–space model obtained for the system at hand, the Simulink model shown in Figure 8.13 is built. Note that double-clicking on each integrator allows you to enter the initial condition for the output signal of that integrator block. Double-clicking on the input block allows you to select a sine wave with amplitude of 1 and frequency of 1 rad/s. Running the simulation and double-clicking on the scope block reveals the response plot, which agrees with that in Figure 8.12. For better access to the plot, simply type

```
>> plot(tout,yout)
```

It should be mentioned that the portion of the model that includes the gain of $\frac{1}{3}$ and the two feedback loops can be replaced with the State–Space block from the Continuous

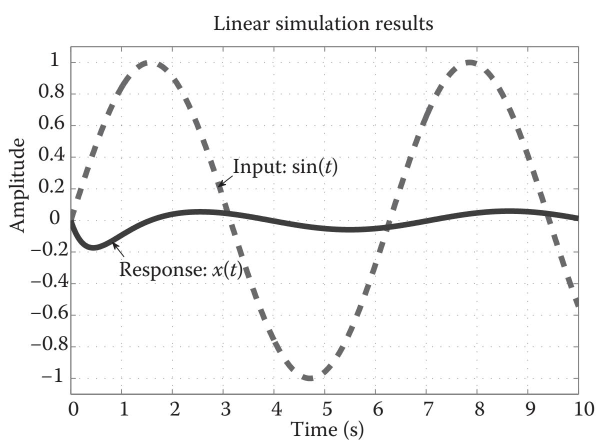


FIGURE 8.12 Simulation result (Example 8.7).

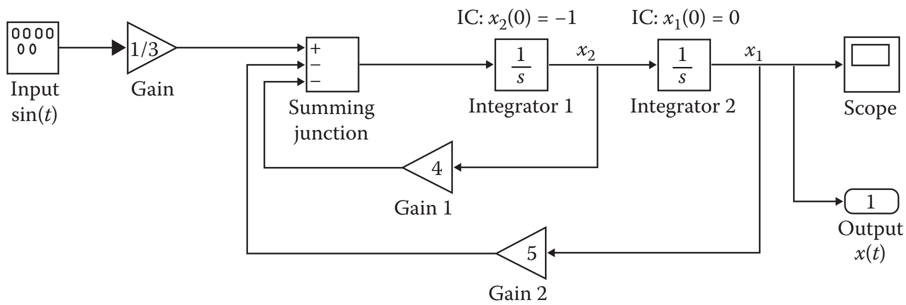


FIGURE 8.13 Simulink model of the system in Example 8.7.

library. Double-clicking on the block allows you to define the proper matrices, as well as the initial conditions.

8.3.3.2.2 Impulse Response Using Simulink

Even though there is no Impulse block in the Sources library of Simulink, the impulse response of a state–space model, subjected to zero initial conditions, may still be obtained using the Simulink model of the system. The model is built as follows. Drag the State–Space block from the Continuous library into the model. Double-clicking on this block allows you to enter the state–space matrices **A**, **B**, **C**, and **D**. For the initial conditions use the input matrix **B**. Use a Constant block (Commonly Used Blocks) of zero for input. Complete the model by adding the Out and Scope blocks. Running the simulation yields the impulse response of the system.

Example 8.8: Impulse Response

A system model is described by $3\ddot{x} + \dot{x} + 2x = \delta(t)$ subject to zero initial conditions. Plot $x(t)$ for $0 \le t \le 10$ by

a. Using the `impulse` command.
b. By simulating a Simulink model of the system.

Solution

a. With state variables $x_1 = x$, $x_2 = \dot{x}$ we obtain the state–space matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

```
>> A= [0 1;-2/3 -1/3]; B= [0;1/3]; C= [1 0];
>> sys=ss(A,B,C,[]);
>> impulse(sys)      % Figure 8.14
```

b. The model is built as shown in Figure 8.15. As mentioned earlier, the initial conditions in the state–space block are specified as the input matrix **B**. Running the simulation yields exactly the response shown in Figure 8.14.

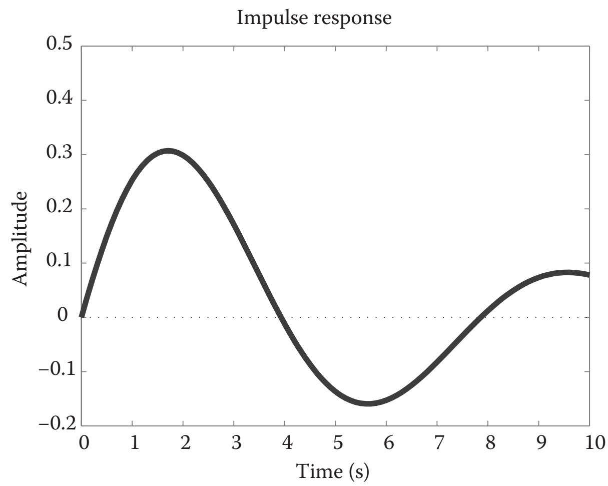


FIGURE 8.14 Impulse response in Example 8.8.

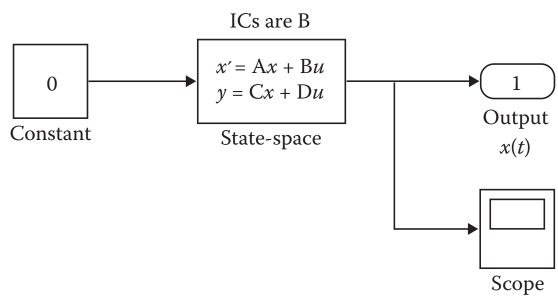


FIGURE 8.15 Simulink model in Example 8.8.

Example 8.9: MIMO System (Step Input)

Consider the mechanical system shown in Figure 8.16, in which all parameter values are in consistent physical units. Assume that the applied forces f_1 and f_2 are unit-step functions and that the system is subject to zero initial conditions. Plot two response curves for x_2: one for the case when only f_1 is the input, and another for the case when only f_2 is the input. Find the superposition of the two plots to determine the total response x_2.

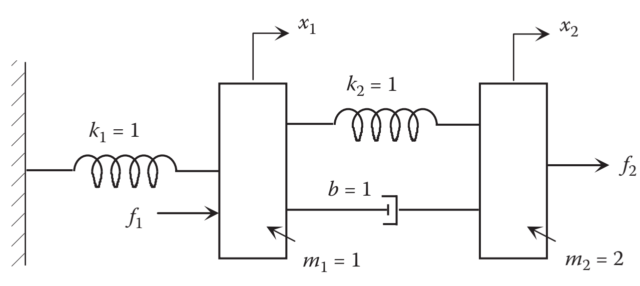


FIGURE 8.16 System in Example 8.9.

Solution

The equations of motion are

$$\begin{cases} \ddot{x}_1 + x_1 - (x_2 - x_1) - (\dot{x}_2 - \dot{x}_1) = f_1 \\ 2\ddot{x}_2 + (x_2 - x_1) + (\dot{x}_2 - \dot{x}_1) = f_2 \end{cases}$$

Selecting state variables $x_1 = x_1$, $x_2 = x_2$, $x_3 = \dot{x}_1$, $x_4 = \dot{x}_2$, the state equation is formed as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -1 & 1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \mathbf{u}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}, \quad \mathbf{u} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

The following script produces a pair of plots, one showing the contribution of f_1 to x_2 (labeled x_{21}), the other reflecting the contribution of f_2 to x_2 (labeled x_{22}). Because the inputs are step functions, and initial conditions are zero, we will use the `step` command in MATLAB.

```
>> A=[0 0 1 0;0 0 0 1;-2 1 -1 1;1/2 -1/2 1/2 -1/2];
>> B=[0 0;0 0;1 0;0 1/2];
>> C=[0 1 0 0];    % x2 is to be plotted
>> sys=ss(A,B,C,[]);     % Define system
>> [x2,t]=step(sys);     % Results suppressed (see Notes below!)
>> subplot(1,2,1), plot(t,x2(:,:,1)),   % Initiate Figure 8.17
>> title('Contribution of f1 to x2')
>> subplot(1,2,2), plot(t,x2(:,:,2)),
>> title('Contribution of f2 to x2')    % Complete Figure 8.17
>> x2=x2(:,:,1)+x2(:,:,2);     % Superposition
>> plot(t,x2)      % Figure 8.18
```

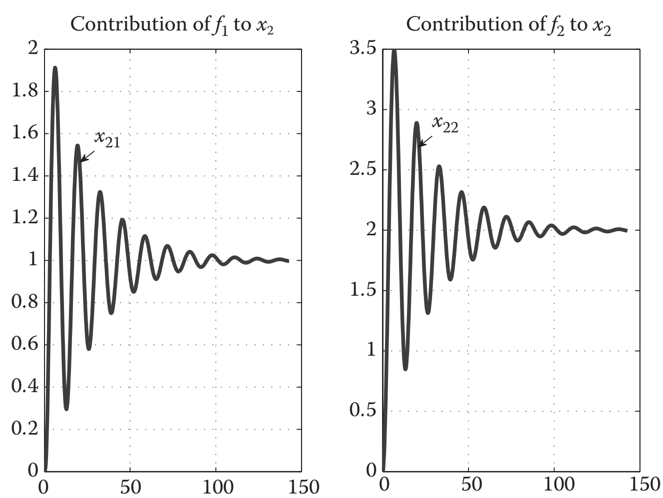


FIGURE 8.17 Contributions of the two inputs to x_2 (Example 8.9).

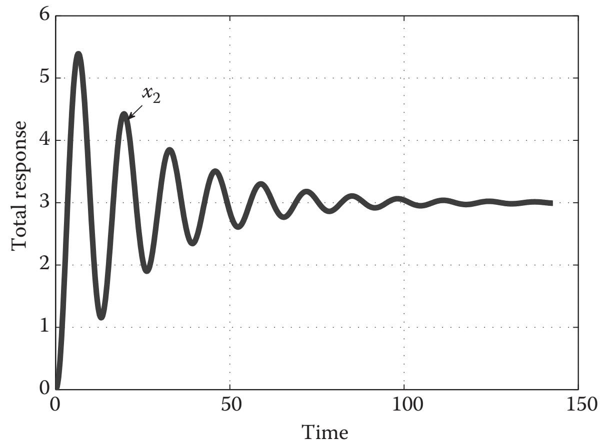


FIGURE 8.18 Time variations of response x_2 (Example 8.9).

Notes

- If we are only interested in the plots in Figure 8.17, and not the returned data, we can simply type

```
>> step(sys)
```

- As always, the `step` command automatically determines the suitable time range. A user-specified time range may also be imposed. For example, to generate the plots for up to 50 units of time, type

```
>> step(sys,50)
```

Example 8.10: MIMO System (General Input)

The mechanical system in Example 8.9 is subjected to $f_1 = 0$, $f_2 = e^{-t/2}\sin t$ and initial conditions $x_1(0) = 0$, $x_2(0) = \frac{1}{2}$, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = -\frac{1}{2}$. Plot x_2 versus $0 \le t \le 20$.

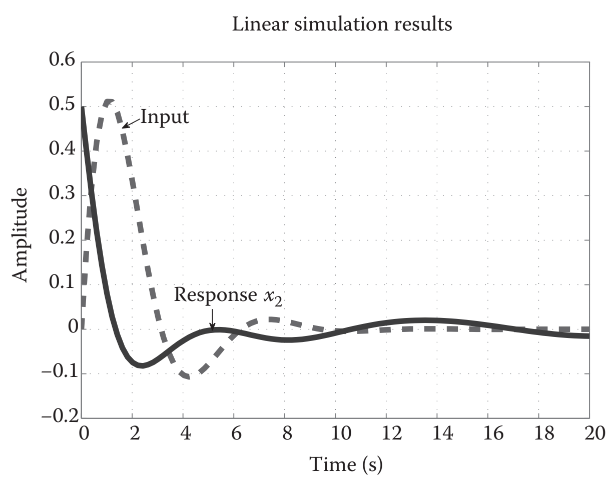


FIGURE 8.19 Response x_2 in Example 8.10.

Solution

```
>> A=[0 0 1 0;0 0 0 1;-2 1 -1 1;1/2 -1/2 1/2 -1/2];
>> B2=[0;0;0;1/2];        % Input f1 is not contributing
>> C=[0 1 0 0];        % x2 is to be plotted
>> t=linspace(0,20);       x0=[0;1/2;0;-1/2];
>> sys=ss(A,B2,C,[]);       % Define system
>> u=exp(-t/2).*sin(t);       % Define input f2
>> lsim(sys,u,t,x0)    % Figure 8.19
```

PROBLEM SET 8.2

1. Show that the free response of an overdamped, second-order system stabilizes at zero after a sufficiently long time.

In Problems 2 through 6, for each given system model,

a. Identify the damping type and find the free response.
b. Find and plot the free response using the `initial` command.

2. $2\ddot{x}+2\dot{x}+x=0,\quad x(0)=0,\quad \dot{x}(0)=1$
3. $\ddot{x}+3\dot{x}+9x=0,\quad x(0)=1,\quad \dot{x}(0)=0$
4. $2\ddot{x}+5\dot{x}+3x=0,\quad x(0)=\frac{1}{2},\quad \dot{x}(0)=1$
5. $4\ddot{x}+4\dot{x}+x=0,\quad x(0)=1,\quad \dot{x}(0)=1$
6. $4\ddot{x}+4\dot{x}+5x=0,\quad x(0)=0,\quad \dot{x}(0)=1$
7. Show that the impulse response of an underdamped, second-order system stabilizes at zero after a sufficiently long time.
8. Show that the response of an underdamped, second-order system to an impulsive input $\delta(t-a)$, a = const > 0, and zero initial conditions, is described by

$$\left[\frac{1}{\omega_d}e^{-\zeta\omega_n(t-a)}\sin\omega_d(t-a)\right]u(t-a)$$

where $u(t)$ is the unit-step function. Use this result to find the response $x(t)$ of

$$2\ddot{x}+2\dot{x}+x=\delta(t-1),\quad x(0)=0,\quad \dot{x}(0)=0$$

In Problems 9 and 10, assuming zero initial conditions,

a. Find the response $x(t)$ in closed form.
b. Plot the response using the `impulse` command.
c. Plot the response through the simulation of the Simulink model of the system.

9. $2\ddot{x}+2\dot{x}+3x=3\delta(t)$
10. $\frac{1}{2}\ddot{x}+2\dot{x}+x=10\delta(t)$
11. Consider a mass–spring–damper system as in Figure 8.6 of this section, in which

$$m = 2\text{ kg},\quad k = 10\text{ N/m}$$

Assuming zero initial conditions, plot (in a single figure) the response x to a unit-impulse force for two cases of c = 1 N-s/m and c = 2 N-s/m, and discuss the results.

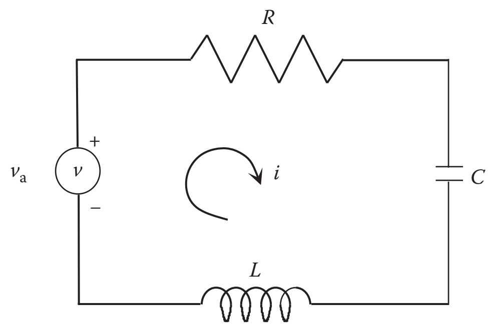


FIGURE 8.20 Problem 12.

12. The governing equation for the RLC circuit in Figure 8.20 is derived as

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v_a(t)$$

where $L = 4$ H, $R = 4\ \Omega$ and $C = \frac{1}{2}$F.

a. Write the governing equation in terms of the electric charge q, where $i = \dot{q}$.

b. Plot q and i versus t (same figure) when the applied voltage v_a is a unit-impulse and initial conditions are zero.

13. Consider the RLC circuit in Problem 12.

a. Write the governing equation in terms of the electric charge q, where $i = \dot{q}$.

b. Plot q and i versus t (same figure) when the applied voltage v_a is a unit-impulse and initial conditions are $q(0) = 0$, $\dot{q}(0) = 1$. *Hint*: Find the superposition of the data returned by `impulse` and `initial` commands.

14. Consider a mass–spring–damper system as in Figure 8.6 of this section, in which

$$m = 3\text{ kg},\quad c = 2\text{ N-s/m},\quad k = 10\text{ N/m}$$

Plot (in a single figure) the responses x and $\dot{x}$ to a unit-impulse force and initial conditions $x(0) = 0.3$, $\dot{x}(0) = 0$.

In Problems 15 and 16, $u(t)$ denotes the unit-step. Assuming zero initial conditions,

a. Find the response $x(t)$ in closed form.

b. Plot the response using the `step` command.

15. $3\ddot{x} + 12\dot{x} + 10x = u(t)$
16. $2\ddot{x} + 3\dot{x} + \frac{25}{8}x = 5u(t)$

In Problems 17 through 20, $u(t)$ denotes the unit-step. Given the nonzero initial conditions,

a. Find the response $x(t)$ in closed form.

b. Plot the response using the `step` and `initial` commands.

c. Plot the response using `lsim`.

17. $\ddot{x} + \dot{x} + 4x = 3u(t)$, $x(0) = 0$, $\dot{x}(0) = 1$
18. $9\ddot{x} + 6\dot{x} + x = 10u(t)$, $x(0) = 0$, $\dot{x}(0) = \frac{1}{5}$
19. $2\ddot{x} + 7\dot{x} + 6x = 8u(t)$, $x(0) = 1$, $\dot{x}(0) = 0$

20. $4\ddot{x}+12\dot{x}+13x=10u(t),\ x(0)=1,\ \dot{x}(0)=0$
21. Repeat Example 8.9 when $\dot{x}_1$ is the output.
22. The equations of motion of a mechanical system are given below, in which $\delta(t)$ denotes the unit-impulse. Assuming zero initial conditions, plot the response $x_1(t)$ by
    a. Simulating the Simulink model of the system.
    b. Using the `impulse` command.

$$\begin{cases} 2\ddot{x}_1+\dot{x}_1+2(x_1-x_2)=\delta(t) \\ \dot{x}_2-2(x_1-x_2)+x_2=0 \end{cases}$$

23. In Example 8.9, plot the time variations of the response x_2 by simulating the Simulink model of the system, using the State–Space block.
24. In Example 8.9, suppose the initial conditions are $x_1(0)=0,\ x_2(0)=1,\ \dot{x}_1(0)=0=\dot{x}_2(0)$. Plot the time variations of the response x_2 by simulating the Simulink model of the system, using the State–Space block.

In Problems 25 through 28, the governing equations and initial conditions of a dynamic system are provided. Plot the specified output(s) by using the `lsim` command.

25. $2\ddot{x}+\dot{x}+3x=e^{-t/2},\ x(0)=\frac{1}{2},\ \ \dot{x}(0)=0,\ \ 0\le t\le 10$
    Output: $x(t)$
26. $8\ddot{x}+\dot{x}+2x=10e^{-t/4}\sin t,\ x(0)=0,\ \dot{x}(0)=\frac{2}{3},\ \ 0\le t\le 20$
    Output: $x(t)$
27. $\begin{cases} 2\ddot{x}_1+x_1-2(x_2-x_1)-\frac{1}{2}(\dot{x}_2-\dot{x}_1)=e^{-t/3} \\ \ddot{x}_2+2(x_2-x_1)+\frac{1}{2}(\dot{x}_2-\dot{x}_1)=\sin(\frac{1}{2}t) \end{cases},\quad \begin{matrix} x_1(0)=0,\ \dot{x}_1(0)=0 \\ x_2(0)=1,\ \dot{x}_2(0)=0 \end{matrix},\quad 0\le t\le 10$
    Outputs: $x_1(t)$, $x_2(t)$
28. $\begin{cases} 3\ddot{x}_1+2x_1-\frac{2}{3}(x_2-x_1)-\frac{1}{2}(\dot{x}_2-\dot{x}_1)=\sin t \\ 2\ddot{x}_2+\frac{2}{3}(x_2-x_1)+\frac{1}{2}(\dot{x}_2-\dot{x}_1)=e^{-2t/3}\sin t \end{cases},\quad \begin{matrix} x_1(0)=1,\ \dot{x}_1(0)=0 \\ x_2(0)=1,\ \dot{x}_2(0)=0 \end{matrix},\quad 0\le t\le 10$
    Outputs: $x_1(t)$, $x_2(t)$

8.4 FREQUENCY RESPONSE

When a LTI system is subjected to a sinusoidal input, characterized by its amplitude and forcing frequency, its response will contain two portions: one that vibrates at the natural frequency of the system, and another that follows the forcing frequency. In the presence of damping, the portion vibrating at the system's natural frequency will eventually die out, as discussed earlier in this chapter. Therefore, the response at steady-state is sinusoidal and it has the same frequency as the input (forcing frequency). The steady-state response to a sinusoidal input is known as the frequency response (Figure 8.21).

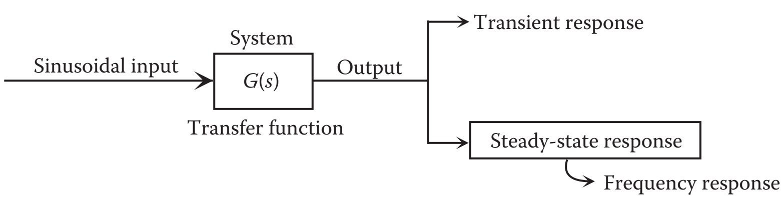


FIGURE 8.21 Frequency response of an LTI system.

8.4.1 Frequency Response of Stable, Linear Systems

Consider a stable, linear time-invariant system described by its transfer function $G(s)$ as in Figure 8.21. Let the input and the output be $f(t)$ and $x(t)$, respectively. Assume that the input is sinusoidal in the form $f(t) = F_0 \sin\omega t$ with amplitude F_0 and forcing frequency ω. The frequency response is the steady-state portion of the response, denoted by x_{ss}, and is determined as follows.

Suppose the transfer function is a ratio of two polynomials in s, that is,

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}, \quad K = \text{const} \tag{8.26}$$

Note that because x_{ss} is independent of the initial conditions, they are simply omitted. Because $G(s) = X(s)/F(s)$, we have $X(s) = G(s)F(s)$, in which $F(s) = F_0\omega/(s^2 + \omega^2)$. If $G(s)$ has distinct poles only, then partial-fraction expansion provides

$$\begin{aligned} X(s) = G(s)F(s) &= \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \cdot \frac{F_0\omega}{s^2+\omega^2} \\ &= \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n} + \frac{d}{s+j\omega} + \frac{\bar{d}}{s-j\omega} \end{aligned} \tag{8.27}$$

where c_i ($i = 1, 2, \ldots, n$) and d are constants, and $\bar{d}$ is the complex conjugate of d. Inverse Laplace transformation of $X(s)$ yields

$$x(t) = c_1 e^{-p_1 t} + c_2 e^{-p_2 t} + \cdots + c_n e^{-p_n t} + d e^{-j\omega t} + \bar{d} e^{j\omega t}$$

Because the system is assumed stable, the poles $-p_1, -p_2, \ldots, -p_n$ all have negative real parts. Therefore, except for the last two, all terms on the right side die out at steady state. The same result holds for the case of one or more multiple poles. If a typical pole $-p_k$ has multiplicity q, then $x(t)$ will contain terms $t^r e^{-p_k t}$ $(r = 0, 1, \ldots, q-1)$ so that they too will die out at steady state. Therefore, regardless of the multiplicity of the poles, we have

$$x_{ss}(t) = d e^{-j\omega t} + \bar{d} e^{j\omega t} \tag{8.28}$$

The constants d and $\bar{d}$ are evaluated from Equation 8.27 via

$$d = (s+j\omega)X(s)\big|_{s=-j\omega} = \left[(s+j\omega)G(s)\frac{F_0\omega}{s^2+\omega^2}\right]_{s=-j\omega} = -\frac{F_0}{2j}G(-j\omega)$$

$$\bar{d} = (s-j\omega)X(s)\big|_{s=j\omega} = \left[(s-j\omega)G(s)\frac{F_0\omega}{s^2+\omega^2}\right]_{s=j\omega} = \frac{F_0}{2j}G(j\omega)$$

These results clearly confirm that $\bar{d}$ and d are complex conjugates. The quantity $G(j\omega)$ is known as the frequency response function (FRF) and is obtained by replacing s with $j\omega$ in $G(s)$. Because $G(j\omega)$ is generally complex, it can be expressed in polar form as

$$G(j\omega) = |G(j\omega)|\, e^{j\phi}, \quad \phi = \angle G(j\omega) = \tan^{-1}\left\{\frac{\text{imaginary part of } G(j\omega)}{\text{real part of } G(j\omega)}\right\}$$

where ϕ is measured from the positive real axis and is considered positive in the counterclockwise direction. Similarly,

$$G(-j\omega) = \left|G(-j\omega)\right| e^{-j\phi} = \left|G(j\omega)\right| e^{-j\phi}$$

Using these in the expressions for d and $\bar{d}$, and inserting the outcomes into Equation 8.28, we find

$$\begin{aligned} x_{ss}(t) &= -\frac{F_0}{2j}\left|G(j\omega)\right| e^{-j\phi} e^{-j\omega t} + \frac{F_0}{2j}\left|G(j\omega)\right| e^{j\phi} e^{j\omega t} \\ &= F_0\left|G(j\omega)\right| \frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} = F_0\left|G(j\omega)\right| \sin(\omega t + \phi) \end{aligned}$$

In summary, if the input of a stable, linear time-invariant system with transfer function $G(s)$ is $F_0 \sin\omega t$, the system's frequency response is

$$x_{ss}(t) = F_0\left|G(j\omega)\right| \sin(\omega t + \phi) \tag{8.29}$$

where $\left|G(j\omega)\right|$ and ϕ denote, respectively, the magnitude and phase of the FRF $G(j\omega)$.

8.4.1.1 Frequency Response of First-Order Systems

Linear, first-order systems (Section 8.1) are described by

$$\tau\dot{x} + x = f(t), \quad \tau = \text{time constant} > 0$$

When the system is subjected to $f(t) = F_0 \sin\omega t$, its frequency response is obtained as follows. The system's transfer function and the ensuing FRF are found as

$$G(s) = \frac{1}{\tau s + 1} \quad \xrightarrow{\text{FRF}} \quad G(j\omega) = \frac{1}{1 + \tau\omega j}$$

The magnitude of the FRF is

$$\left|G(j\omega)\right| = \frac{1}{\sqrt{1 + (\tau\omega)^2}}$$

To calculate the phase, however, it is advised that $G(j\omega)$ be expressed in standard rectangular form to determine its location in the complex plane. That is,

$$G(j\omega) = \frac{1}{1 + \tau\omega j} = \frac{1 - \tau\omega j}{1 + (\tau\omega)^2} = \frac{1}{1 + (\tau\omega)^2} - j\frac{\tau\omega}{1 + (\tau\omega)^2}$$

Because $\tau\omega > 0$, this indicates that $G(j\omega)$ is always located in the fourth quadrant. As a result, the calculation of phase is straightforward, and it is given by

$$\phi = \angle G(j\omega) = \tan^{-1}\left\{\frac{\text{imaginary part of } G(j\omega)}{\text{real part of } G(j\omega)}\right\} = \tan^{-1}(-\tau\omega) = -\tan^{-1}(\tau\omega)$$

Therefore, by Equation 8.29, the frequency response is found as

$$x_{ss}(t) = \frac{F_0}{\sqrt{1+(\tau\omega)^2}}\sin(\omega t - \tan^{-1}(\tau\omega)) \tag{8.30}$$

Example 8.11: Frequency Response of a First-Order System

Find the frequency response corresponding to

$$\dot{x} + 2x = 10\sin 2t$$

Solution

Rewrite in standard form as $\frac{1}{2}\dot{x} + x = 5\sin 2t$, so that $\tau = \frac{1}{2}$, $F_0 = 5$, and $\omega = 2$ rad/s. By Equation 8.30, the frequency response is

$$x_{ss}(t) = \frac{5}{\sqrt{1+1^2}}\sin(2t - \tan^{-1}1) = \frac{5}{\sqrt{2}}\sin\left(2t - \frac{\pi}{4}\right)$$

8.4.1.2 Frequency Response of Second-Order Systems

Frequency-response analysis of second-order systems is best understood when applied to a single–degree-of-freedom mechanical system. To that end, consider the mechanical system shown in Figure 8.22, in which x is measured from the static equilibrium position and the applied force is $f(t) = F_0 \sin\omega t$. The frequency response x_{ss} is obtained as follows.

Because the equation of motion is $m\ddot{x} + b\dot{x} + kx = f(t)$, the system's transfer function is

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \quad \overset{\text{FRF}}{\quad} \quad G(j\omega) = \frac{1}{k - m\omega^2 + b\omega j}$$

Calculation of phase requires writing $G(j\omega)$ in rectangular form to decide its location in the complex plane. That is,

$$G(j\omega) = \frac{k - m\omega^2}{(k-m\omega^2)^2 + (b\omega)^2} - \frac{b\omega}{(k-m\omega^2)^2 + (b\omega)^2}\, j$$

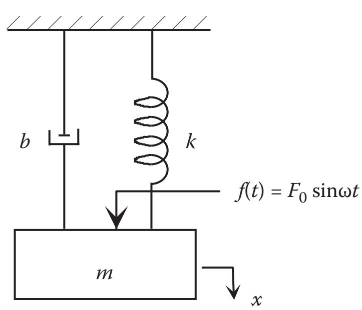


FIGURE 8.22 Second-order mechanical system.

But $(k - m\omega^2)^2 + (b\omega)^2 > 0$ and $b\omega > 0$. Therefore, $G(j\omega)$ is always below the real axis. In particular, $G(j\omega)$ is in the fourth quadrant if $k - m\omega^2 > 0$, and in the third quadrant if $k - m\omega^2 < 0$. Therefore, it is necessary to perform this analysis for each given situation. The magnitude and phase of the FRF are subsequently determined as

$$\left|G(j\omega)\right| = \frac{1}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}}, \quad \phi = \angle G(j\omega) = -\tan^{-1}\frac{b\omega}{k - m\omega^2}$$

where ϕ must be calculated based on the location of $G(j\omega)$. For a fourth quadrant location, the calculation is straightforward and is simply given by the above expression for ϕ. For a third quadrant location, the phase needs to be adjusted accordingly. From Section 8.2, we know that $\omega_n = \sqrt{k/m}$ and $\zeta = b/2\sqrt{mk}$, hence

$$\frac{k}{m} = \omega_n^2, \quad \frac{b}{k} = \frac{2\zeta}{\omega_n}$$

Multiply and divide the fractions involved in $\left|G(j\omega)\right|$ and ϕ, and use the above relations in the resulting expressions, then insert into Equation 8.29 to obtain

$$x_{ss}(t) = \frac{x_{st}}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} \sin\left(\omega t - \tan^{-1}\frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2}\right) \tag{8.31}$$

where $x_{st} = F_0/k$ is the static deflection and the dimensionless ratio ω/ω_n is the normalized frequency. If we let X denote the amplitude of x_{ss} in Equation 8.31, then

$$X = \frac{x_{st}}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} \qquad \frac{X}{x_{st}} = \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}$$

The dimensionless ratio X/x_{st} is called the normalized amplitude. Further details of the above findings, and in particular, applications in mechanical vibrations, will be presented in Section 9.2.

Example 8.12: Frequency Response of a Second-Order System

Find the frequency response of a mechanical system such as that in Figure 8.22 with the equation of motion

$$\ddot{x} + 2\dot{x} + 4x = 0.4\sin t$$

Solution

We first observe that $F_0 = 0.4$, $m = 1$, $b = 2$, $k = 4$, $\omega = 1$ rad/s, all in consistent physical units. Thus,

$$G(s) = \frac{1}{s^2 + 2s + 4} \quad \stackrel{\text{FRF}}{\Rightarrow} \quad G(j\omega)\Big|_{\omega=1} = \frac{1}{3 + 2j} = \frac{3 - 2j}{13}$$

which indicates a fourth quadrant location. We next find

$$x_{st} = \frac{F_0}{k} = \frac{0.4}{4} = 0.1, \quad \omega_n = 2 \text{ rad/s}, \quad \zeta = \frac{1}{2}, \quad \frac{\omega}{\omega_n} = \frac{1}{2}, \quad 2\zeta\frac{\omega}{\omega_n} = \frac{1}{2}$$

The phase is then calculated as

$$\phi = -\tan^{-1}\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2} = -\tan^{-1}\frac{\frac{1}{2}}{1-(\frac{1}{2})^2} = -\tan^{-1}\tfrac{2}{3} = -0.5880 \text{ rad}$$

Finally, substitution into Equation 8.31 yields the frequency response

$$x_{ss}(t) = 0.1109 \sin(t - 0.5880)$$

Note: The frequency response may be obtained directly from Equation 8.29. In our current example,

$$G(s) = \frac{1}{s^2+2s+4} \quad \xrightarrow{\text{FRF}} \quad G(j\omega)\Big|_{\omega=1} = \frac{1}{3+2j} = \underbrace{\boxed{\frac{3-2j}{13}}}_{\text{Fourth quadrant}}$$

The magnitude is $\left|G(j\omega)\right| = \dfrac{1}{\sqrt{13}}$. Due to a fourth quadrant location, the phase is found as

$$\phi = -\tan^{-1}\left\{\frac{\frac{2}{13}}{\frac{3}{13}}\right\} = -\tan^{-1}\left\{\frac{2}{3}\right\} = -0.5880 \text{ rad}$$

Noting that $F_0 = 0.4$, by Equation 8.29, we have

$$x_{ss}(t) = \left\{F_0\left|G(j\omega)\right|\sin(\omega t+\phi)\right\}_{\omega=1} = \frac{0.4}{\sqrt{13}}\sin(t-0.5880) = 0.1109\sin(t-0.5880)$$

8.4.2 Bode Diagram

The FRF of a dynamic system may be represented by a pair of plots, one displaying the magnitude versus frequency, the other the phase angle (in degrees) versus frequency. A very specific presentation of this pair of plots is known as a Bode diagram. For a given FRF $G(j\omega)$ with magnitude $\left|G(j\omega)\right|$ and phase angle ϕ, a Bode diagram consists of two plots: a curve of the (base 10) logarithm of $\left|G(j\omega)\right|$ and a plot of the phase ϕ, both versus $\log\omega$. It is common to represent the logarithmic magnitude of $G(j\omega)$ by $20\log\left|G(j\omega)\right|$ using units in decibels (dB). In a Bode diagram, the curves are sketched on semilog paper, using a linear scale for magnitude (in decibels) and phase (in degrees) and a logarithmic scale for frequency (in radians per second). The logarithmic presentation described here allows the low-frequency and high-frequency behavior of the transfer function to be shown in a single diagram. This is quite valuable because of the significance of low-frequency characteristics of systems in practical situations. Bode diagrams play an important role in control systems design, as discussed in Section 10.6.

8.4.2.1 Plotting Bode Diagrams in MATLAB

The MATLAB command `bode` generates the Bode diagram for an LTI system. For a system `sys`, the command

```
>> bode(sys)
```

will plot the Bode diagram, composed of a magnitude (decibels) plot and a phase (degrees) plot. If the numerical values for magnitude and phase that are returned by `bode` are desired, we define a range of frequency in the form of vector w, and type

```
>> [mag,phase,w] = bode(sys,w)
```

This returns matrices `mag` and `phase` but not a plot. The desired plots may be generated by executing the following script:

```
>> w=linspace(0.01,100,1000);  % 1000 points for smoothness of curves
>> [mag,phase,w]=bode(sys,w);
>> for i=1:1000,
m(i)=mag(1,1,i);      % Extract the magnitude column
p(i)=phase(1,1,i);    % Extract the phase column
end
>> subplot(2,1,1), semilogx(w,20*log10(m))  % Create the magnitude (dB) plot
>> subplot(2,1,2), semilogx(w,p)             % Complete Bode diagram
```

8.4.2.2 Bode Diagram of First-Order Systems

Consider a first-order system model in its standard form,

$$\tau\dot{x} + x = f(t), \quad \tau > 0$$

so that the transfer function is $G(s) = \dfrac{1}{\tau s + 1}$ and the FRF is formed as

$$G(j\omega) = \frac{1}{1 + \tau\omega j}$$

As shown earlier in this section, the magnitude and phase of the FRF are then

$$\left|G(j\omega)\right| = \frac{1}{\sqrt{1 + (\tau\omega)^2}}, \quad \phi = -\tan^{-1}(\tau\omega)$$

The Bode diagram is obtained as follows. By definition, the logarithmic magnitude is

$$20\log\left|G(j\omega)\right| = 20\log\frac{1}{\sqrt{1 + (\tau\omega)^2}} = 20\log[1 + (\tau\omega)^2]^{-1/2} = -10\log[1 + (\tau\omega)^2]\ \text{dB}$$

The plots of this quantity and the phase ϕ of the FRF versus $\tau\omega$ (logarithmic scale) will make up the Bode diagram shown in Figure 8.23. Because τ, the time constant, is in seconds and ω is in radians per second, $\tau\omega$ is dimensionless and is treated as a normalized frequency. The asymptotic approximations of magnitude (in decibels) and phase for low-frequency and high-frequency ranges

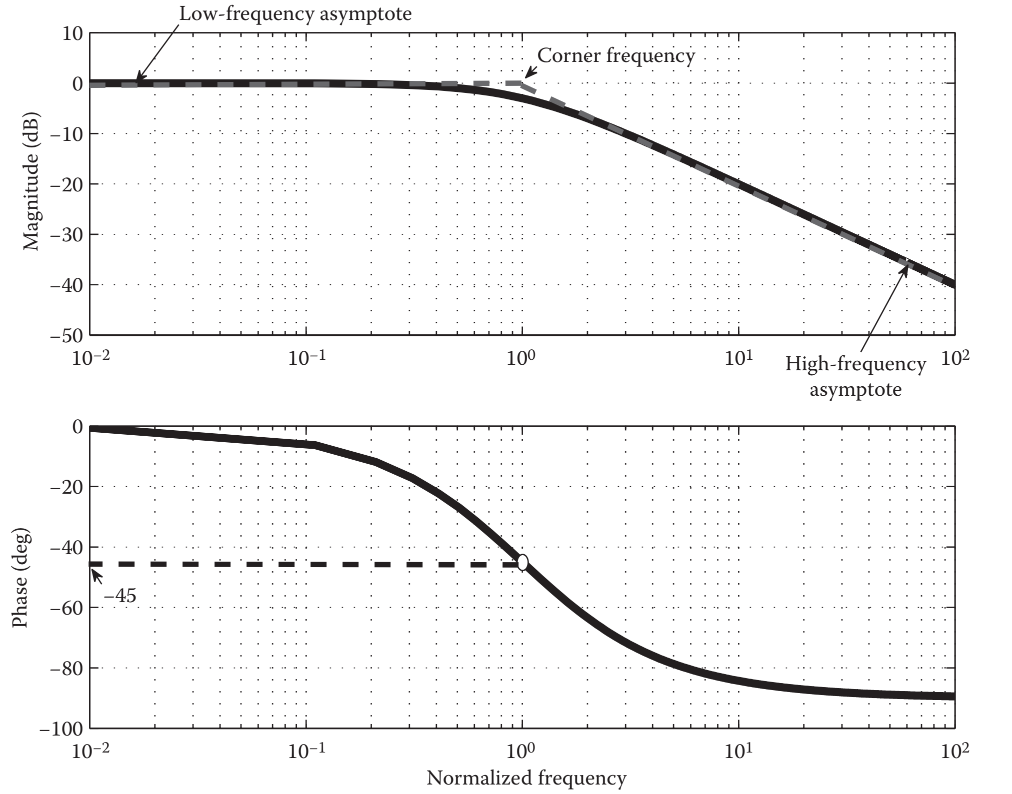


FIGURE 8.23 Bode diagram for $G(j\omega) = 1/(1 + \tau\omega j)$.

are determined as follows. In a low-frequency range, we have $\tau\omega \ll 1$ so that $(\tau\omega)^2 \ll 1$ and the logarithmic magnitude is approximated by $-10\log 1$ dB, or 0 dB. The phase is approximated by $-\tan^{-1}(0)$, or 0 degrees. In a high-frequency range, we have $\tau\omega \gg 1$ so that $(\tau\omega)^2 \gg 1$ and the logarithmic magnitude is approximated by $-10\log(\tau\omega)^2$ dB, or $-20\log\tau\omega$ dB. This represents a straight line with a slope of −20 dB/decade. The phase will approach $-\tan^{-1}(\infty)$, or −90°. The low-frequency and high-frequency asymptotes intersect when $\tau\omega = 1$, so that $\omega = 1/\tau$ is the corner frequency. The phase at this frequency is $\phi = -\tan^{-1}1 = -45°$.

Example 8.13: First-Order System Bode Diagram

Draw the Bode diagram for

$$G(s) = \frac{10}{2s+3}$$

Solution

We first rewrite $G(s)$ as

$$G(s) = \frac{\frac{10}{3}}{\frac{2}{3}s+1} \quad \xrightarrow{\text{FRF}} \quad G(j\omega) = \frac{\frac{10}{3}}{1+\frac{2}{3}\omega j} = \frac{10}{3}\cdot\frac{1}{1+\frac{2}{3}\omega j}$$

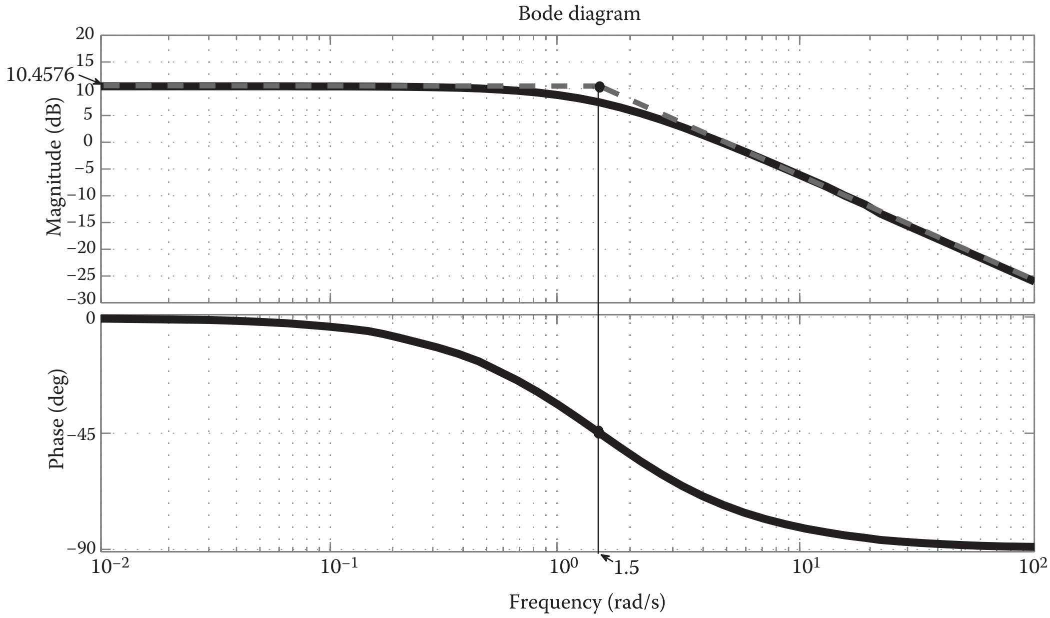


FIGURE 8.24 Bode diagram (Example 8.13).

This agrees with the standard form $\frac{1}{1+\tau\omega j}$, with $\tau = \frac{2}{3}$, except for the constant multiple of $\frac{10}{3}$. The magnitude is

$$\left|G(j\omega)\right| = \frac{10}{3}\cdot\frac{1}{\sqrt{1+\left(\frac{2}{3}\omega\right)^2}} \quad \xrightarrow[20\log|G(j\omega)|]{\text{Logarithmic magnitude}} \quad 20\log\frac{10}{3} + 20\log\frac{1}{\sqrt{1+\left(\frac{2}{3}\omega\right)^2}}\ \text{dB}$$

The second term is the standard magnitude (in decibels) for first-order systems discussed previously. Because the first term is $20\log\frac{10}{3} = 10.4576$, the complete magnitude (in decibels) for the problem at hand is obtained by raising the standard magnitude curve by 10.4576 dB. The constant $\frac{10}{3}$ does not affect the phase angle because its phase is zero. The following script generates the desired Bode diagram:

```
>> n=10; d=[2 3]; sys=tf(n,d);
>> bode(sys)  % Figure 8.24
```

As mentioned earlier in this section, the same pair of plots can also be generated by retaining the numerical values of magnitude and phase using the function call `[mag,phase,w] =bode(sys,w)` and then plotting them versus frequency. In the low-frequency range, the magnitude for the standard case is 0 dB as in Figure 8.23. Raising that by $20\log\frac{10}{3} = 10.4576$ dB yields the result depicted in Figure 8.24. The corner frequency is $\omega = 1/\tau = 1.5$ rad/s. As expected, the phase plot is clearly unchanged compared with the standard case in Figure 8.23.

8.4.2.3 Bode Diagram of Second-Order Systems

Consider a second-order system in the standard (normalized) form*

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 f(t) \tag{8.32}$$

* This differs slightly from the form given in Section 8.3. In the current form, the forcing term is normalized.

so that the standard, second-order transfer function is obtained as

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

and the FRF is subsequently formed as

$$G(j\omega) = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + 2\zeta\omega_n\omega j} \overset{\text{Divide by } \omega_n^2}{=} \frac{1}{1 - (\omega/\omega_n)^2 + 2\zeta(\omega/\omega_n)j}$$

As a result, the magnitude and phase of the FRF are calculated as

$$\left|G(j\omega)\right| = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}, \quad \phi = -\tan^{-1}\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2} \tag{8.33}$$

The logarithmic magnitude is

$$20\log\left|G(j\omega)\right| = 20\log\frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} \text{ dB}$$

For a fixed damping ratio ζ, both the magnitude (in decibels) and phase angle ϕ are functions of the normalized frequency ω/ω_n and thus may be plotted versus ω/ω_n. Figure 8.25 shows several

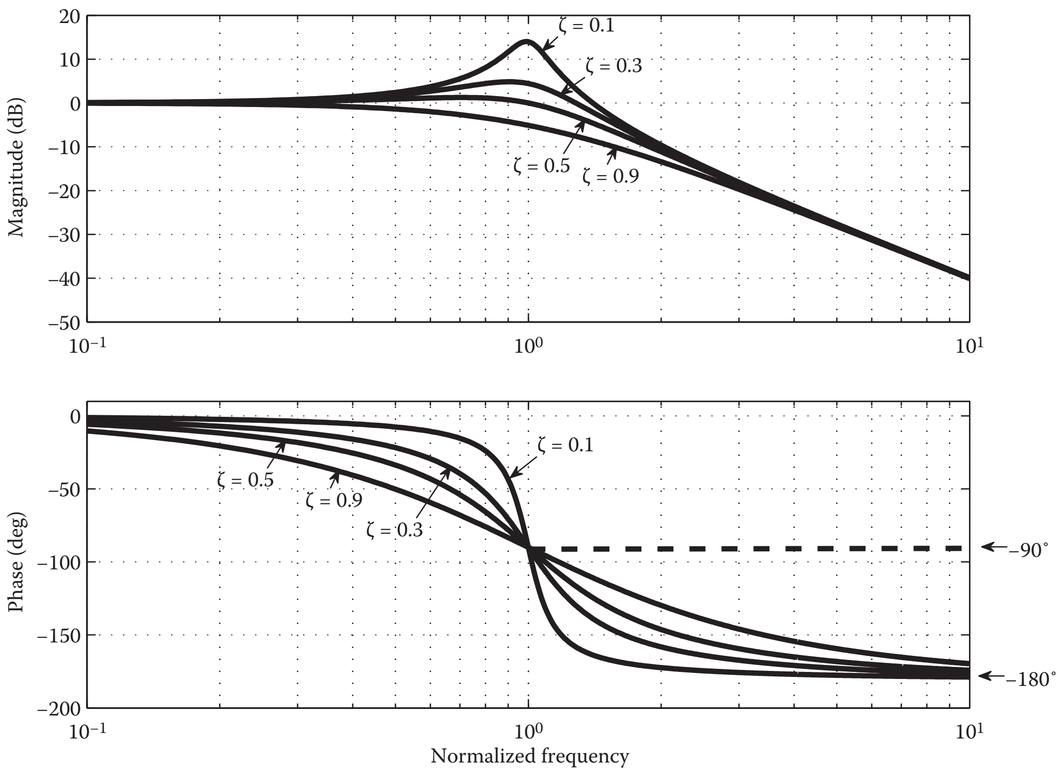


FIGURE 8.25 Bode diagram for $G(j\omega) = 1/[1 - (\omega/\omega_n)^2 + 2\zeta(\omega/\omega_n)j]$.

pairs of curves, each pair corresponding to a fixed ζ. It turns out that some magnitude curves attain a maximum peak, whereas others do not. To find the frequency at which a maximum peak occurs, we proceed as follows. Maximizing the logarithmic magnitude is equivalent to maximizing $|G(j\omega)|$, which is equivalent to minimizing the denominator of $|G(j\omega)|$, that is,

$$\sqrt{(1-r^2)^2+(2\zeta r)^2}\ ,\ r=\frac{\omega}{\omega_n}$$

It can be easily shown that minimization of this quantity with respect to r leads to the critical value

$$r=\frac{\omega}{\omega_n}=\sqrt{1-2\zeta^2}$$

With this result, we define the resonant frequency ω_r as

$$\omega_r=\omega_n\sqrt{1-2\zeta^2} \tag{8.34}$$

Based on this, the magnitude curves that attain a maximum peak correspond to $0 \le \zeta \le 0.7071$, where 0.7071 is an approximate value for $\sqrt{2}/2$. The maximum peak of the magnitude itself is then found as

$$|G(j\omega)|_{\max}=|G(j\omega)|_{\omega/\omega_n=\sqrt{1-2\zeta^2}}=\frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{8.35}$$

For example, the maximum peak for the case of $\zeta = 0.1$ occurs at $\omega/\omega_n=\sqrt{1-2(0.1)^2}=0.9899$ and

$$|G(j\omega)|_{\max}=\left.\frac{1}{2\zeta\sqrt{1-\zeta^2}}\right|_{\zeta=0.1}=5.0252\ \xrightarrow{\text{Logarithmic magnitude}}\ 20\log(5.0252)=14.0231\ \text{dB}$$

All these are readily verified via Figure 8.25.

Example 8.14: Second-Order System Bode Diagram

Draw the Bode diagram for

$$G(s)=\frac{3}{s^2+2s+4}$$

Solution

Rewrite $G(s)$ to extract the standard, second-order transfer function, as

$$G(s)=\frac{3}{4}\cdot\underbrace{\boxed{\frac{4}{s^2+2s+4}}}_{\text{Standard 2nd-order TF}}\ \xrightarrow{\text{FRF}}\ G(j\omega)=\frac{3}{4}\cdot\frac{4}{4-\omega^2+2\omega j}$$

The magnitude is

$$\left|G(j\omega)\right| = 3/4 \cdot \frac{4}{\sqrt{(4-\omega^2)^2+(2\omega)^2}} \quad \xrightarrow{\text{Logarithmic magnitude}} \quad 20\log 3/4 + 20\log\frac{4}{\sqrt{(4-\omega^2)^2+(2\omega)^2}} \text{ dB}$$

The second term is the standard magnitude (in decibels) for second-order systems. Because the first term is $20\log\frac{3}{4} = -2.50$, the complete logarithmic magnitude is obtained by lowering the standard magnitude curve by 2.50 dB. Once again, the constant $\frac{3}{4}$ does not affect the phase angle. The following script generates the Bode diagram:

```
>> n=3; d=[1 2 4]; sys=tf(n,d);
>> bode(sys) % Figure 8.26
```

Using the standard, second-order transfer function, we find ω_n = 2 rad/s and ζ = 0.5. Because $\zeta \le 0.7071$, the corresponding magnitude (in decibels) curve will have a maximum peak, which occurs at the resonant frequency

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}\Big|_{\omega_n=2,\zeta=0.5} \cong 1.4 \text{ rad/s}$$

The corresponding peak value is calculated as

$$\left|G(j\omega)\right|_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}\Bigg|_{\zeta=0.5} = 1.16 \quad \xrightarrow{\text{Logarithmic magnitude}} \quad 20\log(1.16) = 1.25 \text{ dB}$$

However, because the magnitude curve has been lowered by 2.50 dB, the maximum peak for the case at hand is adjusted to 1.25 – 2.50 = –1.25 dB (Figure 8.26).

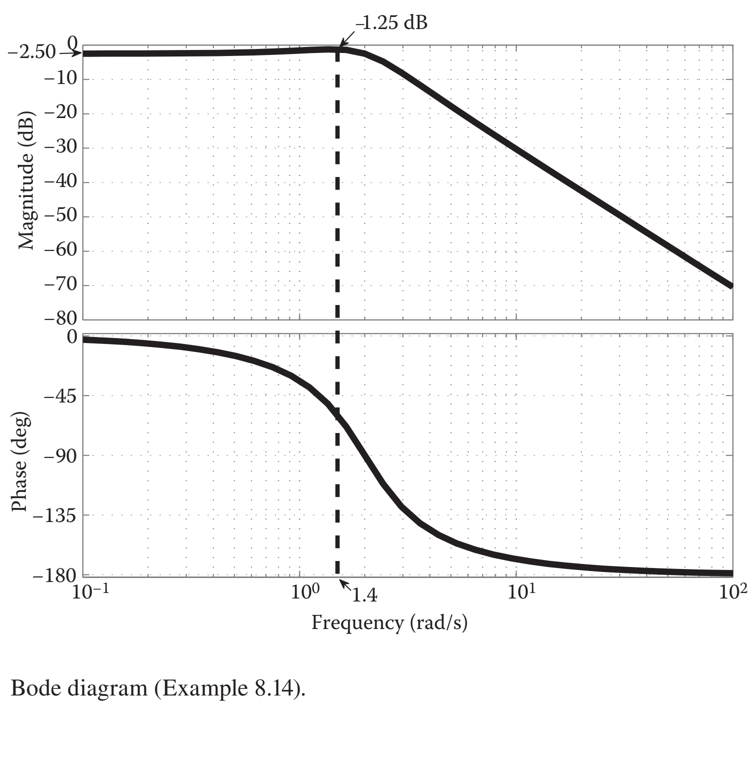


FIGURE 8.26 Bode diagram (Example 8.14).

PROBLEM SET 8.3

In Problems 1 through 8, find the frequency response of the system.

1. $3\dot{x} + 4x = 12\sin t$
2. $2\dot{x} + \frac{1}{2}x = 10\sin 2t$
3. $\dot{y} + 4y = 0.9\sin(t/2)$
4. $\frac{2}{3}\dot{y} + 3y = 19.8\sin(t/3)$
5. $4\ddot{x} + 12\dot{x} + 13x = 25\sin 2t$
6. $2\ddot{x} + 5\dot{x} + 8x = 10\sin 3t$
7. $4\ddot{x} + 2\dot{x} + 10x = 3.8\sin(t/2)$
8. $9\ddot{x} + 0.9\dot{x} + 20x = 20\sin(t/3)$
9. The equation of motion for the mechanical system in Figure 8.27 is derived as

$$m\ddot{x} + b\dot{x} + 2kx = F_0 \sin\omega t$$

   where x is measured from the static equilibrium. Assuming m = 15 kg, b = 20 N-s/m, k = 200 N/m, F_0 = 50 N, and ω = 6 rad/s, find the system's frequency response.
10. Find the frequency response of the mechanical system in Figure 8.28, assuming m = 15 kg, k = 50 N/m, F_0 = 100 N, and ω = 2 rad/s.
11. For the RLC circuit in Figure 8.29, show that the frequency response is described by

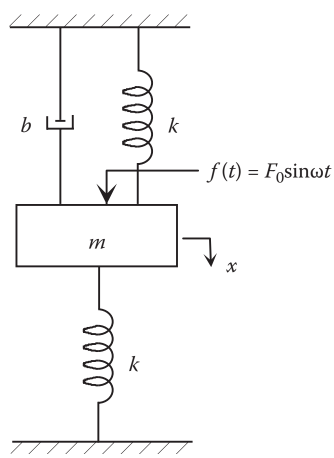


FIGURE 8.27 Problem 9.

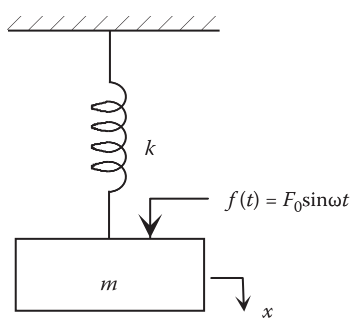


FIGURE 8.28 Problem 10.

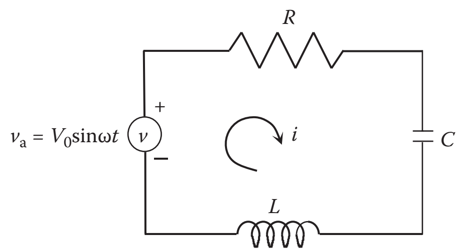


FIGURE 8.29 Problem 11.

$$i_{ss}(t) = \frac{\omega C V_0}{\sqrt{(1 - LC\omega^2)^2 + (RC\omega)^2}} \cos\left(\omega t - \tan^{-1}\frac{RC\omega}{1 - LC\omega^2}\right)$$

In Problems 12 through 15 draw the Bode diagram and identify the corner frequency, as well as the low-frequency magnitude (in decibels).

12. $G(s) = \dfrac{2}{3s + 1}$
13. $G(s) = \dfrac{9}{\frac{1}{2}s + 2}$
14. $G(s) = \dfrac{3.5}{3s + 5}$
15. $G(s) = \dfrac{6}{9s + \frac{2}{3}}$
16. Show that the magnitude $|G(j\omega)| = \dfrac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}$ attains a maximum when

$$\frac{\omega}{\omega_n} = \sqrt{1 - 2\zeta^2}$$

In Problems 17 through 20 draw the Bode diagram, identify the resonant frequency and find the peak magnitude (in decibels), if applicable, and give the approximate low-frequency decibel value.

17. $G(s) = \dfrac{2}{s^2 + 2s + 9}$
18. $G(s) = \dfrac{27.04}{s^2 + 2.6s + 6.76}$
19. $G(s) = \dfrac{10.58}{3s^2 + 11.04s + 15.87}$
20. $G(s) = \dfrac{7.35}{2s^2 + 5.6s + 24.5}$

8.5 SOLVING THE STATE EQUATION

In Section 4.2, we learned that using the state variables, the mathematical model of a linear dynamic system can be expressed in the form of the state equation

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{8.36}$$

where $\mathbf{x}(t)$ is the state vector, $\mathbf{A}$ is the state matrix, $\mathbf{B}$ is the input matrix, $\mathbf{u}(t)$ is the vector of the inputs, and $\mathbf{x}_0$ is the initial state vector. In this section, we will learn how to solve the state equation to derive the state vector in closed form.

8.5.1 Formal Solution of the State Equation

The scalar counterpart of Equation 8.36 is the initial-value problem

$$\dot{x}(t) = ax(t) + bu(t), \quad x(0) = x_0$$

whose solution is obtained as

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)\, d\tau$$

Inspired by this, the formal solution of the state equation $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$ is expressed as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)\, d\tau \tag{8.37}$$

where $\mathbf{x}_0$ is the initial state vector. The quantity $e^{\mathbf{A}t}$ is called the matrix exponential of $\mathbf{A}$. Note that, technically, $e^{\mathbf{A}}$ is the matrix exponential but $e^{\mathbf{A}t}$ is what is often involved in dynamic systems analysis.

8.5.1.1 Matrix Exponential

When a is scalar, e^{at} is defined as

$$e^{at} = \sum_{k=0}^{\infty} \frac{1}{k!}(at)^k = 1 + ta + \frac{1}{2!}t^2a^2 + \cdots$$

This idea can be extended to the case involving a square matrix $\mathbf{A}_{n\times n}$, as

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{1}{k!}(t\mathbf{A})^k = \mathbf{I} + t\mathbf{A} + \frac{1}{2!}t^2\mathbf{A}^2 + \cdots \tag{8.38}$$

where $\mathbf{I}$ is the $n\times n$ identity matrix and $\mathbf{A}^k = \underbrace{\mathbf{AA}\ldots\mathbf{A}}_{k\ \text{times}}$. Note that $e^{\mathbf{A}t}$ is an $n\times n$ matrix.

Properties of the Matrix Exponential

1. Given $\mathbf{A}_{n\times n}$ and scalars t and τ

$$e^{\mathbf{A}(t+\tau)} = e^{\mathbf{A}t}e^{\mathbf{A}\tau}$$

which can be verified as follows:

$$\begin{aligned} e^{\mathbf{A}(t+\tau)} &= \sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{A}^k(t+\tau)^k = \sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{A}^k\sum_{m=0}^{\infty}\binom{k}{m}t^m\tau^{k-m} = \sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{A}^k\sum_{m=0}^{\infty}\frac{k!}{m!(k-m)!}t^m\tau^{k-m} \\ &= \sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\mathbf{A}^k\frac{t^m\tau^{k-m}}{m!(k-m)!} \underset{k-m=r}{=} \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\mathbf{A}^{r+m}\frac{t^m\tau^r}{m!r!} \\ &= \left(\sum_{m=0}^{\infty}\frac{1}{m!}\mathbf{A}^m t^m\right)\left(\sum_{r=0}^{\infty}\frac{1}{r!}\mathbf{A}^r\tau^r\right) = e^{\mathbf{A}t}e^{\mathbf{A}\tau} \end{aligned}$$

Here, $(t+\tau)^k = \sum_{m=0}^{\infty}\binom{k}{m}t^m\tau^{k-m}$ is the binomial series and $\binom{k}{m} = \frac{k!}{m!(k-m)!}$ is the binomial coefficient.

2. Because the series in Equation 8.38 is absolutely convergent for finite values of t, term-by-term differentiation of the series yields

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$$

3. If $\mathbf{A}_{n\times n}$ and $\mathbf{B}_{n\times n}$ are similar matrices such that $\mathbf{S}^{-1}\,\mathbf{AS} = \mathbf{B}$, then $\mathbf{A}t = \mathbf{S}[\mathbf{B}t]\mathbf{S}^{-1}$ and

$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{B}t}\mathbf{S}^{-1}$$

4. If matrix $\mathbf{D} = [d_{ii}]_{i=1,\,2,\ldots,n}$ is diagonal, then

$$e^{\mathbf{D}t} = [e^{d_{ii}t}]_{i=1,2,\ldots,n}$$

We now present a systematic procedure to find $e^{\mathbf{A}t}$ where $\mathbf{A}_{n\times n}$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. In Section 3.3, it was shown that the modal matrix $\mathbf{V} = [\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \ldots \;\; \mathbf{v}_n]$ diagonalizes $\mathbf{A}$ via

$$\mathbf{V}^{-1}\mathbf{AV} = \mathbf{D} = [\lambda_i]_{i=1,\,2,\ldots,n} \quad\Rightarrow\quad \mathbf{A} = \mathbf{VDV}^{-1} \quad\Rightarrow\quad \mathbf{A}t = \mathbf{V}[\mathbf{D}t]\mathbf{V}^{-1}$$

Then, by property 3 above, we find

$$e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{D}t}\mathbf{V}^{-1} \tag{8.39}$$

Noting that the modal matrix $\mathbf{V}$ is known and $e^{\mathbf{D}t}$ is easy to calculate (property 4), $e^{\mathbf{A}t}$ will be readily available.

8.5.1.2 Formal Solution in MATLAB

The command `expm` calculates the matrix exponential. The integration portion of the formal solution, Equation 8.37, is handled by the `int` command.

Example 8.15: Matrix Exponential

Given $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$, find $e^{\mathbf{A}t}$ using

a. The `expm` command
b. Equation 8.39

Solution

a.

```
>> A = [1 0;-1 -2];
>> syms t
>> expm(A*t)
ans =
[                  exp(t),         0]
[1/3*exp(-2*t)-1/3*exp(t), exp(-2*t)]
```

b.

```
% Find the modal matrix V and the diagonal matrix of the eigenvalues of A

>> [V,D] = eig(A)
V =
     0         0.9487
     1.0000   -0.3162
D =
    -2         0
     0         1
>> V*expm(D*t)*inv(V) % Apply Eq. (8.39)

ans =
[                  exp(t),         0]
[1/3*exp(-2*t)-1/3*exp(t), exp(-2*t)]
```

The result agrees with that in Part (a).

Example 8.16: Formal Solution

Find the formal solution of

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{-t}, \quad \mathbf{x}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

Solution

```
>> A = [1 0;-1 -2]; B = [1;3]; x0 = [1;0];
>> syms t tau
>> x = simple(expm(A*t)*x0+int(expm(A*(t-tau))*B*exp(-tau),tau,0,t))
x =
                     exp(t) + sinh(t)
7/(2*exp(t)) - 3/exp(2*t) - exp(t)/2
```

The `simple` command is employed to ensure that the returned result is in simplified form.

8.5.2 Solution of the State Equation via Laplace Transformation

We now solve the state equation using Laplace transformation, noting that

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ \dots \\ x_n \end{Bmatrix} \quad \xrightarrow{\text{Laplace transform}} \quad \mathcal{L}\{\mathbf{x}\} = \mathbf{X}(s) = \begin{Bmatrix} \mathcal{L}\{x_1\} \\ \dots \\ \mathcal{L}\{x_n\} \end{Bmatrix}$$

Using the Laplace transform of Equation 8.36, and taking into account the initial state vector $\mathbf{x}_0$, yields

$$s\mathbf{X}(s) - \mathbf{x}_0 = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \quad \xrightarrow{\text{Rearrange}} \quad (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}_0 + \mathbf{B}\mathbf{U}(s)$$

Premultiply by $(s\mathbf{I} - \mathbf{A})^{-1}$ to obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 + (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\mathbf{U}(s)$$

so that

$$\mathbf{x}(t) = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}\mathbf{x}_0 + \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\mathbf{U}(s)\} \tag{8.40}$$

We will next manipulate the two terms in Equation 8.40. First note that

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s}\mathbf{I} + \frac{1}{s^2}\mathbf{A} + \frac{1}{s^3}\mathbf{A}^2 + \cdots$$

Therefore,

$$\mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathbf{I} + \frac{1}{s^2}\mathbf{A} + \frac{1}{s^3}\mathbf{A}^2 + \cdots\right\} = \mathbf{I} + t\mathbf{A} + \frac{1}{2!}t^2\mathbf{A}^2 + \cdots \overset{\text{By Equation 8.38}}{=} e^{\mathbf{A}t}$$

This implies that the first term in Equation 8.40 is simply $e^{\mathbf{A}t}\mathbf{x}_0$. The second term may be worked out using the convolution integral (Section 2.3) as

$$\mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)\} = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Combining the two terms yields

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

which exactly agrees with the formal solution derived earlier.

Example 8.17: Laplace-Transform Approach

Solve the state equation in Example 8.16 using the Laplace transform approach.

Solution

```
>> A=[1 0;-1 -2]; B=[1;3]; x0=[1;0];
>> syms s t
>> U = laplace(exp(-t));
>> x = simple(ilaplace(inv(s*eye(2)-A))*x0 + ilaplace(inv(s*eye(2)-A)*B*U))

x =
       (3*exp(t))/2 - 1/(2*exp(t))
7/(2*exp(t)) - 3/exp(2*t) - exp(t)/2
```

Noting $\sinh t = \frac{1}{2}(e^t - e^{-t})$, the above result matches that in Example 8.16.

8.5.3 Solution of the State Equation via State-Transition Matrix

The solution of the homogeneous state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

can be expressed as

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 \tag{8.41}$$

so that $\mathbf{x}(t)$ is obtained by a transformation of the initial state vector $\mathbf{x}_0$. For this reason, $\Phi(t)$ is known as the state-transition matrix. We claim that $\Phi(t)$ is the unique solution of

$$\dot{\Phi} = \mathbf{A}\Phi, \quad \Phi(0) = \mathbf{I}$$

and prove it as follows. First, using Equation 8.41, we have

$$\mathbf{x}(0) = \Phi(0)\mathbf{x}_0 \quad \Rightarrow \quad \Phi(0) = \mathbf{I}$$

Next, differentiation of Equation 8.41 with respect to t, and further manipulation, yields

$$\dot{\mathbf{x}}(t) = \dot{\Phi}(t)\mathbf{x}_0 \overset{\dot{\Phi}=\mathbf{A}\Phi}{=} \mathbf{A}\Phi(t)\mathbf{x}_0 \overset{\Phi(t)\mathbf{x}_0=\mathbf{x}}{=} \mathbf{A}\mathbf{x}$$

which completes the proof of the claim. Comparing Equation 8.41 with the homogeneous portions of Equations 8.40 and 8.37 reveals that

$$\Phi(t) = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\} = e^{\mathbf{A}t}$$

Based on this finding, the solution of the nonhomogeneous state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

is expressed as

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 + \int_0^t \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau \tag{8.42}$$

PROBLEM SET 8.4

In Problems 1 through 4, find $e^{\mathbf{A}t}$, where $\mathbf{A}$ is the matrix provided and t is scalar, using

a. The `expm` command.
b. The inverse Laplace-transform approach.

1. $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$
2. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$
3. $\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$
4. $\mathbf{A} = \begin{bmatrix} 1 & -6 & 10 \\ 2 & -7 & 10 \\ 1 & -6 & 10 \end{bmatrix}$

In Problems 5 through 10, determine the state vector via the formal-solution approach.

5. $\dot{\mathbf{x}} = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u, \quad u = \text{unit-step function}, \quad \mathbf{x}(0) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$
6. $\dot{\mathbf{x}} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}u, \quad u = \sin t, \quad \mathbf{x}(0) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$
7. $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u, \quad u = e^{-t}, \quad \mathbf{x}(0) = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$
8. $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 3 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{u}, \quad \mathbf{u} = \begin{Bmatrix} 1 \\ e^{-t} \end{Bmatrix}, \quad \mathbf{x}(0) = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$
9. $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 4 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}u, \quad u = \text{unit-step function}, \quad \mathbf{x}(0) = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$
10. $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u, \quad u = \text{unit-ramp function}, \quad \mathbf{x}(0) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$

In Problems 11 and 12, the state–space representation of a system model is provided. Using the formal-solution approach, find the response $y(t)$.

11. $\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u \\ y = \mathbf{Cx} \end{cases}$, $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathbf{C} = [1\ 1]$, $u = \sin t$, $\mathbf{x}_0 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

12. $\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \\ y = \mathbf{Cx} \end{cases}$, $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{C} = [1\ 0\ 1]$, $\mathbf{u} = \begin{Bmatrix} 1 \\ e^{-t} \end{Bmatrix}$, $\mathbf{x}_0 = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$

8.6 RESPONSE OF NONLINEAR SYSTEMS

Up to this point, we have been mainly concerned with the response analysis of linear systems. In Section 4.6 we learned how to derive a linearized model of a nonlinear system, and that such approximation is reasonably accurate only in a small neighborhood of an operating (equilibrium) point. To avoid the obvious limitations that go along with this approach, we may tackle the nonlinear model directly, either numerically or via the simulation of the Simulink model of the system.

8.6.1 Numerical Solution of the State-Variable Equations

The mathematical models of dynamic systems of any order can be expressed in terms of state-variable equations, which may generally be nonlinear (see Section 4.2). If a dynamic system has n state variables $x_1, x_2, \ldots, x_n$ and m inputs $u_1, u_2, \ldots, u_m$, the state-variable equations take the general form

$$\begin{cases} \dot{x}_1 = f_1(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \\ \dot{x}_2 = f_2(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \\ \qquad \cdots \\ \dot{x}_n = f_n(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \end{cases}$$

where $f_1, f_2, \ldots, f_n$ are algebraic functions of the state variables and inputs, and are generally nonlinear. These equations may be conveniently expressed in vector form, as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \tag{8.43}$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{Bmatrix}_{n\times 1}, \quad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ \ldots \\ u_m \end{Bmatrix}_{m\times 1}, \quad \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ \ldots \\ f_n \end{Bmatrix}_{n\times 1}$$

If the system is linear, then Equation 8.43 simplifies to the familiar state equation.

8.6.1.1 Fourth-Order Runge–Kutta Method

Among several methods designed for the numerical solution of Equation 8.43, the most commonly used is the very efficient and robust fourth-order Runge–Kutta (RK4) method. To introduce this method, we reformulate the problem as

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(a) = \mathbf{x}_0, \quad a \le t \le b \tag{8.44}$$

with all variables as defined previously. Define an integer $N > 0$ and let $h = (b - a)/N$ denote the step size. The mesh points $t_i = a + ih$ ($i = 0, 1, 2, \ldots, N - 1$) partition the interval $[a,b]$ into N subintervals. Knowing the initial state vector $\mathbf{x}_0$, the solution vector $\mathbf{x}_i$ at each of the subsequent mesh points t_i is obtained via

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{1}{6}h(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4), \quad i = 0,\ 1,\ 2, \ldots, N-1 \tag{8.45}$$

where

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(t_i, \mathbf{x}_i) \\
\mathbf{k}_2 &= \mathbf{f}\left(t_i + \frac{1}{2}h, \mathbf{x}_i + \frac{1}{2}h\mathbf{k}_1\right) \\
\mathbf{k}_3 &= \mathbf{f}\left(t_i + \frac{1}{2}h, \mathbf{x}_i + \frac{1}{2}h\mathbf{k}_2\right) \\
\mathbf{k}_4 &= \mathbf{f}(t_i + h, \mathbf{x}_i + h\mathbf{k}_3)
\end{aligned}$$

The following user-defined function numerically solves a system of state-variable equations subjected to a prescribed initial state over a specified time interval. Note that the function is also capable of handling a scalar initial-value problem.

```
function x = RK4System(f,t,x0)
%
% RK4System uses the RK4 method to solve a system of first-order ODEs
% given in the form x' = f(t,x) subject to initial condition vector x0.
%
%      x = RK4System(f,t,x0) where
%
%         f is an inline (m-dim. vector) function representing f(t,x),
%         t is an (n+1)-dim. vector representing the mesh points,
%         x0 is an m-dim. vector representing the initial condition of x,
%
%         x is an m-by-(n+1) matrix, each column being the vector of
%         solution estimates at a mesh point.
x(:,1) = x0;     % The first column is set to be the initial vector x0
h = t(2) - t(1);

for i = 1:length(t)-1,
       k1 = f(t(i),x(:,i));
       k2 = f(t(i)+h/2,x(:,i)+h*k1/2);
       k3 = f(t(i)+h/2,x(:,i)+h*k2/2);
       k4 = f(t(i)+h,x(:,i)+h*k3);
       x(:,i+1) = x(:,i)+h*(k1+2*k2+2*k3+k4)/6;
end
```

Example 8.18: Nonlinear System Response via RK4

The mechanical system shown in Figure 8.30 contains a nonlinear spring identified by the spring force $f_{spr} = x|x|$ and is subjected to a unit-step externally applied force $u(t)$ and initial conditions $x(0) = 0,\ \dot{x}(0) = 1$.

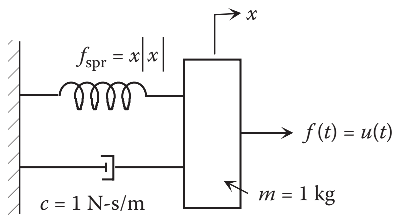


FIGURE 8.30 A nonlinear mechanical system (Example 8.18).

a. Derive the state-variable equations.
b. Using RK4, find and plot the displacement $x(t)$ versus $0 \le t \le 10$ s.

Solution

a. The system's motion is described by

$$\ddot{x} + \dot{x} + x|x| = u(t), \quad x(0) = 0, \quad \dot{x}(0) = 1$$

With state variables $x_1 = x$ and $x_2 = \dot{x}$, the nonlinear state-variable equations are obtained as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 - x_1|x_1| + u(t) \end{cases}, \quad \begin{matrix} x_1(0) = 0 \\ x_2(0) = 1 \end{matrix}$$

b. The problem is formulated as

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} x_2 \\ -x_2 - x_1|x_1| + 1 \end{Bmatrix}, \quad \mathbf{x}_0 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \quad 0 \le t \le 10$$

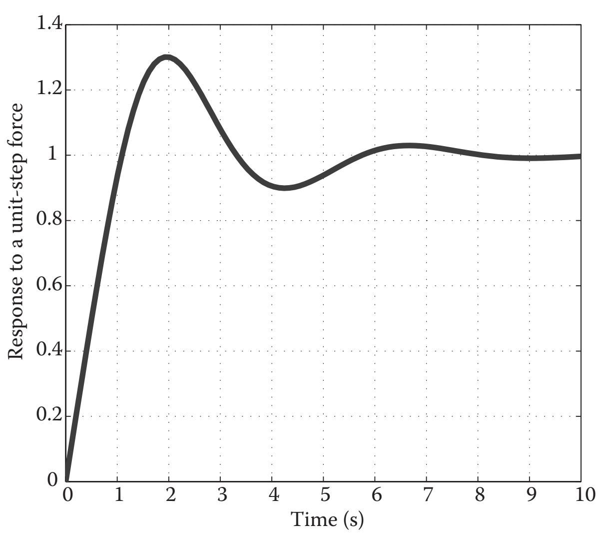


FIGURE 8.31 Nonlinear system response (Example 8.18).

```
>> t =linspace(0,10); x0 = [0;1];
>> f=inline('[x(2,1);-x(2,1)-x(1,1)*abs(x(1,1))+1]','t','x');
>> x=RK4System(f, t, x0);
>> x1=x(1,:);   % Extract x1 from the state vector
>> plot(t,x1)   % Figure 8.31
```

8.6.2 Response via MATLAB Simulink

In Chapter 4, we learned how to methodically construct a block diagram directly from the system model and subsequently build the corresponding Simulink model. Running the simulation of this model will generate the system response to a specified forcing function, as well as prescribed initial conditions.

Example 8.19: Nonlinear System Response in Simulink

Find the response of the nonlinear system in Example 8.18 in Simulink.

Solution

The Simulink model shown in Figure 8.32 is built based on the state-variable equations derived in Example 8.18. Running the simulation and double-clicking on the Scope block reveals the exact same response curve as in Figure 8.31. As always, for direct access to the returned data, simply type

```
>> plot(tout,yout)
```

8.6.3 Response of the Linearized Model

In Chapter 4, we learned how to derive the linearized model of a nonlinear system in the close vicinity of an operating point analytically or in Simulink. Either approach yields the state–space matrices that describe the linear model. Once these matrices are available, the Simulink model of the linearized model may be simulated to find its response. Also, as mentioned above, the response of the original nonlinear model is obtained directly via Simulink. As one would expect, these two response curves tend to agree near the operating point.

Example 8.20: Linearized Model

Consider the nonlinear mechanical system in Examples 8.18 and 8.19. Find the linear model analytically, plot its response, and compare with the response of the original nonlinear system.

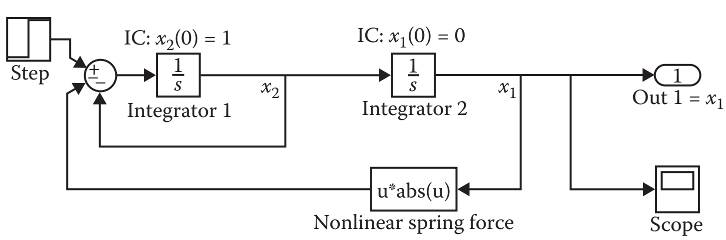


FIGURE 8.32 Simulink model (Example 8.19).

Solution

Based on the developments in Section 4.6, the operating point is characterized by $\bar{x} = 1$ and the linearized model is derived as

$$\ddot{x} + \dot{x} + 2x = 0, \quad x(0) = -1, \quad \dot{x}(0) = 1$$

The relation between Δx and x (in the nonlinear system) is $x = x - \bar{x} = x - 1$. Therefore, we need to first plot $\Delta x(t)$ and then raise the curve by 1 unit to obtain a variable compatible with x. The linearized model is described by its state–space matrices and initial state, as

$$\mathbf{x} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [1 \;\; 0], \quad D = 0, \quad \mathbf{x}(0) = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

We will use these in the State–Space block of the Simulink model (Figure 8.33), and run the simulation. Double-clicking on the Scope block reveals the response Δx versus t. This curve must then be raised by 1 unit to generate the time variations compatible with x in the original model.

```
>> plot(tout,yout+1)       % Figure 8.34
```

Finally, embedding the nonlinear system response (Figure 8.31) obtained in Example 8.18 completes Figure 8.34. It is immediately observed that the nonlinear and linear response curves tend to agree only in a small neighborhood of $\bar{x} = 1$, the operating point.

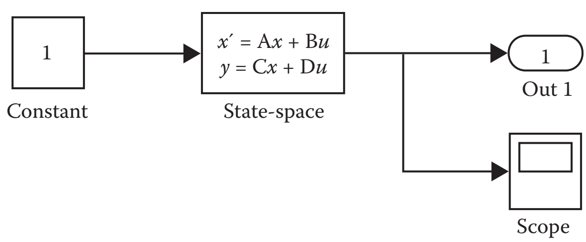


FIGURE 8.33 The linearized model (Example 8.20).

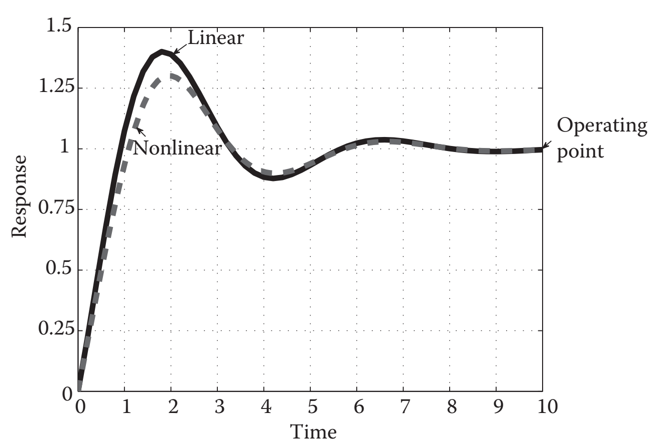


FIGURE 8.34 Nonlinear and linear responses (Example 8.20).

PROBLEM SET 8.5

In Problems 1 through 6, find and plot the specified output using the RK4 method.

1. $\ddot{x}+\dot{x}+2x^3=1, \quad x(0)=0.5, \quad \dot{x}(0)=0, \quad 0 \le t \le 10$
   Output: $x(t)$
2. $\dddot{x}+6\ddot{x}+11\dot{x}+5x^3=\frac{1}{2}e^{-t}, \quad x(0)=0, \quad \dot{x}(0)=-1, \quad \ddot{x}(0)=0, \quad 0 \le t \le 10$
   Output: $x(t)$
3. $3\ddot{x}+2\dot{x}+x\left|x\right|=1+\sin(t/2), \quad x(0)=1, \quad \dot{x}(0)=1, \quad 0 \le t \le 15$
   Output: $x(t)$
4. $\ddot{x}+2\dot{x}+x\sqrt{\left|x\right|}=1+\sin t, \quad x(0)=1, \quad \dot{x}(0)=0, \quad 0 \le t \le 10$
   Output: $x(t)$
5. $\begin{cases}\dot{x}_1=x_2 \\ \dot{x}_2=-x_1-x_1\left|x_1\right|-x_2^3-1+0.2\sin t\end{cases}, \quad \begin{matrix}x_1(0)=0 \\ x_2(0)=1\end{matrix}, \quad 0 \le t \le 10$
   Output: $x_2(t)$
6. $\begin{cases}\dot{x}_1=-x_1-2x_2^3 \\ \dot{x}_2=x_1+8+\sin t\end{cases}, \quad \begin{matrix}x_1(0)=0 \\ x_2(0)=-1\end{matrix}, \quad 0 \le t \le 5$
   Output: $x_2(t)$

7. A nonlinear dynamic system is modeled as

$$\begin{cases}\dot{x}_1=-x_2-x_1\left|x_1\right|-3 \\ \dot{x}_2=x_1-x_2-3\end{cases}, \quad \begin{matrix}x_1(0)=-1 \\ x_2(0)=1\end{matrix}, \quad 0 \le t \le 10$$

   a. Build the Simulink model and use it to generate the plot of $x_2(t)$.
   b. Derive the linearized model analytically. Build a Simulink model and use it to plot the time variations of the variable in the linear model that is compatible with $x_2(t)$. Compare the plots generated in Parts (a) and (b) and comment.

8. Repeat Problem 7 for

$$\begin{cases}\dot{x}_1=x_2-x_1 \\ \dot{x}_2=2x_2^{-1}+1\end{cases}, \quad \begin{matrix}x_1(0)=0 \\ x_2(0)=-1\end{matrix}, \quad 0 \le t \le 10$$

9. A nonlinear system model is derived as

$$\begin{cases}\dot{x}_1=x_2 \\ \dot{x}_2=-x_1-x_1\left|x_1\right|-x_2^3-2+\sin t\end{cases}, \quad \begin{matrix}x_1(0)=-1 \\ x_2(0)=1\end{matrix}, \quad 0 \le t \le 10$$

   Plot $x_2(t)$ using
   a. The Simulink model of the system.
   b. The RK4 method.

10. Consider the nonlinear model

$$\dot{x}+2ax\left|x\right|=e^{-t/3}\sin t, \quad x(0)=0, \quad 0 \le t \le 10$$

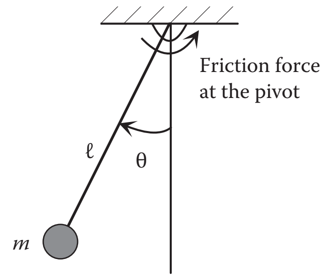


FIGURE 8.35 Problem 11.

where a is a parameter. Use the RK4 method to plot $x(t)$ for $a = 0.1, 0.5, 1$ versus $0 \le t \le 10$ in the same graph.

11. The pendulum system in Figure 8.35 consists of a uniform thin rod of length ℓ and a concentrated mass m at its tip. The friction at the pivot causes the system to be damped. When the angular displacement θ is not very small, the system is described by the nonlinear model

$$\tfrac{2}{3}m\ell^2\ddot{\theta} + 0.09\dot{\theta} + \tfrac{1}{2}mg\ell\sin\theta = 0$$

Assume, in consistent physical units, that $m\ell^2 = 1.3$, $\dfrac{g}{\ell} = 7.5$. Two sets of initial conditions are to be considered: (1) $\theta(0) = 15°$, $\dot{\theta}(0) = 0$, and (2) $\theta(0) = 30°$, $\dot{\theta}(0) = 0$. Using the RK4 method, plot the two angular displacements θ_1 and θ_2 corresponding to the two sets of initial conditions versus $0 \le t \le 5$ in the same graph. Angle measures must be converted to radians. Use at least 100 points for plotting.

12. A nonlinear dynamic system model is derived as

$$\dot{x} + 2x^3 = u(t), \quad x(0) = 0, \quad 0 \le t \le 3$$

where $u(t)$ is the unit-step function.

a. Build the Simulink model and use it to generate the plot of $x(t)$.
b. Derive the linearized model analytically. Build a Simulink model and use it to plot the time variations of the variable in the linear model that is compatible with $x(t)$. Compare the plots generated in Parts (a) and (b) and comment.

8.7 SUMMARY

The total response of a dynamic system can be decomposed into transient response and steady-state response. The transient response consists of those terms in the total response that decay to zero eventually. The portion of the total response that remains after the transient terms have vanished is the steady-state response. Linear, first-order systems are modeled as

$$\tau\dot{x} + x = f(t), \quad \tau = \text{const} > 0, \quad x(0) = x_0$$

where τ = time constant provides a measure of how quickly the system reaches steady state. Linear, second-order systems are modeled as

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0$$

where ζ and ω_n are the damping ratio and (undamped) natural frequency, respectively.

The MATLAB function `initial` finds system response to initial conditions, whereas `impulse` and `step` find the response to impulse and step inputs assuming zero initial conditions. For more general cases, the `lsim` command is used. Response analysis may also be performed through simulation of the Simulink model of the system at hand.

The steady-state response to a sinusoidal input is called the frequency response. The frequency response of a system with transfer function $G(s)$ to input $F_0 \sin\omega t$ is

$$F_0 \left|G(j\omega)\right| \sin(\omega t + \phi), \quad \phi = \angle G(j\omega)$$

where $G(j\omega)$ is the FRF.

A Bode diagram consists of two curves versus frequency: $20\log\left|G(j\omega)\right|$ with unit in decibels, and the phase ϕ in degrees. The curves are sketched on semilog paper, using a linear scale for magnitude (in decibels) and phase (in degrees) and a logarithmic scale for frequency (in radians per second). The formal solution of the state equation

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

is expressed as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)\,d\tau$$

where $e^{\mathbf{A}t}$ is the matrix exponential of $\mathbf{A}$.

The RK4 method is designed to numerically solve any set of (linear or nonlinear) state-variable equations

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(a) = \mathbf{x}_0, \quad a \le t \le b$$

where $\mathbf{x}_0$ is the initial state vector and $[a,b]$ is the interval in which the system is solved. For an integer $N > 0$ and step size $h = (b - a)/N$, the mesh points $t_i = a + ih$, $i = 0, 1, 2,\ldots, N - 1$, partition $[a,b]$ into N subintervals. Knowing $\mathbf{x}_0$, the solution vector $\mathbf{x}_i$ at each of the subsequent mesh points t_i is obtained via

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{1}{6}h(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4), \quad i = 0,1,2,\ldots,N-1$$

where

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(t_i, \mathbf{x}_i) \\
\mathbf{k}_2 &= \mathbf{f}\left(t_i + \frac{1}{2}h, \mathbf{x}_i + \frac{1}{2}h\mathbf{k}_1\right) \\
\mathbf{k}_3 &= \mathbf{f}\left(t_i + \frac{1}{2}h, \mathbf{x}_i + \frac{1}{2}h\mathbf{k}_2\right) \\
\mathbf{k}_4 &= \mathbf{f}(t_i + h, \mathbf{x}_i + h\mathbf{k}_3)
\end{aligned}$$

REVIEW PROBLEMS

1. A system is modeled as

$$\dot{y} + \tfrac{1}{2}y = u(t) + 2u_r(t), \quad y(0) = 0$$

where $u(t)$ and $u_r(t)$ denote the unit-step and unit-ramp functions, respectively.

a. Find the response $y(t)$ in closed form.

b. Find and plot $y(t)$, $0 \le t \le 10$, using the `lsim` command.

2. A first-order system is modeled as

$$\tfrac{1}{2}\dot{y} + 3y = 2u(t), \quad y(0) = 0$$

where $u(t)$ is the unit-step function.

a. Determine the response $y(t)$ in closed form and find y_{ss}.

b. Find and plot $y(t)$ using the `step` command.

3. The model of a second-order system is given as

$$4\ddot{x} + 4\dot{x} + 17x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 1$$

a. Identify the damping type and find the response in closed form.

b. Find and plot the response using the `initial` command.

4. A second-order dynamic system is modeled as

$$9\ddot{x} + 12\dot{x} + 5x = 10\delta(t), \quad x(0) = 0, \quad \dot{x}(0) = \tfrac{1}{2}$$

a. Find the response $x(t)$ in closed form.

b. Find and plot the response using the `impulse` and `initial` commands.

5. The mathematical model of a dynamic system is described by

$$\ddot{x} + \tfrac{3}{2}\dot{x} + x = \tfrac{5}{2}u(t), \quad x(0) = 1, \quad \dot{x}(0) = 0$$

where $u(t)$ is the unit-step. Plot the response $x(t)$ by

a. Using the `initial` and `step` commands.

b. Simulating the Simulink model of the system.

6. The mathematical model of a dynamic system is described by

$$4\ddot{x} + 4\dot{x} + 5x = \tfrac{1}{3}u_r(t), \quad x(0) = 0, \quad \dot{x}(0) = \tfrac{1}{2}$$

where $u_r(t)$ is the unit-ramp. Find and plot the response $x(t)$ by

a. Using the `lsim` command over $0 \le t \le 5$.

b. Simulating the Simulink model of the system.

7. A dynamic system model is given as

$$\begin{cases} \ddot{x}_1 + \dot{x}_1 - \frac{2}{3}(x_2 - x_1) = 2u(t) \\ \dot{x}_2 + x_2 + \frac{2}{3}(x_2 - x_1) = 5u(t) \end{cases}, \quad x_1(0) = 1, \quad x_2(0) = 0, \quad \dot{x}_1(0) = 0$$

where $u(t)$ is the unit-step. Find and plot x_1, $0 \le t \le 10$, by
   a. Using the `lsim` command.
   b. Simulating the Simulink model of the system.

8. The governing equations for a system are derived as

$$\begin{cases} \ddot{x}_1 + \dot{x}_1 + 3x_1 - 2x_2 = \delta(t) \\ \dot{x}_2 - 3x_1 + 3x_2 = 0 \end{cases}, \quad x_1(0) = 0, \quad x_2(0) = 1, \quad \dot{x}_1(0) = 0$$

where $\delta(t)$ is the unit-impulse. Find and plot x_2, $0 \le t \le 10$, by using the `impulse` and `initial` commands.

9. Find the frequency response of a system whose model is

$$36\ddot{x} + 24\dot{x} + 5x = 10\sin(t/2)$$

10. Determine the damping ratio associated with a second-order system in the standard form of Equation 8.32, which corresponds to a maximum logarithmic magnitude of 12.25 dB.

11. Decide whether the magnitude plot (in decibels) for the following transfer function attains a maximum peak. If so, calculate the maximum value and the corresponding frequency:

$$\frac{1}{4s^2 + 4s + 3}$$

12. The state–space form of a system model is given as

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} = \mathbf{Cx} \end{cases}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{Bmatrix} e^{-t} \\ 2 \end{Bmatrix}, \quad \mathbf{x}_0 = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

Find the output vector $\mathbf{y}$ using the formal solution of the state vector.

13. Consider

$$\begin{cases} \dot{x}_1 = -2x_1|x_1| + ax_2 - 1 + \frac{1}{2}\cos t \\ \dot{x}_2 = -x_1 - x_2 - 1 \end{cases}, \quad \begin{matrix} x_1(0) = 1 \\ x_2(0) = 1 \end{matrix}$$

where a is a parameter. Using RK4, solve the system for $a = 1$, $a = 2$, and plot the two resulting $x_1(t)$, $0 \le t \le 10$, in the same graph.

14. The nonlinear state-variable equations for a dynamic system are derived as

$$\begin{cases} \dot{x}_1 = -2x_2 + 0.7\sin t \\ \dot{x}_2 = x_1^3 - 3x_2 + 8 \end{cases}, \quad \begin{matrix} x_1(0) = 1 \\ x_2(0) = 0 \end{matrix}$$

Plot $x_2(t)$, $0 \le t \le 10$, by

a. Using the RK4 method.
b. Simulating the Simulink model of the system.

9 Introduction to Vibrations

Vibration can be regarded as a subset of dynamics, in which a system subjected to restoring forces oscillates about an equilibrium position. The restoring forces are due to elasticity or gravity. Two different types of excitations cause a system to vibrate: initial excitation and external excitation. The vibration of a system caused by nonzero initial excitations, including initial displacements or initial velocities (or both), is known as free vibration. The vibration of a system caused by externally applied forces is known as forced vibration.

In this chapter, we first extend the knowledge learned in Chapters 5 and 8 to the analysis of free vibration and forced vibration of single-degree-of-freedom or two-degree-of-freedom systems. Section 9.1 introduces the free vibration of Coulomb damped systems, in which energy is dissipated via dry friction. Section 9.2 considers the forced vibration of systems with rotating eccentric masses and systems with harmonically moving supports. Sections 9.1 and 9.2 also cover topics such as logarithmic decrement and bandwidth, both of which can be used to estimate the widely used viscous damping model. To reduce the effects of undesired vibration, Section 9.3 discusses the design of vibration suppression systems, including vibration isolators and vibration absorbers. For multi-degree-of-freedom systems, it is convenient to use the matrix-based method to perform vibration analyses. In Section 9.4, the concepts of the eigenvalue problem, natural modes, and orthogonality of modes are presented using matrix algebra. The modal analysis method is developed and used to obtain the response to initial or harmonic excitations. The chapter concludes with coverage of vibration measurement technology. Section 9.5 introduces the hardware available for vibration testing and the methods used to identify system parameters, such as natural frequencies and damping ratios.

9.1 FREE VIBRATION

Consider a single-degree-of-freedom, viscously damped system subjected to nonzero initial excitations. Its governing differential equation is given by

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0 \tag{9.1}$$

or

$$\ddot{x}(t) + 2\zeta\omega_{\rm n}\dot{x}(t) + \omega_{\rm n}^2 x(t) = 0, \tag{9.2}$$

with $x(0) = x_0$ and $\dot{x}(0) = v_0$. As indicated in Section 8.2, the nature of the system response to initial excitations depends on the value of the damping ratio ζ. For an overdamped system, $\zeta > 1$, the response represents aperiodic decay, which is also true for a critically damped system, $\zeta = 1$. For an underdamped system, $0 < \zeta < 1$, the response represents oscillatory decay. For an undamped system, $\zeta = 0$, the response represents harmonic oscillation with natural frequency $\omega_{\rm n}$.

Viscous damping is a widely used damping model, but not the only one. Damping is a very complex phenomenon and a wide range of damping models can be found in the literature. In this section, we first discuss the measurement of the viscous damping ratio ζ based on transient time response plots and then introduce another damping model, known as Coulomb damping.

9.1.1 Logarithmic Decrement

Unlike mass m and spring stiffness k, which could both be easily measured with static tests, the viscous damping coefficient b has to be measured with a dynamic test. A common way is to use the free response of the whole system to measure the damping ratio ζ and then determine the damping coefficient b by using $b = 2\zeta\sqrt{mk}$.

As discussed in Section 8.2, the free response of an underdamped single-degree-of-freedom system is

$$x(t) = \mathrm{e}^{-\zeta\omega_\mathrm{n}t}\left(x_0 \cos(\omega_\mathrm{d}t) + \frac{\zeta\omega_\mathrm{n}x_0 + v_0}{\omega_\mathrm{d}}\sin(\omega_\mathrm{d}t)\right), \tag{9.3}$$

where the damped natural frequency is $\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2}$. Equation 9.3 can also be written as

$$x(t) = A\mathrm{e}^{-\zeta\omega_\mathrm{n}t}\cos(\omega_\mathrm{d}t - \phi), \tag{9.4}$$

where the amplitude A and the phase ϕ are given by

$$A = \sqrt{x_0^2 + \left(\frac{\zeta\omega_\mathrm{n}x_0 + v_0}{\omega_\mathrm{d}}\right)^2} \tag{9.5}$$

and

$$\phi = \tan^{-1}\frac{\zeta\omega_\mathrm{n}x_0 + v_0}{\omega_\mathrm{d}x_0}. \tag{9.6}$$

Figure 9.1 shows the system response to initial excitations, where $T = 2\pi/\omega_\mathrm{d}$ is the period of damped oscillation.

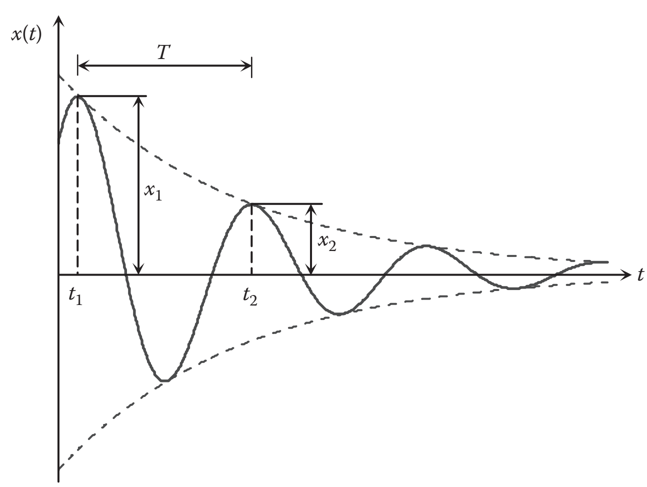


FIGURE 9.1 Free response of an underdamped single-degree-of-freedom system to initial excitations.

Note that the peak drops after one cycle of vibration, and the ratio between the first and the second peaks is

$$\frac{x_1}{x_2} = \frac{x(t_1)}{x(t_2)} = \frac{A\mathrm{e}^{-\zeta\omega_\mathrm{n}t_1}\cos(\omega_\mathrm{d}t_1 - \phi)}{A\mathrm{e}^{-\zeta\omega_\mathrm{n}t_2}\cos(\omega_\mathrm{d}t_2 - \phi)} = \frac{\mathrm{e}^{-\zeta\omega_\mathrm{n}t_1}\cos(\omega_\mathrm{d}t_1 - \phi)}{\mathrm{e}^{-\zeta\omega_\mathrm{n}(t_1+T)}\cos[\omega_\mathrm{d}(t_1 + T) - \phi]}. \tag{9.7}$$

Because $\omega_\mathrm{d}T = 2\pi$, we have $\zeta\omega_\mathrm{n}T = \zeta\omega_\mathrm{n}(2\pi/\omega_\mathrm{d}) = 2\pi\zeta/\sqrt{1-\zeta^2}$ and $\cos[\omega_\mathrm{d}(t_1 + T) - \phi] = \cos(\omega_\mathrm{d}t_1 + 2\pi - \phi) = \cos(\omega_\mathrm{d}t_1 - \phi)$. Thus, Equation 9.7 reduces to

$$\frac{x_1}{x_2} = \mathrm{e}^{2\pi\zeta/\sqrt{1-\zeta^2}}. \tag{9.8}$$

Taking the natural logarithm of both sides of Equation 9.8 yields

$$\delta = \ln\frac{x_1}{x_2} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}, \tag{9.9}$$

where δ is called the logarithmic decrement. It turns out that the ratio of any two consecutive displacement peaks gives the same result as Equation 9.8,

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \cdots = \frac{x_n}{x_{n+1}} = \mathrm{e}^{2\pi\zeta/\sqrt{1-\zeta^2}}. \tag{9.10}$$

Thus, the logarithmic decrement δ can be determined by measuring any two consecutive displacement peaks. The damping ratio ζ can then be found from Equation 9.9 as

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}. \tag{9.11}$$

For a more accurate estimation, the damping ratio ζ can be determined by measuring the displacements of two peaks separated by a number of periods instead of two consecutive peaks. If we denote the peak displacements at time t_1 and t_{n+1} as x_1 and x_{n+1}, respectively, where $t_{n+1} = t_1 + nT$, then the ratio between the two peak displacements can be written as

$$\frac{x_1}{x_{n+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\cdots\frac{x_n}{x_{n+1}}. \tag{9.12}$$

Taking the natural logarithm of both sides of Equation 9.12, we have

$$\ln\frac{x_1}{x_{n+1}} = \ln\left(\frac{x_1}{x_2}\frac{x_2}{x_3}\cdots\frac{x_n}{x_{n+1}}\right) = \ln\frac{x_1}{x_2} + \ln\frac{x_2}{x_3} + \cdots + \ln\frac{x_n}{x_{n+1}} = n\delta, \tag{9.13}$$

which yields the logarithmic decrement as

$$\delta = \frac{1}{n}\ln\frac{x_1}{x_{n+1}}. \tag{9.14}$$

Substituting Equation 9.14 into Equation 9.11 gives the damping ratio. If the damping is very small, that is, $\zeta << 1$ and $\sqrt{1-\zeta^2} \approx 1$, then Equation 9.9 gives

$$\zeta \approx \frac{\delta}{2\pi}. \tag{9.15}$$

Example 9.1: Logarithmic Decrement

A vibrating system consisting of a mass of 4.5 kg and a spring of stiffness 5250 N/m is viscously damped. The ratio of any two consecutive amplitudes is 4/3.

a. Determine the logarithmic decrement δ.
b. Determine the exact value of the damping ratio ζ.
c. Determine the damping coefficient *b*.
d. Assuming small damping, recalculate the damping ratio ζ and determine the percentage of error.

Solution

a. The logarithmic decrement can be determined by measuring the displacements of any two consecutive peaks, as

$$\delta = \ln\frac{x_j}{x_{j+1}} = \ln\frac{4}{3} = 0.2877.$$

b. The exact value of the damping ratio is

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2+\delta^2}} = \frac{0.2877}{\sqrt{(2\pi)^2+(0.2877)^2}} = 0.0457.$$

c. The viscous damping coefficient is

$$b = 2\zeta\sqrt{mk} = 2(0.0457)\sqrt{4.5(5250)} = 14.0512\,\text{N·s/m}.$$

d. For small damping,

$$\zeta \approx \frac{\delta}{2\pi} = \frac{0.2877}{2\pi} = 0.0458$$

and the percentage of error is 0.22%. Note that the error is close to zero and thus the assumption of small damping is valid.

9.1.2 Coulomb Damping

The linear viscous damping model was adopted in the previous chapters. The viscous damping force is linearly dependent on velocity, simplifying the analysis. Although this is widely used in vibration, it is not the only damping model. Coulomb damping is another type of damping in which energy is dissipated via dry friction.

Figure 9.2 shows a mass–spring system subject to Coulomb damping, in which N is the normal force and F_f is the dry friction force. Denote the kinetic friction coefficient as μ_k. Note that the

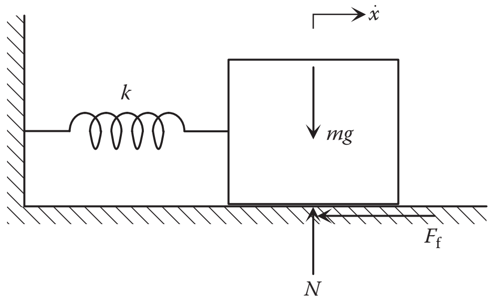


FIGURE 9.2 A mass–spring system subject to Coulomb damping.

friction force remains constant in magnitude, which is $\mu_k mg$, and the force is opposite in direction to the motion or the velocity $\dot{x}$. Introducing the sign function sgn (·),

$$\text{sgn}(\dot{x}) = \begin{cases} 1, & \dot{x} > 0, \\ -1, & \dot{x} < 0, \end{cases} \tag{9.16}$$

the friction force can be expressed as

$$\mathbf{F}_\text{f} = -F_\text{f}\text{sgn}(\dot{x}) = \begin{cases} -F_\text{f}, & \dot{x} > 0, \\ F_\text{f}, & \dot{x} < 0, \end{cases} \tag{9.17}$$

where $F_\text{f} = \mu_k mg$. Equation 9.17 implies that the friction force points to the left if the mass moves to the right and points to the right if the mass moves to the left.

Using Equation 9.17, we can write the dynamics equation of motion as

$$m\ddot{x} + F_\text{f}\text{sgn}(\dot{x}) + kx = 0, \tag{9.18}$$

which is a nonlinear equation that can be separated into two linear equations,

$$m\ddot{x} + F_\text{f} + kx = 0, \quad \dot{x} > 0, \tag{9.19}$$

$$m\ddot{x} - F_\text{f} + kx = 0, \quad \dot{x} < 0. \tag{9.20}$$

Without loss of generality, let us assume the initial conditions to be $x(0) = x_0 > 0$ and $\dot{x}(0) = v_0 = 0$. Due to the restoring spring force, the mass first moves from right to left as shown in Figure 9.3a. The velocity is negative and the dynamics of the system is expressed by Equation 9.20, which can be rewritten as

$$m\ddot{x} + k(x - \Delta) = 0, \tag{9.21}$$

where $\Delta = F_\text{f}/k$. For a given system, the mass m, spring stiffness k, and kinetic friction coefficient μ_k are all constants. Thus, Δ is a constant and $\ddot{\Delta} = 0$. Equation 9.21 can then be rewritten as

$$m(\ddot{x} - \ddot{\Delta}) + k(x - \Delta) = 0. \tag{9.22}$$

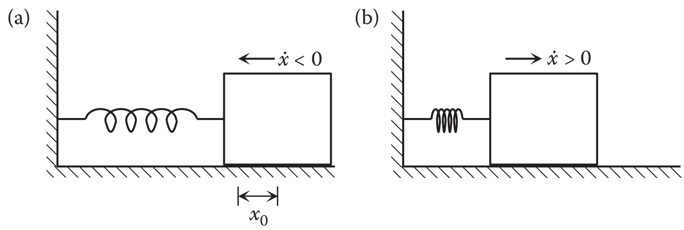


FIGURE 9.3 Motion of the mass with (a) negative velocity and (b) positive velocity.

Note that Equation 9.22 has the same format as $m\ddot{x} + kx = 0$ if we replace $x - \Delta$ with x. Therefore, the free response of the system can be determined by using Equations 9.4 through 9.6, and neglecting damping. If we let $x' = x - \Delta$, the initial conditions can be expressed as $x'(0) = x(0) - \Delta = x_0 - \Delta$ and $\dot{x}'(0) = \dot{x}(0) - \dot{\Delta} = 0$. Substituting the initial conditions into Equations 9.4 through 9.6 yields

$$x(t) - \Delta = (x_0 - \Delta)\cos(\omega_n t) \tag{9.23}$$

or

$$x(t) = (x_0 - \Delta)\cos(\omega_n t) + \Delta. \tag{9.24}$$

When the spring reaches maximum compression as shown in Figure 9.3b, the velocity of the mass reduces to zero, that is, $\dot{x}(t) = -(x_0 - \Delta)\omega_n \sin(\omega_n t) = 0$, which yields $t = \pi/\omega_n = T/2$. The corresponding displacement is $-(x_0 - 2\Delta)$. The mass then starts to move from left to right and the velocity becomes positive. Thus, Equation 9.24 is only valid for $0 \le t < T/2$.

For $t \ge T/2$, the dynamics of the system is expressed by Equation 9.19, which can be rewritten as

$$m\ddot{x} + k(x + \Delta) = 0 \tag{9.25}$$

or

$$m(\ddot{x} + \ddot{\Delta}) + k(x + \Delta) = 0. \tag{9.26}$$

The response of the system can also be determined by using Equations 9.4 through 9.6 and neglecting damping. Let $x' = x + \Delta$. Note that the initial values of x and $\dot{x}$ in Equation 9.26 are the displacement and the velocity at time $T/2$, respectively, that is, $x(T/2) = -(x_0 - 2\Delta)$ and $\dot{x}(T/2) = 0$. As a result, $x'(0) = -(x_0 - 3\Delta)$ and $\dot{x}'(0) = 0$. The solution of Equation 9.26 is

$$x(t) + \Delta = (x_0 - 3\Delta)\cos(\omega_n t) \tag{9.27}$$

or

$$x(t) = (x_0 - 3\Delta)\cos(\omega_n t) - \Delta. \tag{9.28}$$

When the spring reaches maximum elongation, the velocity of the mass reduces to zero once again. By differentiating Equation 9.28 and equating it to zero, we obtain the time corresponding to the maximum elongation, that is, T. The displacement at $t = T$ is $x_0 - 4\Delta$. After this point, the motion reverses in direction and the mass moves from right to left. Thus, Equation 9.28 is only valid for $T/2 \le t < T$.

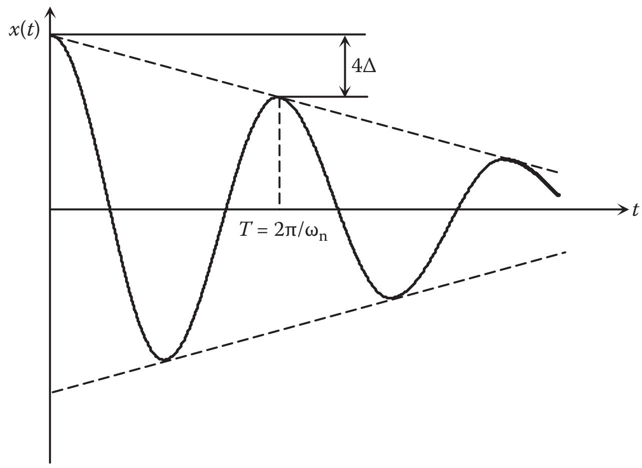


FIGURE 9.4 Free response of a mass–spring system subject to Coulomb damping.

The above discussion gives the displacement magnitudes at $t = 0$, $T/2$, and T, which are x_0, $x_0 - 2\Delta$, and $x_0 - 4\Delta$, respectively. It can be concluded that the displacement magnitude is reduced by 2Δ after every half-cycle. This process is repeated as the mass oscillates back and forth about its equilibrium position. The motion stops when the displacement is not large enough for the restoring spring force to overcome the static friction force. The free response of the mass–spring system with Coulomb damping is shown in Figure 9.4. Different from viscously damped systems, the envelope of the response for a vibrating system with Coulomb damping is a straight line instead of an exponential decay curve, as shown in Figure 9.1.

Example 9.2: Coulomb Damping

For a Coulomb damped system, it is observed that the first three consecutive maximum displacement amplitudes x_0, x_1, and x_2 for a free vibration are 15, 12.55, and 10.10 cm, respectively. The time duration between any two of these amplitudes is 0.7 s.

a. Determine the value of the kinetic friction coefficient μ_k.
b. Determine the position when the oscillation stops.

Solution

a. The decay per cycle is

$$4\Delta = 15 - 12.55 = 12.55 - 10.10 = 2.45 \text{ cm},$$

which gives $\Delta = 6.125 \times 10^{-3}$ m. The time duration between any two of these amplitudes is 0.7 s. This implies the period of vibration is $T = 0.7$ s and

$$\omega_n = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} = \frac{2\pi}{0.7} = 8.98 \text{ rad/s}.$$

Thus, the kinetic friction coefficient is

$$\mu_k = \frac{F_f}{mg} = \frac{k\Delta}{mg} = \frac{8.98^2(6.125\times10^{-3})}{9.81} = 0.05.$$

b. Note that the displacement magnitude is reduced by 2Δ after every half-cycle. The motion stops at the end of the half-cycle for which the displacement magnitude is smaller than 2Δ. This can be expressed mathematically as

$$x_0 - n(2\Delta) < 2\Delta,$$

where n denotes the number of half-cycles before stopping. Solving for n and substituting the appropriate values, we find

$$n > \frac{x_0}{2} - 1 = \frac{0.15}{2(6.125\times10^{-3})} - 1 = 11.24.$$

The smallest integer satisfying the inequality is n = 12. Thus, the oscillation stops after 12 half-cycles, or 6 cycles, with

$$x(t = 12T/2) = x_0 - 12(2\Delta) = 0.15 - 12(2\times 6.125\times 10^{-3}) = 3\times 10^{-3}\ \text{m}.$$

PROBLEM SET 9.1

1. A lightly damped single-degree-of-freedom system is subjected to free vibration. The response of the system is shown in Figure 9.5. Estimate the value of the viscous damping ratio ζ.
2. An underdamped single-degree-of-freedom vibrating system is viscously damped. It is observed that the maximum displacement amplitude during the third cycle is 60% of the first. Calculate the damping ratio ζ and determine the maximum displacement amplitude during the fifth cycle as a fraction of the first.

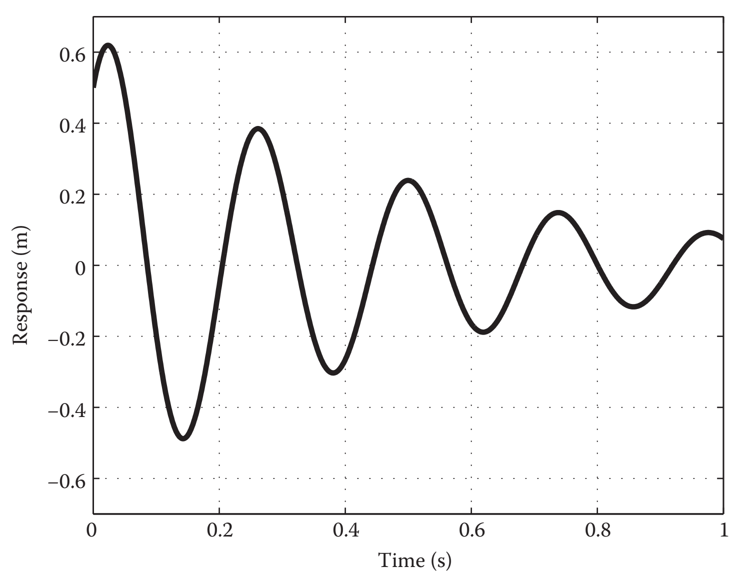


FIGURE 9.5 Problem 1.

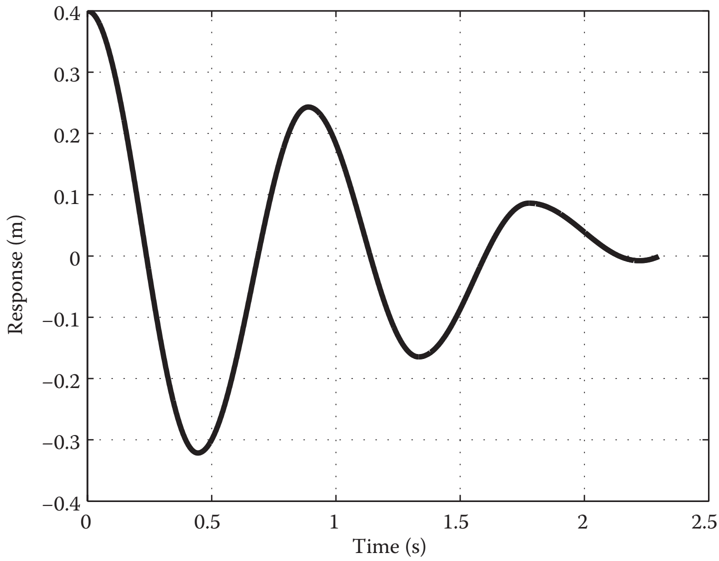


FIGURE 9.6 Problem 3.

3. Figure 9.6 shows the free response of a single-degree-of-freedom system subjected to Coulomb damping. The parameters of the system include the mass (m = 40 kg) and the spring stiffness (k = 2000 N/m). Estimate the value of the kinetic friction coefficient μ_k.
4. A Coulomb damped vibrating system consists of a mass of 8 kg and a spring of stiffness 6000 N/m. The kinetic friction coefficient μ_k is 0.15. The initial conditions are x_0 = 0.025 m and v_0 = 0 m/s.
   a. Determine the decay per cycle.
   b. Determine the position when the oscillation stops.

9.2 FORCED VIBRATION

The vibration of a system caused by externally applied forces is known as forced vibration. A very important class of external excitations involves harmonic forces. Consider a single-degree-of-freedom, viscously damped system subjected to a harmonic excitation, $f(t) = F_0\sin(\omega t)$. Its governing differential equation is given by

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F_0\sin(\omega t) \tag{9.29}$$

or

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = \omega_n^2\frac{F_0}{k}\sin(\omega t). \tag{9.30}$$

As discussed in Section 8.3, the steady-state response of a system subjected to a harmonic excitation $f(t)$ may be obtained using the frequency response, which is a very important concept in vibration. The steady-state response of the system described by Equation 9.29 or 9.30 is

$$x(t) = X\sin(\omega t + \phi), \tag{9.31}$$

where the amplitude and the phase angle are given by

$$X = \frac{F_0/k}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \tag{9.32}$$

and

$$\phi = -\tan^{-1}\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}. \tag{9.33}$$

Note that the value of X depends on the driving frequency ω, and X is called the dynamic amplitude. The term F_0/k in Equation 9.32 has units of displacement and is known as the static deflection. If we use x_{st} to denote the static deflection, we can obtain the dimensionless ratio

$$\frac{X}{x_{st}} = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}. \tag{9.34}$$

When compared with other excitations, more information on the steady-state response to harmonic excitations can be extracted using the frequency domain technique rather than the time domain technique. In this section, we first discuss how to measure the viscous damping coefficient ζ based on frequency response plots. Systems subjected to two types of harmonic excitations, including rotating unbalance and base excitation, are then introduced and their responses are determined using the frequency response method.

9.2.1 Half-Power Bandwidth

Figure 9.7 shows the frequency response of the dimensionless ratio X/x_{st} near the natural frequency for a viscously damped system. As proven in Section 8.3, the maximum peak occurs at $\omega/\omega_n = \sqrt{1-2\zeta^2}$. If the system is lightly damped, the peak occurs in the immediate neighborhood

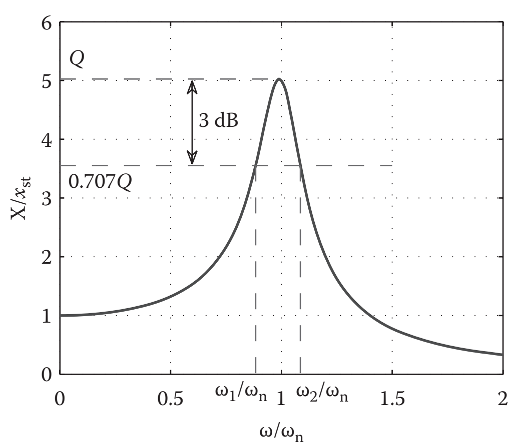


FIGURE 9.7 Magnitude of the frequency response for a viscously damped system.

of $\omega/\omega_n = 1$, as shown in Figure 9.7. From Equation 9.34, the corresponding value of the peak amplitude can be approximated as

$$Q = \left(\frac{X}{x_{st}}\right)_{max} \approx \frac{1}{2\zeta} \tag{9.35}$$

for small damping. The symbol Q introduced in Equation 9.35 is known as the quality factor or Q factor, which is usually used in electrical engineering applications and is related to the amplitude at resonance. Thus, the damping ratio ζ can be obtained from Equation 9.35 as

$$\zeta \approx \frac{1}{2Q}, \tag{9.36}$$

which can be used as a quick way of estimating the viscous damping ratio by measuring the peak amplitude Q.

For a more accurate estimation, the damping ratio ζ can be determined by measuring the frequencies at two half-power points instead of the peak amplitude. Note that for a viscously damped system subjected to a harmonic force, the velocity response is

$$\dot{x}(t) = \omega X \cos(\omega t + \phi), \tag{9.37}$$

which is obtained from Equation 9.31 by taking the time derivative. The maximum kinetic energy is

$$T_{max} = \frac{1}{2} m\omega^2 X_{max}^2 = \frac{1}{2} m\omega^2 x_{st}^2 Q^2, \tag{9.38}$$

which is proportional to the square of Q. Half-power points occur at frequencies in which the power drops to half its maximum level. As shown in Figure 9.7, points 1 and 2, at which the amplitude decreases to $Q/\sqrt{2} \approx 0.707Q$, or drops by 3 dB down from the peak, are half-power points.

Let $\nu = \omega/\omega_n$. At the half-power points, combining Equations 9.34 and 9.35 gives

$$\frac{1}{\sqrt{(1-\nu^2)^2 + (2\zeta\nu)^2}} \approx \frac{1}{\sqrt{2}} \frac{1}{2\zeta} \tag{9.39}$$

or

$$\nu^4 - 2(1 - 2\zeta^2)\nu^2 + (1 - 8\zeta^2) = 0, \tag{9.40}$$

which has the solutions

$$\nu^2 = (1 - 2\zeta^2) \pm 2\zeta\sqrt{1 + \zeta^2}. \tag{9.41}$$

If the system is lightly damped, ζ^2 can be ignored and Equation 9.41 can be approximated as

$$\nu^2 = 1 \pm 2\zeta \tag{9.42}$$

or

$$\left(\frac{\omega_1}{\omega_n}\right)^2 = 1 - 2\zeta, \tag{9.43}$$

$$\left(\frac{\omega_2}{\omega_n}\right)^2 = 1 + 2\zeta. \tag{9.44}$$

Subtracting Equation 9.43 from 9.44 gives

$$\frac{\omega_2^2 - \omega_1^2}{\omega_n^2} = 4\zeta. \tag{9.45}$$

In general, the natural frequency ω_n is between the half-power points and $\omega_n \approx \frac{1}{2}(\omega_1 + \omega_2)$ for light damping. Thus, we have

$$\frac{\omega_2 - \omega_1}{\omega_n} \approx 2\zeta \tag{9.46}$$

or

$$\zeta \approx \frac{\Delta\omega}{2\omega_n}, \tag{9.47}$$

where $\Delta\omega = \omega_2 - \omega_1$ is referred to as the bandwidth of the system.

Example 9.3: Half-Power Bandwidth

A viscously damped system consisting of a mass of 5 kg and a spring of stiffness 12,500 N/m is subjected to a harmonic force excitation. The frequency ratios ω_1/ω_n and ω_2/ω_n at half-power points are observed to be 0.9093 and 1.0713, respectively. Assume that the system is lightly damped.

a. Determine the bandwidth of the system.
b. Determine the damping ratio ζ.

Solution

a. Noting the undamped natural frequency is $\omega_n = \sqrt{k/m} = \sqrt{12{,}500/5} = 50\ \text{rad/s}$, the bandwidth of the system is calculated as

$$\Delta\omega = \omega_2 - \omega_1 = \omega_n\left(\frac{\omega_2}{\omega_n} - \frac{\omega_1}{\omega_n}\right) = 50(1.0713 - 0.9093) = 8.1\ \text{rad/s},$$

b. For small damping, the damping ratio is

$$\zeta \approx \frac{\Delta\omega}{2\omega_n} = \frac{8.1}{2(50)} = 0.081.$$

9.2.2 Rotating Unbalance

Unbalance in rotating machines is a common source of harmonic excitation. The unbalance is caused by the rotating part, for which the mass center does not coincide with the center of rotation. Figure 9.8 shows a system with an unbalanced mass m rotating at a constant angular velocity of ω. The distance between the unbalanced mass and the center of rotation is e, which represents the eccentricity. The mass of the entire system is M and its motion is assumed to be constrained along the vertical direction only. The entire system can therefore be considered as a single-degree-of-freedom system.

Choose the static equilibrium position of the entire system as the coordinate origin. Denote the mass of the nonrotating part as $M - m$ and the corresponding vertical displacement as the generalized coordinate x. Thus, the vertical displacement of the rotating unbalanced mass is $x + e\sin(\omega t)$. The free-body diagrams of $M - m$ and m are shown in Figure 9.9.

Applying Newton's second law to $M - m$ and m along the vertical direction yields

$$(M-m)\ddot{x} = -F_V - kx - b\dot{x}, \tag{9.48}$$

$$m\frac{d^2}{dt^2}[x + e\sin(\omega t)] = F_V, \tag{9.49}$$

where F_V is the vertical component of the internal force between the rotating and the nonrotating parts. Combining Equations 9.48 and 9.49, and eliminating F_V, gives the differential equation of the system with rotating unbalance,

$$(M-m)\ddot{x} + m\frac{d^2}{dt^2}[x + e\sin(\omega t)] + b\dot{x} + kx = 0. \tag{9.50}$$

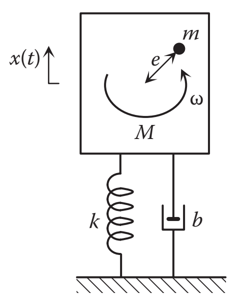


FIGURE 9.8 A system with rotating unbalance.

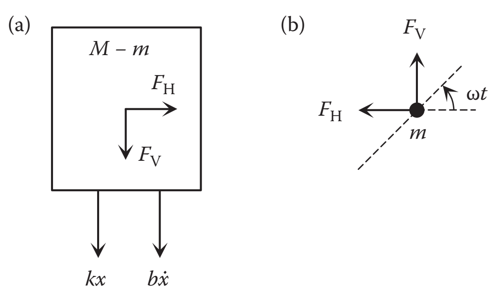


FIGURE 9.9 Free-body diagram of the system shown in Figure 9.8: (a) nonrotating part $M - m$ and (b) rotating unbalance m.

Differentiating $x + e\sin(\omega t)$ with respect to time twice, we have

$$M\ddot{x} + b\dot{x} + kx = me\omega^2\sin(\omega t), \tag{9.51}$$

which implies that the effect of a rotating unbalance mass is to exert a harmonic force $me\omega^2\sin(\omega t)$ on the system.

Note that Equation 9.51 is similar to Equation 9.29, except that the magnitude of the harmonic force is $me\omega^2$ instead of F_0. Thus, with this modification, we can obtain the steady-state solution of the system using Equations 9.31 through 9.33. The dynamic amplitude is given by

$$X = \frac{me\omega^2/k}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}, \tag{9.52}$$

where $\omega_n = \sqrt{k/M}$. Replacing k in Equation 9.52 with $M\omega_n^2$ yields

$$X = \frac{me(\omega/\omega_n)^2/M}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}. \tag{9.53}$$

For a system with rotating unbalance, the dimensionless ratio

$$\frac{MX}{me} = \frac{(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \tag{9.54}$$

is usually plotted versus the dimensionless ratio ω/ω_n, known as normalized frequency or frequency ratio. Figure 9.10 shows the magnitude of the frequency response for a system with rotating unbalance, where the vertical axis is $MX/(me)$. For low-speed rotations, that is, $\omega/\omega_n \ll 1$, the magnitude

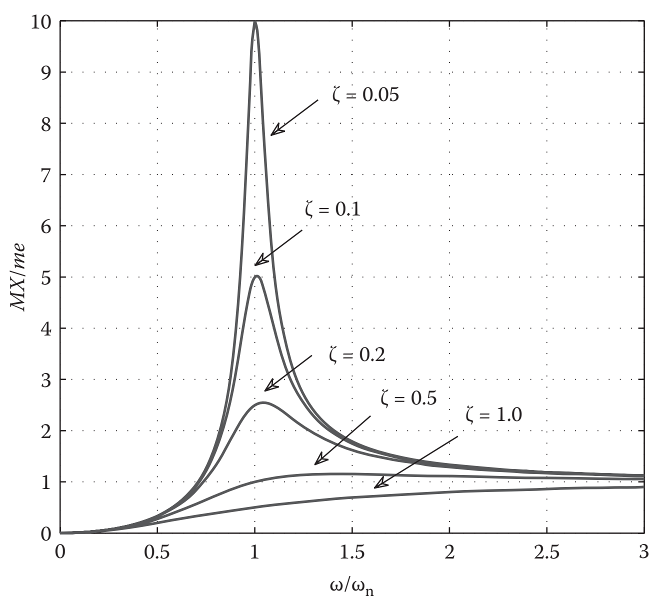


FIGURE 9.10 Magnitude of the frequency response for a system with rotating unbalance.

of the response $MX/(me)$ is very small and thus the vibration amplitude X is close to zero. Resonance peaks occur in the neighborhood of $\omega/\omega_n = 1$. For high-speed rotations, that is, $\omega/\omega_n \gg 1$, the magnitude of the response $MX/(me)$ tends to 1. This implies that the dynamic amplitude X is me/M, which is constant regardless of the driving frequency and the amount of damping.

Example 9.4: Rotating Unbalance

An electric motor of mass M = 500 kg is mounted on a simply supported beam with negligible mass. Assume that the supporting beam is equivalent to a spring of stiffness k = 4500 kN/m and the damping ratio of the system is 0.1. The unbalance in the rotor of the motor is me = 0.5 kg·m. Determine the dynamic amplitude X of the motor when it runs at a speed of 950 rpm.

Solution

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{4500\times 10^3}{500}} = 94.87 \text{ rad/s.}$$

When the motor runs at a speed of 950 rpm, the frequency ratio is

$$\frac{\omega}{\omega_n} = \frac{950(2\pi)}{60(94.87)} = 1.05.$$

Thus,

$$\frac{MX}{me} = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta(\omega/\omega_n)\right]^2}} = \frac{1.05^2}{\sqrt{(1-1.05^2)^2 + [2(0.1)(1.05)]^2}} = 4.72,$$

which gives the dynamic amplitude as

$$X = \frac{me}{M}\left(\frac{MX}{me}\right) = \frac{0.5(4.72)}{500} = 4.72\times 10^{-3} \text{ m.}$$

9.2.3 Harmonic Base Excitation

Many applications in vibration involve systems with displacement as the input. Examples include a machine placed on a foundation undergoing vibration and a vehicle traveling on a wavy road. Assume that each of the two systems mentioned can be modeled as a single-degree-of-freedom system as shown in Figure 9.11, where x and y are the displacements of the mass and the base, respectively.

Applying Newton's second law gives the differential equation of motion as

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = b\dot{z}(t) + kz(t) \tag{9.55}$$

or

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 2\zeta\omega_n\dot{z}(t) + \omega_n^2 z(t). \tag{9.56}$$

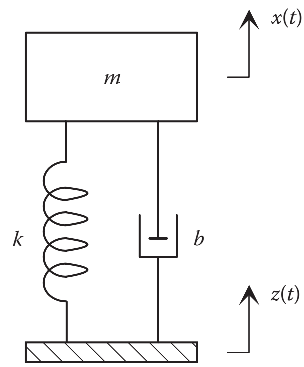


FIGURE 9.11 A single-degree-of-freedom system undergoing base excitation.

As discussed in Section 8.3, the system's frequency response function is

$$G(j\omega) = \frac{X(j\omega)}{Z(j\omega)} = \frac{j2\zeta\omega_n\omega + \omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2} = \frac{1 + j2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2 + j2\zeta\omega/\omega_n}, \tag{9.57}$$

where the magnitude and phase of the frequency response function are

$$|G(j\omega)| = \frac{\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} \tag{9.58}$$

and

$$\phi = \angle G(j\omega) = -\tan^{-1}\frac{2\zeta(\omega/\omega_n)^3}{1-(\omega/\omega_n)^2 + (2\zeta\omega/\omega_n)^2}. \tag{9.59}$$

Assume that the motion of the base is harmonic, for example, $z(t) = Z_0\sin(\omega t)$. Then, the steady-state response of the system is also harmonic, $x(t) = X\sin(\omega t + \phi)$, where the dynamic amplitude X is given by

$$X = Z_0|G(j\omega)| = Z_0\frac{\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}. \tag{9.60}$$

Thus, the dimensionless ratio between the dynamic amplitude X and the amplitude of the base displacement is

$$\frac{X}{Z_0} = \frac{\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}, \tag{9.61}$$

which is known as the transmissibility. Figure 9.12 shows the magnitude of the frequency response for a system with harmonic excitation, where the vertical axis is X/Z_0. Note that $X/Z_0 = 1$ for all

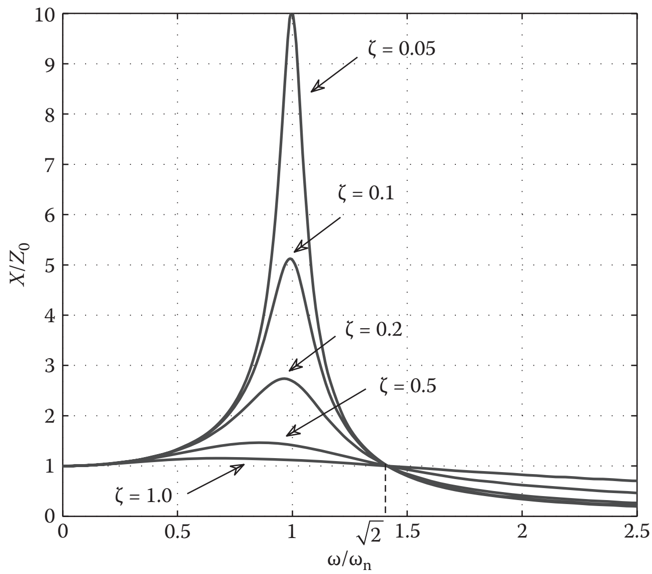


FIGURE 9.12 Magnitude of the frequency response for a system with harmonic base excitation.

curves when the frequency ratio is $\omega/\omega_n = \sqrt{2}$. This implies that the response has the same magnitude as the excitation. The response is amplified when $\omega/\omega_n < \sqrt{2}$ and is reduced when $\omega/\omega_n > \sqrt{2}$.

Example 9.5: Harmonic Base Excitation

The mass–spring–damper system in Example 5.18 represents a vehicle traveling on a rough road. Assume that the surface of the road can be approximated as a sine wave $z = Z_0\sin(\omega t)$, where $Z_0 = 0.01$ m and $\omega = 3.5$ rad/s. The mathematical model of the system is given by an ordinary differential equation $m\ddot{x} + b\dot{x} + kx = b\dot{z} + kz$, where $m = 3000$ kg, $b = 2000$ N·s/m, and $k = 50$ kN/m. Determine the dynamic amplitude of the vehicle.

Solution

The natural frequency and the damping ratio of the vehicle are

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{50 \times 10^3}{3000}} = 4.08 \text{ rad/s}$$

and

$$\zeta = \frac{b}{2\sqrt{mk}} = \frac{2000}{2\sqrt{(3000)(50 \times 10^3)}} = 0.08.$$

The frequency ratio is

$$\frac{\omega}{\omega_n} = \frac{3.5}{4.08} = 0.86,$$

and the transcribability can be calculated via Equation 9.61, as

$$\frac{X}{Z_0} = \frac{X}{0.01} = \frac{\sqrt{1+[2(0.08)(0.86)]^2}}{\sqrt{(1-0.86^2)^2+[2(0.08)(0.86)]^2}} = 3.43.$$

Thus, the dynamic amplitude of the system is $X = 3.43 \times 10^{-2}$ m.

PROBLEM SET 9.2

1. Figure 9.13 shows the experimental data for the frequency response of a single-degree-of-freedom system. Use the half-power bandwidth method to estimate the damping ratio ζ of the system.
2. A viscously damped single-degree-of-freedom system is subjected to a harmonic force excitation. It is observed that the amplitude of the frequency response X/x_{st} reaches its maximum when the driving frequency is 120 rpm. The peak value is 50. Assume the system to be lightly damped.
   a. Determine the damping ratio ζ.
   b. Determine the bandwidth of the system.
3. An industrial machine of mass M = 450 kg is supported by a spring with a static deflection x_{st} = 0.5 cm. If the machine has a rotating unbalance me = 0.25 kg·m, determine the dynamic amplitude X at 1200 rpm. Assume damping to be negligible.
4. Tires must be balanced so that no periodic forces develop during operation. Figure 9.14 shows a tire with an eccentric mass because of uneven wear. The parameters are given as follows: the mass of the tire is M = 11.75 kg, the unbalanced mass is m = 0.1 kg, the radius of the tire is r = 22.5 cm, and the eccentric distance is e = 15 cm. Assume that the stiffness of the tire is 120 kN/m. Neglect the damping of the system. Determine the

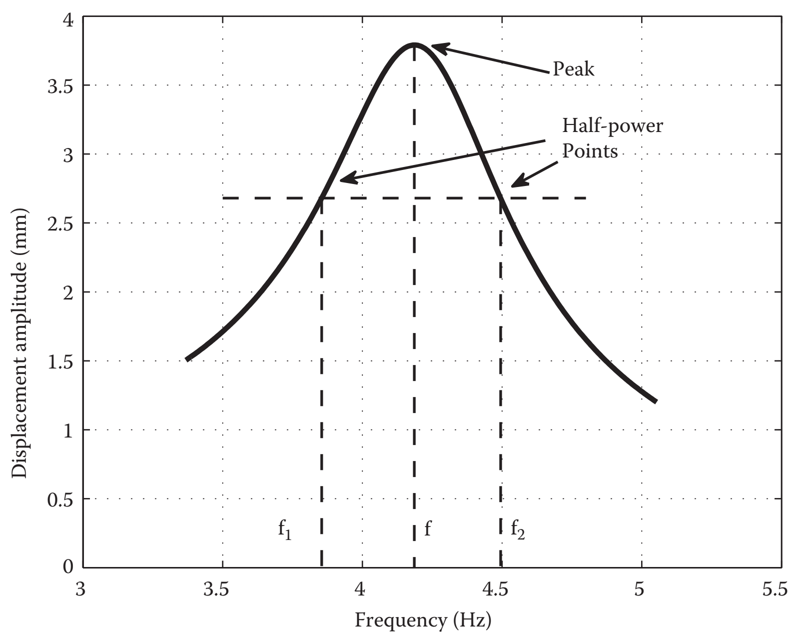


FIGURE 9.13 Problem 1.

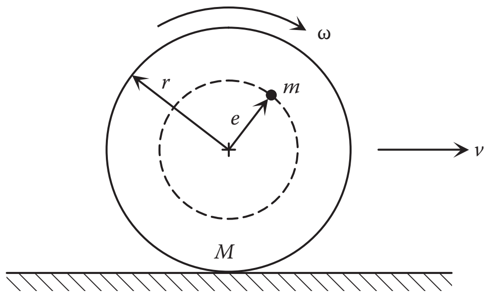


FIGURE 9.14 Problem 4.

amplitude of the steady-state response of the tire caused by mass unbalance when the car moves at 100 km/h.

5. Reconsider Example 9.5, in which the mathematical model of a vehicle is given by an ordinary differential equation $m\ddot{x} + b\dot{x} + kx = b\dot{z} + kz$ with m = 3000 kg, b = 2000 N·s/m, and k = 50 kN/m. It is observed that the base excitation due to the roughness of the road surface is also related to the speed of the vehicle, that is, $z = 0.01\sin(0.2\pi vt)$. Determine the transmissibility and the dynamic amplitude X of the vehicle when it moves at a speed of (a) 25 km/h and (b) 105 km/h.
6. Precision instruments must be placed on rubber mounts, which act as springs and dampers, to reduce the effects of base vibration. Consider a precision instrument of mass 110 kg mounted on a rubber block. For the entire assembly, the spring stiffness is 280 kN/m and the damping ratio is 0.10. Assume that the base undergoes vibration, and the displacement of the base is expressed as $y(t) = Y_0\sin(\omega t)$. Determine the dynamic amplitude of the system if the acceleration amplitude of the base excitation is 0.15 m/s^2 and the excitation frequency is 20 Hz.

9.3 VIBRATION SUPPRESSIONS

Vibrations are undesirable in most cases, particularly in cars, machining tools, precision instruments, buildings in an active seismic zone, and so on. To protect these systems and enhance their life, it is necessary to reduce vibration. This can be achieved with vibration isolators or vibration absorbers.

9.3.1 Vibration Isolators

To isolate an object from the source of vibrations, two types of vibration isolation systems are used: passive and active. Passive vibration isolation systems consist of springs and dampers. Active vibration isolation systems contain (along with the springs) piezoelectric accelerometers, electromagnetic actuators, and control circuits. The topic of active vibration isolation is beyond the scope of this text, and therefore will not be discussed.

A vibration isolation system attempts either to protect delicate equipment from vibration transmitted to it from its support system or to prevent transmission of the vibratory force generated by a machine to its surroundings. The essence of these two objectives is the same. The concept of transmissibility introduced in Section 9.2 can be used for either displacement isolation design or force isolation design.

As shown in Equation 9.61, for a system placed on a support undergoing harmonic vibration, the dynamic response depends on the natural frequency ω_n and the damping ratio ζ. This fact is also shown in Figure 9.12. When the natural frequency ω_n is much less than the excitation frequency ω,

more specifically, $\omega/\omega_n > \sqrt{2}$, the displacement transmissibility X/Z_0 is less than 1. This implies that the magnitude of the system response is reduced. Thus, it is desirable to design a vibration isolator such that the natural frequency of the entire assembly is within the region of $\omega/\omega_n > \sqrt{2}$. This can be achieved by placing the system on a spring-damper system. Note that the value of damping should not be too large because the displacement transmissibility increases when the damping ratio increases for $\omega/\omega_n > \sqrt{2}$.

Example 9.6: Displacement Isolation

A machine of mass 50 kg is mounted on a rubber isolator to protect it from the ground's vibration caused by the operation of other machines nearby. Assume that the ground vibrates at 10 Hz. Determine the stiffness of the rubber isolation spring if only 10% of the ground's motion is transmitted to the machine.

a. Neglect damping.
b. Assume that the damping coefficient of the rubber isolator is 0.1.

Solution

a. If only 10% of the ground's motion is transmitted to the machine, then the transmissibility given by Equation 9.61, with damping neglected, is

$$\frac{X}{Z_0} = \frac{1}{\left|1-(\omega/\omega_n)^2\right|} = 0.10,$$

from which the frequency ratio ω/ω_n is solved to be $\sqrt{11}$. For the excitation frequency ω = 10 Hz, the natural frequency is

$$\omega_n = \frac{2\pi(10)}{\sqrt{11}} = 18.94 \text{ rad/s}.$$

Thus, the stiffness of the rubber isolation spring is

$$k = \omega_n^2 m = 18.94^2(50) \approx 17.94 \text{ kN/m}.$$

b. If the damping coefficient of the rubber isolator is 0.1, then the transmissibility is

$$\frac{X}{Z_0} = \frac{\sqrt{1+[2(0.1)\omega/\omega_n]^2}}{\sqrt{[1-(\omega/\omega_n)^2]^2+[2(0.1)\omega/\omega_n]^2}} = 0.10,$$

which can be rewritten as

$$(\omega/\omega_n)^4 - 5.96(\omega/\omega_n)^2 - 99 = 0.$$

Solving for $(\omega/\omega_n)^2$, we find

$$(\omega/\omega_n)^2 = 13.37 \text{ or } (\omega/\omega_n)^2 = -7.41.$$

The negative value is not valid. Thus, the frequency ratio is ω/ω_n is $\sqrt{13.37}$. Repeating the calculation in Part (a) gives $k \approx 14.76$ kN/m.

For a machine placed on a rigid base, the machine itself is a vibration source. The vibratory force generated by the machine will be transmitted to the base and then affect the surrounding equipment. To reduce the damaging effect of the vibratory machine on its surroundings, it is necessary to isolate it from the base.

Figure 9.15a shows a machine placed on a rigid foundation through a spring and damper system. The machine is subjected to a harmonic excitation force $f(t)$. Recall that the displacement response $x(t)$ to harmonic excitation is given by Equations 9.31 through 9.33. Differentiating Equation 9.31 with respect to time yields the velocity response as given in Equation 9.37, in which X is the amplitude of the displacement, and the amplitude of the velocity is ωX. Moreover, the velocity leads the displacement by the phase angle $\pi/2$ because $\cos(\omega t + \phi)$ in Equation 9.37 can be expressed as $\sin(\omega t + \phi + \pi/2)$.

From the free-body diagram in Figure 9.15b, it is clear that the force transmitted to the base includes two parts: the spring force kx and the damping force $b\dot{x}$. The amplitudes of the spring force and the damping force are kX and $b\omega X$, respectively. The force vectors are shown in Figure 9.15c. The angle between the two force vectors is $\pi/2$. Thus, the amplitude of the force transmitted to the base is

$$F_T = \sqrt{(kX)^2 + (b\omega X)^2}. \tag{9.62}$$

Note that $b/k = (b/m)(m/k) = 2\zeta\omega_n/\omega_n^2 = 2\zeta/\omega_n$. Then, Equation 9.62 can be rewritten as

$$F_T = kX\sqrt{1 + (2\zeta\omega/\omega_n)^2}. \tag{9.63}$$

Substituting Equation 9.32 into Equation 9.63 yields

$$F_T = F_0 \frac{\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}. \tag{9.64}$$

The dimensionless ratio F_T/F_0 is the transmissibility given by Equation 9.61, and is a measure of the force transmitted to the base. Thus, the plot of F_T/F_0 versus ω/ω_n is the same as the plot of X/Z_0 versus ω/ω_n, as shown in Figure 9.12. When the frequency ratio is $\omega/\omega_n = \sqrt{2}$, we have $F_T/F_0 = 1$ and the excitation force is fully transmitted to the base. For $\omega/\omega_n > \sqrt{2}$, the force transmissibility F_T/F_0 is less than 1 and the force transmitted reduces with increasing excitation frequency ω.

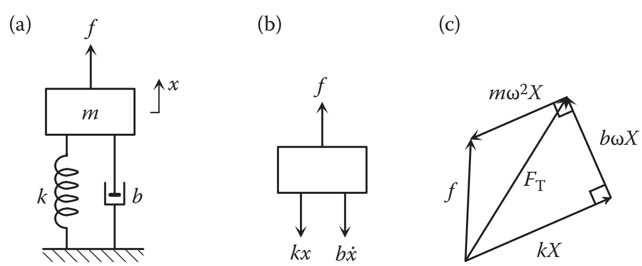


FIGURE 9.15 Force isolation: (a) physical system, (b) free-body diagram, and (c) force vector diagram.

Example 9.7: Force Isolation

A rotating machine of mass 2000 kg is mounted on an isolator block of stiffness 500 kN/m and damping ratio $\zeta = 0.1$. The machine is subjected to a harmonic disturbance force, for which the frequency is the same as the rotational speed of the machine. Assuming that 20% of the disturbance force is transmitted to the base, determine the rotational speed of the machine.

Solution

By Equation 9.64,

$$\frac{F_T}{F_0} = \frac{\sqrt{1+[2(0.1)(\omega/\omega_n)]^2}}{\sqrt{[1-(\omega/\omega_n)^2]^2+[2(0.1)(\omega/\omega_n)]^2}} = 0.2.$$

Solving for the frequency ratio yields $\omega/\omega_n = 2.57$. The natural frequency of the system is $\omega_n = \sqrt{500{,}000/2000} = 15.81\,\text{rad/s}$. Thus, the excitation frequency is

$$\omega = 2.57(15.81) = 40.63 \text{ rad/s},$$

which is also the rotational speed of the machine.

9.3.2 Vibration Absorbers

As discussed in Section 8.3, for a single-degree-of-freedom system subjected to harmonic excitation, violent vibration is induced when the excitation frequency is close to the natural frequency of the system. To protect the system, we can change either the mass or the spring stiffness so that the natural frequency is not too close to the excitation frequency. However, this may not always be possible. To circumvent this issue, a vibration absorber consisting of a second mass and spring can be added to the system to protect the original single-degree-of-freedom system from harmonic excitation.

Consider the two-degree-of-freedom system shown in Figure 9.16, where m_1 and k_1 are the mass and the spring stiffness of the primary system, and m_2 and k_2 are the mass and the spring stiffness of the absorber. A harmonic force $F_1\sin(\omega t)$ is applied to the primary system, which would undergo violent vibrations if the absorber is not installed.

For the combined system, the equations of motion in matrix form are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \sin(\omega t). \tag{9.65}$$

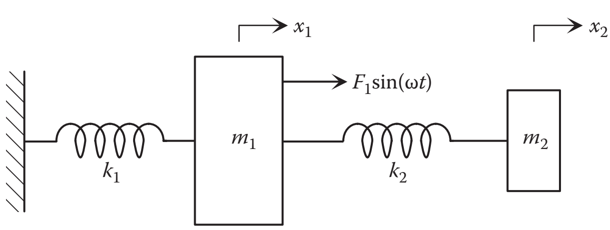


FIGURE 9.16 A vibration absorber.

Because the system is undamped, the steady-state response to a sinusoidal input is still sinusoidal, which has the same frequency as the excitation frequency and a zero phase angle. We can express the steady-state response as

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t). \tag{9.66}$$

Substituting Equation 9.66 into Equation 9.65 yields

$$\begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}. \tag{9.67}$$

Solving for X_1 and X_2 gives

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{1}{\Delta(\omega)} \begin{bmatrix} k_2 - m_2\omega^2 & k_2 \\ k_2 & k_1 + k_2 - m_1\omega^2 \end{bmatrix} \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}, \tag{9.68}$$

where

$$\Delta(\omega) = (k_2 - m_2\omega^2)(k_1 + k_2 - m_1\omega^2) - k_2^2. \tag{9.69}$$

Thus, the amplitudes are

$$X_1 = \frac{(k_2 - m_2\omega^2)F_1}{(k_2 - m_2\omega^2)(k_1 + k_2 - m_1\omega^2) - k_2^2}, \tag{9.70}$$

$$X_2 = \frac{k_2 F_1}{(k_2 - m_2\omega^2)(k_1 + k_2 - m_1\omega^2) - k_2^2}. \tag{9.71}$$

Dividing the numerators and denominators in Equations 9.70 and 9.71 by $k_1 k_2$, we obtain

$$X_1 = \frac{[1 - (m_2/k_2)\omega^2](F_1/k_1)}{[1 - (m_2/k_2)\omega^2][1 + k_2/k_1 - (m_1/k_1)\omega^2] - (k_2/k_1)}, \tag{9.72}$$

$$X_2 = \frac{F_1/k_1}{[1 - (m_2/k_2)\omega^2][1 + k_2/k_1 - (m_1/k_1)\omega^2] - (k_2/k_1)}. \tag{9.73}$$

From Equation 9.72, it is clear that the amplitude X_1 of the primary system reduces to zero when the excitation frequency ω equals $\sqrt{k_2/m_2}$. Correspondingly, the amplitude X_2 of the absorber becomes

$$X_2 = -\frac{F_1}{k_2}, \tag{9.74}$$

where the negative sign indicates that the absorber moves in the opposite direction of the force F_1. Thus, the force exerted on the primary mass m_1 by the spring of the absorber is

$$k_2x_2 = k_2X_2\sin\omega t = -F_1\sin\omega t, \tag{9.75}$$

which exactly balances the externally applied force $F_1\sin(\omega t)$.

Let us denote the natural frequency of the original single-degree-of-freedom system alone as $\omega_1 = \sqrt{k_1/m_1}$ and the natural frequency of the absorber alone as $\omega_2 = \sqrt{k_2/m_2}$. To determine the parameters of the absorber, we introduce the following notations, $x_{st} = F_1/k_1$, $\mu = m_2/m_1$, and $\nu = \omega_2/\omega_1$, where x_{st} is the static deflection of the original system, μ is the mass ratio, and ν is the natural frequency ratio. Then, Equations 9.72 and 9.73 can be rewritten as

$$X_1 = \frac{[1-(\omega/\omega_2)^2]x_{st}}{[1-(\omega/\omega_2)^2][1+k_2/k_1-(\omega/\omega_1)^2]-(k_2/k_1)}, \tag{9.76}$$

$$X_2 = \frac{x_{st}}{[1-(\omega/\omega_2)^2][1+k_2/k_1-(\omega/\omega_1)^2]-(k_2/k_1)}. \tag{9.77}$$

Note that $\omega/\omega_2 = (\omega/\omega_1)(1/\nu)$ and $k_2/k_1 = (m_2/m_1)(k_2/m_2)(m_1/k_1) = \mu\nu^2$. Thus,

$$X_1 = \frac{[\nu^2-(\omega/\omega_1)^2]x_{st}}{[\nu^2-(\omega/\omega_1)^2][1+\mu\nu^2-(\omega/\omega_1)^2]-\mu\nu^4}, \tag{9.78}$$

$$X_2 = \frac{\nu^2 x_{st}}{[\nu^2-(\omega/\omega_1)^2][1+\mu\nu^2-(\omega/\omega_1)^2]-\mu\nu^4}. \tag{9.79}$$

From Equations 9.78 and 9.79, it is clear that the values of X_1 and X_2 depend on the mass ratio μ and the natural frequency ratio ν. Note that the natural frequency ratio ν must be very close to 1. Because adding the absorber aims to alleviate the vibration of the original system at resonance, that is, $\omega = \omega_1$, and the absorber can only be effective when its own natural frequency is the same as the excitation frequency, that is, $\omega_2 = \omega$. Thus, we have $\omega_2 = \omega_1$ or $\nu = 1$. If ν is very close to one, the motion of m_1 is not zero, but its amplitude is still very small.

Figure 9.17 shows two curves, X_1/x_{st} versus ω/ω_1 (Figure 9.17a) and X_2/x_{st} versus ω/ω_1 (Figure 9.17b) for $\mu = 0.2$ and $\nu = 1$. Note that the horizontal axis can be replaced by ω/ω_2 because $\nu = 1$ or $\omega_2 = \omega_1$. As observed from Figure 9.17a, when the excitation frequency ω is in the immediate neighborhood of ω_2, the amplitude X_1 is very small. However, when the excitation frequency ω shifts slightly away from ω_2, the amplitude X_1 increases significantly. This shows that the absorber is only effective when the excitation frequency ω is close to ω_2.

To design a vibration absorber, we first need to select the operation frequency ω, at which the displacement amplitude of the primary mass will be tuned to zero. Then the relation between the mass and the spring stiffness of the absorber is obtained by $\omega^2 = \omega_2^2 = k_2/m_2$. Select appropriate values for m_2 and k_2 by considering restrictions on the motion of the absorber mass. Once the absorber is designed, the mass ratio μ is checked, for which the recommended value is $\mu < 0.25$.

It should be pointed out that one disadvantage of the vibration absorber is that two new resonant frequencies in the neighborhood of the excitation frequency are created as seen from Figure 9.18. Because the values of X_1/x_{st} and X_2/x_{st} change dramatically, the vertical axis in Figure 9.18 is given in the decibel units, which can be used to conveniently represent very large or small numbers. The details on how to determine natural frequencies for multi-degree-of-freedom systems will be discussed in the next section.

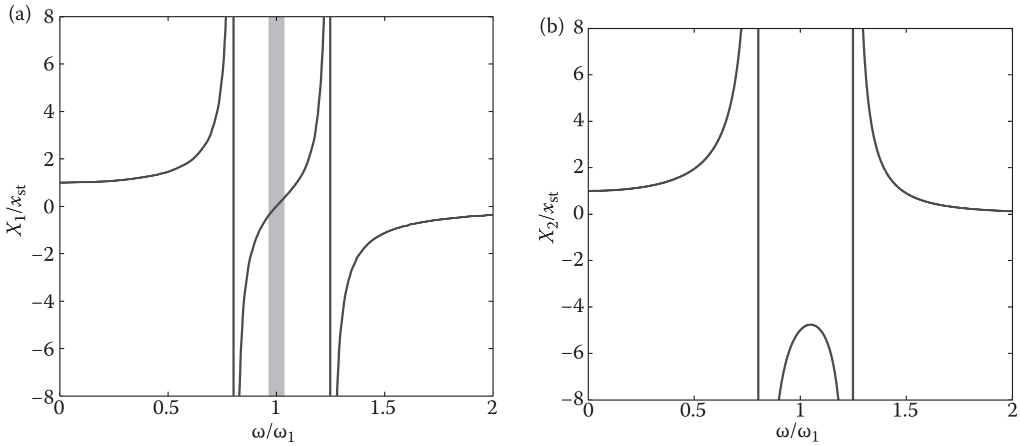


FIGURE 9.17 Frequency response curves: (a) main mass and (b) absorber mass.

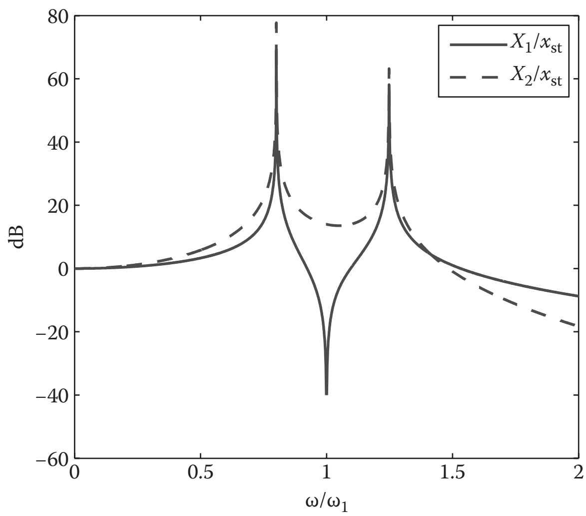


FIGURE 9.18 Frequency response curves (in dB) for the main mass and the absorber mass.

Example 9.8: Vibration Absorber Design

When an engine operates at a speed of 6000 rpm, vibration is induced through its pedestal mount. The amplitude of the excitation force is 300 N. Design a vibration absorber that will reduce the vibration when mounted on the pedestal. Assume that the maximum allowable displacement of the absorber is 2 mm.

Solution

The natural frequency of the absorber is required to be the same as the excitation frequency,

$$\omega_2 = \omega = \frac{6000(2\pi)}{60} = 200\pi \text{ rad/s.}$$

The maximum allowable displacement amplitude of the absorber is 2 mm. From Equation 9.74, considering the magnitude only, the spring stiffness of the absorber can be obtained as

$$k_2 = \frac{F_1}{X_2} = \frac{300}{2\times 10^{-3}} = 150{,}000\ \text{N/m}.$$

Thus, the mass of the absorber is

$$m_2 = \frac{k_2}{\omega_2^2} = \frac{150{,}000}{(200\pi)^2} = 0.38\ \text{kg}.$$

PROBLEM SET 9.3

1. A 90-kg instrument is suspended from a ceiling by four springs, each of which has a stiffness of 4 kN/m. The ceiling vibrates with a frequency of 2 Hz and amplitude of 0.05 mm due to the air conditioning compressor and chiller mounted on the roof. Neglecting the damping, determine the maximum displacement amplitude of the instrument.
2. Consider a 12-kg instrument placed on a floor that vibrates with a frequency of 3500 rpm and amplitude of 2 mm due to nearby machinery. A vibration isolator is designed to protect the instrument from the vibration of the floor. Assume that the damping ratio of the isolator is 0.05 and the maximum allowable acceleration amplitude of the instrument is 2 g.
   a. Determine the stiffness of the isolator.
   b. Determine the maximum displacement amplitude of the instrument.
3. A 9000-kg air conditioning compressor mounted on a roof is supported by four springs. The static deformation of each spring is 8 cm. Assuming negligible damping, determine the force transmissibility when the compressor works at 60 Hz.
4. Consider the single-degree-of-freedom system shown in Figure 9.8. The excitation force due to the rotating unbalance is $me\omega^2\sin(\omega t)$. Assume that the system has the following parameters: $m = 5$ kg, $M = 100$ kg, $e = 0.1$ m, $k = 5{,}000$ N/m, and $b = 200$ N·s/m.
   a. Determine the force transmitted to the support when the system runs at the rotating speed $\omega = 2\omega_n$.
   b. Determine the force transmissibility.
   c. The spring is replaced to decrease the force transmissibility to 20%. Determine the stiffness of the new spring.
5. A rotating machine has a mass of 6 kg and a natural frequency of 5 Hz. Due to a rotating unbalanced mass, the machine is subjected to a harmonic disturbance force $me\omega^2\sin(\omega t)$. When the machine operates at a frequency of 3.5 Hz, the amplitude of the disturbance force is 40 N.
   a. Design a vibration absorber assuming that the maximum allowable displacement of the absorber is 5 cm.
   b. Using MATLAB®, write an m-file to plot X_1/me versus ω.
6. The pendulum in Figure 9.19 is known as a tuned mass damper. It is mounted in a building, which is simplified as a block of mass m_1 supported by a spring of stiffness k. The mass of the pendulum is m_2 and the length l is tunable. The tunable pendulum is used to control the vibration of the building under extreme wind loads. Assume that the force due to gusty winds can be modeled as a harmonic force $F_1\sin(\omega t)$, and the excitation frequency ω is very close to the natural frequency of the building. Prove that the forced vibration of the building can be eliminated when the length of the pendulum is tuned such that $\omega = \sqrt{g/l}$, where g is the gravitational acceleration.

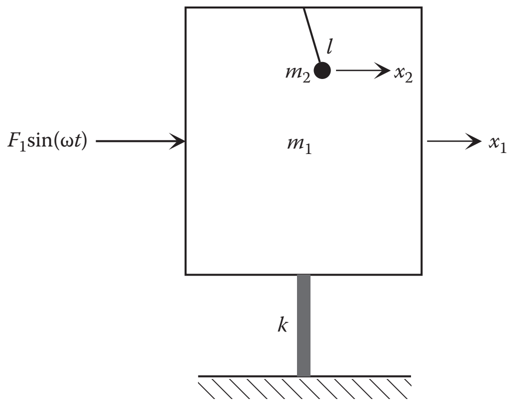


FIGURE 9.19 Problem 6.

9.4 MODAL ANALYSIS

The systems discussed in the previous sections are mainly single-degree-of-freedom systems, for which the vibration can be studied using elementary methods presented in Chapter 8. However, for a multi-degree-of-freedom system, more advanced mathematical tools are required to solve the equations of motion due to coordinate coupling. In this section, we first introduce key concepts, such as the eigenvalue problem, natural modes, and orthogonality of modes. Then, we develop modal analysis in a rigorous manner to decouple coordinates and use it to obtain the response to initial excitations or external forces. All derivations in this section are presented in matrix form.

9.4.1 Eigenvalue Problem

Consider a three-degree-of-freedom mass–spring system as shown in Figure 9.20, in which the motion of the system is described by the displacement coordinates x_1, x_2, and x_3. In the absence of damping and external forces, the system undergoes undamped free vibration, and the differential equations of motion in matrix form are

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{9.80}$$

Note that the mass matrix is diagonal but the stiffness matrix is not. Thus, Equation 9.80 represents a set of three simultaneous (or coupled) second-order differential equations. It is generally difficult to obtain the analytical solution of Equation 9.80 because of coordinate coupling. If we can find a coordinate transformation that simultaneously diagonalizes the mass and stiffness matrices,

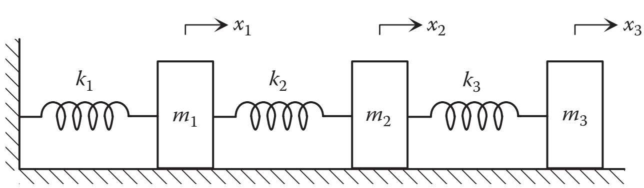


FIGURE 9.20 A three-degree-of-freedom mass–spring system.

then the system dynamics can be decoupled into a set of three independent second-order differential equations, each in the form of $m\ddot{x} + kx = 0$, which can be easily solved as a single-degree-of-freedom system. Such coordinate transformation, which is not unique, indeed exists and may be found by solving the eigenvalue problem.

The eigenvalue problem is a problem associated with free, undamped vibration, for which the set of differential equations in the general matrix form is given by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}. \tag{9.81}$$

The solution of Equation 9.81 is harmonic. Assume that the solution is in the form

$$\mathbf{x} = \mathbf{X}e^{j\omega t}, \tag{9.82}$$

where ω is the frequency of the harmonic motion. Note that $\ddot{\mathbf{x}} = -\omega^2\mathbf{X}e^{j\omega t}$. Thus, Equation 9.81 becomes

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{X}e^{j\omega t} = \mathbf{0}. \tag{9.83}$$

Canceling the nonzero term $e^{j\omega t}$ gives

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{X} = \mathbf{0} \tag{9.84}$$

or

$$\mathbf{K}\mathbf{X} = \omega^2\mathbf{M}\mathbf{X}. \tag{9.85}$$

Either one of Equations 9.84 or 9.85 represents the eigenvalue problem associated with matrices **M** and **K**. In particular, it is known as the algebraic eigenvalue problem, whose solution process is similar to that of the eigenvalue problem associated with a matrix **A** as discussed in Section 3.3. First, Equation 9.84 has a nontrivial solution $\mathbf{X} \neq \mathbf{0}$ if and only if the coefficient matrix is singular,

$$\left|\mathbf{K} - \omega^2\mathbf{M}\right| = 0. \tag{9.86}$$

Equation 9.86 is known as the characteristic equation or frequency equation. For an n-degree-of-freedom system, the determinant $\left|\mathbf{K} - \omega^2\mathbf{M}\right|$ is a polynomial of degree n in ω^2. The n roots of Equation 9.86 are referred to as eigenvalues and denoted by $\omega_1^2, \omega_2^2, \ldots$, and ω_n^2. Once the eigenvalues are identified, each eigenvector corresponding to each of the eigenvalues can be obtained by solving Equation 9.84 or 9.85. The n eigenvectors are denoted by $\mathbf{X}_1, \mathbf{X}_2, \ldots$, and $\mathbf{X}_n$.

For a vibration system, the eigenvalues and eigenvectors associated with the eigenvalue problem defined in Equations 9.84 or 9.85 have significant physical meanings. The square roots of the eigenvalues are the system's natural frequencies ω_r, where $r = 1, 2, \ldots, n$. The natural frequencies are usually arranged in increasing order of magnitude, that is, $\omega_1 \le \omega_2 \le \cdots \le \omega_n$. The lowest frequency ω_1 is referred to as the fundamental frequency, which is extremely important for many practical problems. The eigenvectors are referred to as modal vectors. Each modal vector represents physically the shape of a normal mode, a certain pattern of motion in which all masses move harmonically with the same natural frequency associated with this modal vector. The following example shows how to solve the eigenvalue problem and describes the physical significance of the eigenvalues and eigenvectors.

Example 9.9: Natural Frequencies and Modal Vectors

Consider the three-degree-of-freedom system shown in Figure 9.20. Assume that $m_1 = m_2 = m_3 = m$, $k_1 = 3k$, $k_2 = 2k$, and $k_3 = k$.

a. Solve the associated eigenvalue problem by hand.
b. Solve the associated eigenvalue problem using MATLAB. Without loss of generality, assume $m = 1$ kg and $k = 1$ N/m.

Solution

a. The differential equations for the system in Figure 9.20 are given by Equation 9.80. Thus, the mass and stiffness matrices are

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 5k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix}.$$

Substituting **M** and **K** into the frequency equation defined by Equation 9.86, we have

$$\begin{vmatrix} 5k-\omega^2 m & -2k & 0 \\ -2k & 3k-\omega^2 m & -k \\ 0 & -k & k-\omega^2 m \end{vmatrix} = 0 .$$

To simplify the calculation, divide the above equation by k and let $\beta = \omega^2 m/k$. Thus, the determinant is

$$\begin{vmatrix} 5-\beta & -2 & 0 \\ -2 & 3-\beta & -1 \\ 0 & -1 & 1-\beta \end{vmatrix} = (5-\beta)[(3-\beta)(1-\beta)-1]-(-2)[(-2)(1-\beta)] = 0,$$

which reduces to

$$\beta^3 - 9\beta^2 + 18\beta - 6 = 0.$$

The three roots are

$$\beta_1 = 0.4158, \quad \beta_2 = 2.2943, \quad \beta_3 = 6.2899,$$

which gives the three eigenvalues

$$\omega_1^2 = 0.4158\frac{k}{m}, \quad \omega_2^2 = 2.2943\frac{k}{m}, \quad \omega_3^2 = 6.2899\frac{k}{m}.$$

Thus, the three natural frequencies are

$$\omega_1 = 0.6448\sqrt{\frac{k}{m}}, \quad \omega_2 = 1.5147\sqrt{\frac{k}{m}}, \quad \omega_3 = 2.5080\sqrt{\frac{k}{m}}.$$

To determine the eigenvectors or modal vectors, we insert $\omega_r^2 (r = 1,2,3)$ into Equation 9.84. For $r = 1$, we have

$$(\mathbf{K} - \omega_1^2 \mathbf{M})\mathbf{X}_1 = \mathbf{0}.$$

Note that the modal vectors are 3×1 column vectors for a three-degree-of-freedom system. The above equation can be written as

$$\begin{bmatrix} 5k-\omega_1^2 m & -2k & 0 \\ -2k & 3k-\omega_1^2 m & -k \\ 0 & -k & k-\omega_1^2 m \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{12} \\ X_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Again, to simplify the calculation, we use β_1 to replace $\omega_1^2 m/k$ and write the equations in scalar form

$$(5-\beta_1)X_{11} - 2X_{12} = 0,$$
$$-2X_{11} + (3-\beta_1)X_{12} - X_{13} = 0,$$
$$-X_{12} + (1-\beta_1)X_{13} = 0.$$

Note that combining the first and the third equations by canceling X_{12} gives

$$\frac{X_{13}}{X_{11}} = \frac{5-\beta_1}{2-2\beta_1}.$$

Then, expressing X_{13} in the second equation in terms of X_{11} yields

$$\frac{X_{12}}{X_{11}} = \frac{9-5\beta_1}{(2-2\beta_1)(3-\beta_1)}.$$

If we assign one element in $\mathbf{X}_1$ an arbitrary value, the other two can be determined uniquely. This implies that we have three equations and two unknowns. Assuming $X_{11} = 1$ leads to $X_{13} = 3.9235$ and $X_{12} = 2.2922$. Thus, the first modal vector is

$$\mathbf{X}_1 = \begin{bmatrix} 1 \\ 2.2922 \\ 3.9235 \end{bmatrix}.$$

Similarly, we can find the other two modal vectors as

$$\mathbf{X}_2 = \begin{bmatrix} 1 \\ 1.3529 \\ -1.0452 \end{bmatrix} \quad \mathbf{X}_3 = \begin{bmatrix} 1 \\ -0.6450 \\ 0.1219 \end{bmatrix}.$$

Here, we set the magnitude of the first component in all modal vectors as 1. The modal vectors can also be normalized so that the largest component in magnitude is equal to 1:

$$\mathbf{X}_1 = \begin{bmatrix} 0.2549 \\ 0.5842 \\ 1 \end{bmatrix} \quad \mathbf{X}_2 = \begin{bmatrix} 0.7392 \\ 1 \\ -0.7726 \end{bmatrix} \quad \mathbf{X}_3 = \begin{bmatrix} 1 \\ -0.6450 \\ 0.1219 \end{bmatrix}.$$

The modal vectors represent physically the shape of the natural modes as shown in Figure 9.21. In mode 1, all masses move in the same direction and oscillate with a frequency

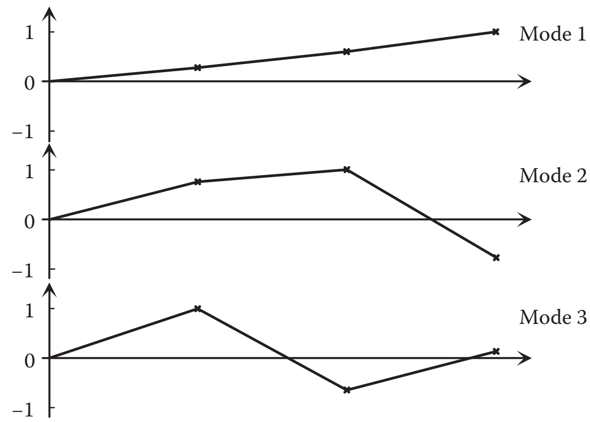


FIGURE 9.21 Natural modes of the three-degree-of-freedom system shown in Figure 9.20.

of $\omega_1 = 0.6448\sqrt{k/m}$. In mode 2, all masses oscillate with a frequency of $\omega_2 = 1.5147\sqrt{k/m}$. Masses 1 and 2 move in the same direction, whereas mass 3 moves in the opposite direction. The direction change is also implied by the one sign change in the second mode $\mathbf{X}_2$. In mode 3, there are two sign changes in $\mathbf{X}_3$. Masses 1 and 3 move in the same direction, whereas mass 2 moves in the opposite direction. All masses oscillate with a frequency of $\omega_3 = 2.5080\sqrt{k/m}$.

b. Given the mass and stiffness matrices, it is easy to use MATLAB to solve the associated algebraic eigenvalue problem. The following is a MATLAB session:

```
>> M = eye(3);
>> K = [5 -2 0; -2 3 -1; 0 -1 1];
>> [V,D] = eig(K,M);
```

The `eig` command returns two matrices named **V** and **D**,

$$\mathbf{V} = \begin{bmatrix} 0.2149 & -0.5049 & -0.8360 \\ 0.4927 & -0.6831 & 0.5392 \\ 0.8433 & 0.5277 & -0.1019 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0.4158 & 0 & 0 \\ 0 & 2.2943 & 0 \\ 0 & 0 & 6.2899 \end{bmatrix},$$

where **D** is a diagonal matrix containing the eigenvalues and the columns in matrix **V** are the corresponding eigenvectors. Note that the modal vectors returned by the MATLAB command `eig` are normalized so that the magnitude of each vector is equal to 1. For example, the first column in matrix **V** is the first modal vector, and we have

$$\sqrt{0.2149^2 + 0.4927^2 + 0.8433^2} = 1.$$

Comparing the eigenvectors obtained by hand and using MATLAB, we observe that the ratios between the elements are the same. For example, for mode 1, $X_{11}/X_{13} = 0.2149/0.8433 = 0.2548$ and $X_{12}/X_{13} = 0.4927/0.8433 = 0.5843$. This implies that the shape of mode is unique but the amplitude is not. Due to round-off error, the ratios are slightly different from those obtained in Part (a).

9.4.2 Orthogonality of Modes

The natural modes possess an important property known as orthogonality. To show this concept, let us consider an n-degree-of-freedom system. Following the same procedure as in Example 9.9, we can obtain n natural frequencies and n modal vectors by solving the eigenvalue problem. Denote any two solutions of the eigenvalue problem as ω_r^2, $\mathbf{X}_r$, and ω_s^2, $\mathbf{X}_s$. Both pairs should satisfy Equation 9.85, that is,

$$\mathbf{KX}_r = \omega_r^2\mathbf{MX}_r, \tag{9.87}$$

$$\mathbf{KX}_s = \omega_s^2\mathbf{MX}_s. \tag{9.88}$$

Premultiplying both sides of Equation 9.87 by $\mathbf{X}_s^T$ and Equation 9.88 by $\mathbf{X}_r^T$ gives

$$\mathbf{X}_s^T\mathbf{KX}_r = \omega_r^2\mathbf{X}_s^T\mathbf{MX}_r, \tag{9.89}$$

$$\mathbf{X}_r^T\mathbf{KX}_s = \omega_s^2\mathbf{X}_r^T\mathbf{MX}_s. \tag{9.90}$$

Taking the transpose of Equation 9.90, we find

$$\mathbf{X}_s^T\mathbf{K}^T\mathbf{X}_r = \omega_s^2\mathbf{X}_s^T\mathbf{M}^T\mathbf{X}_r. \tag{9.91}$$

Recall that the mass and stiffness matrices are symmetric, $\mathbf{M} = \mathbf{M}^T$ and $\mathbf{K} = \mathbf{K}^T$. Then, Equation 9.91 reduces to

$$\mathbf{X}_s^T\mathbf{KX}_r = \omega_s^2\mathbf{X}_s^T\mathbf{MX}_r. \tag{9.92}$$

Subtracting Equation 9.92 from Equation 9.89, we have

$$(\omega_r^2 - \omega_s^2)\mathbf{X}_s^T\mathbf{MX}_r = 0. \tag{9.93}$$

For two distinct modes, the frequencies are different, $\omega_r^2 \neq \omega_s^2$. Thus, Equation 9.93 is satisfied if and only if

$$\mathbf{X}_s^T\mathbf{MX}_r = 0, \quad \mathrm{r} \neq \mathrm{s}. \tag{9.94}$$

Substituting Equation 9.94 into Equation 9.89 gives

$$\mathbf{X}_s^T\mathbf{KX}_r = 0, \quad \mathrm{r} \neq \mathrm{s}. \tag{9.95}$$

Equations 9.94 and 9.95 represent the orthogonality relation for two distinct modal vectors, $\mathbf{X}_r$ and $\mathbf{X}_s$. They are orthogonal with respect to the mass matrix $\mathbf{M}$ as well as the stiffness matrix $\mathbf{K}$. If r = s, then the values of $\mathbf{X}_r^T\mathbf{MX}_r$ and $\mathbf{X}_r^T\mathbf{KX}_r$ are nonzero, and their values depend on the method used to normalize the modal vectors. A convenient scheme is to normalize $\mathbf{X}_r$ such that

$$\mathbf{X}_r^T\mathbf{MX}_r = 1. \tag{9.96}$$

The modal vector $\mathbf{X}_r$ is then called orthonormal. Premultiplying both sides of Equation 9.87 by $\mathbf{X}_r^T$ yields

$$\mathbf{X}_r^T\mathbf{K}\mathbf{X}_r = \omega_r^2. \tag{9.97}$$

Example 9.10: Normalization of Modal Vectors

Consider the three-degree-of-freedom system discussed in Example 9.9. Normalize the modal vectors $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ obtained in Part (a) of Example 9.9 to a set of orthonormal modal vectors satisfying Equations 9.96 and 9.97. Assume $m = 1$ kg and $k = 1$ N/m.

Solution

Note that regardless of how the modal vectors are normalized, they result in the same mode shape but scaled in magnitude. So, we can write the first modal vector in the form

$$\tilde{\mathbf{X}}_1 = \alpha\mathbf{X}_1,$$

where $\tilde{\mathbf{X}}_1$ is the modal vector after normalization and α is a nonzero scaling constant. Substituting $\tilde{\mathbf{X}}_1$ into Equation 9.96 gives

$$\tilde{\mathbf{X}}_1^T\mathbf{M}\tilde{\mathbf{X}}_1 = (\alpha\mathbf{X}_1)^T\mathbf{M}(\alpha\mathbf{X}_1) = \alpha^2\mathbf{X}_1^T\mathbf{M}\mathbf{X}_1,$$

which should be equal to 1 if $\tilde{\mathbf{X}}_1$ is orthonormal. Inserting $\mathbf{X}_1 = \{0.2549\ 0.5842\ 1\}^T$ and $\mathbf{M}$ obtained in Example 9.9 yields

$$\alpha^2\mathbf{X}_1^T\mathbf{M}\mathbf{X}_1 = \alpha^2\begin{bmatrix}0.2549\\0.5842\\1\end{bmatrix}^T\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0.2549\\0.5842\\1\end{bmatrix} = 1.4063\alpha^2 = 1,$$

Thus, $\alpha = 0.8433$ and

$$\tilde{\mathbf{X}}_1 = \alpha\mathbf{X}_1 = \begin{bmatrix}0.2150\\0.4927\\0.8433\end{bmatrix}.$$

Neglecting round-off error, note that $\tilde{\mathbf{X}}_1$ is the same as the first column in matrix $\mathbf{V}$ obtained in Example 9.9, Part (b). Following the same procedure yields the other two orthonormal modal vectors, $\tilde{\mathbf{X}}_2$ and $\tilde{\mathbf{X}}_3$,

$$\tilde{\mathbf{X}}_2 = 0.6831\mathbf{X}_2 = \begin{bmatrix}0.5049\\0.6831\\-0.5278\end{bmatrix}, \quad \tilde{\mathbf{X}}_3 = 0.8360\mathbf{X}_3 = \begin{bmatrix}0.8360\\-0.5392\\0.1019\end{bmatrix},$$

which correspond to the second and third columns in matrix $\mathbf{V}$ with all elements multiplied by –1. Thus, the MATLAB command `eig` returns a set of orthonormal modal vectors.

The modal vectors are generally placed side by side to form a matrix. For example, $\mathbf{\Phi} = [\mathbf{X}_1 \ \cdots \ \mathbf{X}_n]$, which is defined as a modal matrix. If the modal vectors are orthonormal, then we have

$$\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \begin{bmatrix} \tilde{\mathbf{X}}_1^T \\ \vdots \\ \tilde{\mathbf{X}}_n^T \end{bmatrix} \mathbf{M}[\tilde{\mathbf{X}}_1 \ \cdots \ \tilde{\mathbf{X}}_n] = \begin{bmatrix} \tilde{\mathbf{X}}_1^T\mathbf{M}\tilde{\mathbf{X}}_1 & \cdots & \tilde{\mathbf{X}}_1^T\mathbf{M}\tilde{\mathbf{X}}_n \\ \vdots & \ddots & \vdots \\ \tilde{\mathbf{X}}_n^T\mathbf{M}\tilde{\mathbf{X}}_1 & \cdots & \tilde{\mathbf{X}}_n^T\mathbf{M}\tilde{\mathbf{X}}_n \end{bmatrix} \tag{9.98}$$

Due to the orthogonality of modal vectors as defined in Equation 9.94, all off-diagonal entries are zero. Also, the modal vectors from $\tilde{\mathbf{X}}_1$ to $\tilde{\mathbf{X}}_n$ are orthonormal. Following Equation 9.96, each entry along the diagonal should be 1. Thus,

$$\mathbf{\Phi}^T\,\mathbf{M}\mathbf{\Phi} = \mathbf{I}_n \tag{9.99}$$

Similarly, we have

$$\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \begin{bmatrix} \tilde{\mathbf{X}}_1^T \\ \vdots \\ \tilde{\mathbf{X}}_n^T \end{bmatrix} \mathbf{K}[\tilde{\mathbf{X}}_1 \ \cdots \ \tilde{\mathbf{X}}_n] = \begin{bmatrix} \tilde{\mathbf{X}}_1^T\mathbf{K}\tilde{\mathbf{X}}_1 & \cdots & \tilde{\mathbf{X}}_1^T\mathbf{K}\tilde{\mathbf{X}}_n \\ \vdots & \ddots & \vdots \\ \tilde{\mathbf{X}}_n^T\mathbf{K}\tilde{\mathbf{X}}_1 & \cdots & \tilde{\mathbf{X}}_n^T\mathbf{K}\tilde{\mathbf{X}}_n \end{bmatrix} \tag{9.100}$$

Following Equations 9.95 and 9.97 gives

$$\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_n^2 \end{bmatrix} = \mathbf{\Omega} \tag{9.101}$$

Equations 9.99 and 9.101 are very useful for decoupling equations of motion and obtaining responses of vibration systems.

9.4.3 Response to Initial Excitations

Now let us consider the free, undamped vibration problem described by Equation 9.81, in which the initial condition vectors are $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$. Solving the associated eigenvalue problem yields a set of natural frequencies, $\omega_1, \omega_2, \ldots,$ and ω_n, and a set of orthonormal modal vectors, $\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2, \ \ldots,$ and $\tilde{\mathbf{X}}_n$. To decouple the equations of motion, we introduce the coordinate transformation

$$\mathbf{x} = \mathbf{\Phi}\mathbf{q}, \tag{9.102}$$

where $\mathbf{\Phi} = [\tilde{\mathbf{X}}_1\,\tilde{\mathbf{X}}_2 \cdots \tilde{\mathbf{X}}_n]$ is the modal matrix and $\mathbf{q}$ is known as the vector of modal coordinates. Substituting Equation 9.102 into Equation 9.81 and premultiplying by $\mathbf{\Phi}^T$ on both sides gives

$$\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}\ddot{\mathbf{q}} + \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi}\mathbf{q} = \mathbf{0}. \tag{9.103}$$

Due to the orthogonality of modal vectors given by Equations 9.99 and 9.101, Equation 9.103 reduces to

$$\ddot{\mathbf{q}} + \mathbf{\Omega}\mathbf{q} = \mathbf{0}, \tag{9.104}$$

which can be written as n independent modal equations

$$\ddot{q}_r + \omega_r^2 q_r = 0, \quad r = 1,2,\ldots,n. \tag{9.105}$$

Obviously, modal equations are analogous to differential equations of single-degree-of-freedom systems, for which the response was given in Section 8.2. Following the discussion in Chapter 8, the solutions to Equation 9.105 are

$$q_r(t) = q_r(0)\cos(\omega_r t) + \frac{\dot{q}_r(0)}{\omega_r}\sin(\omega_r t), \quad r = 1,2,\ldots,n, \tag{9.106}$$

where $q_r(0)$ and $\dot{q}_r(0)$ are the initial displacements and initial velocities of modal coordinates. Note that the prescribed initial conditions are associated with the physical coordinates $\mathbf{x}$. To transform the initial conditions, we use Equation 9.102, which gives $\mathbf{q} = \mathbf{\Phi}^{-1}\mathbf{x}$. Because of the property of orthonormal modal vectors, $\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \mathbf{I}_n$, we have $\mathbf{\Phi}^{-1} = \mathbf{\Phi}^T\mathbf{M}$. Thus, the initial modal displacements and velocities are

$$\mathbf{q}(0) = \mathbf{\Phi}^T\mathbf{M}\mathbf{x}(0), \tag{9.107}$$

$$\dot{\mathbf{q}}(0) = \mathbf{\Phi}^T\mathbf{M}\dot{\mathbf{x}}(0). \tag{9.108}$$

Expanding the modal matrix yields

$$\begin{Bmatrix} q_1(0) \\ \vdots \\ q_n(0) \end{Bmatrix} = \begin{bmatrix} \tilde{\mathbf{X}}_1^T \\ \vdots \\ \tilde{\mathbf{X}}_n^T \end{bmatrix} \mathbf{M}\mathbf{x}(0) = \begin{Bmatrix} \tilde{\mathbf{X}}_1^T\mathbf{M}\mathbf{x}(0) \\ \vdots \\ \tilde{\mathbf{X}}_n^T\mathbf{M}\mathbf{x}(0) \end{Bmatrix}, \tag{9.109}$$

$$\begin{Bmatrix} \dot{q}_1(0) \\ \vdots \\ \dot{q}_n(0) \end{Bmatrix} = \begin{bmatrix} \tilde{\mathbf{X}}_1^T \\ \vdots \\ \tilde{\mathbf{X}}_n^T \end{bmatrix} \mathbf{M}\dot{\mathbf{x}}(0) = \begin{Bmatrix} \tilde{\mathbf{X}}_1^T\mathbf{M}\dot{\mathbf{x}}(0) \\ \vdots \\ \tilde{\mathbf{X}}_n^T\mathbf{M}\dot{\mathbf{x}}(0) \end{Bmatrix}, \tag{9.110}$$

or

$$q_r(0) = \tilde{\mathbf{X}}_r^T\mathbf{M}\mathbf{x}(0), \quad r = 1,2,\ldots,n, \tag{9.111}$$

$$\dot{q}_r(0) = \tilde{\mathbf{X}}_r^T\mathbf{M}\dot{\mathbf{x}}(0), \quad r = 1,2,\ldots,n. \tag{9.112}$$

Then, the modal responses are

$$q_r(t) = \tilde{\mathbf{X}}_r^T\mathbf{M}\mathbf{x}(0)\cos(\omega_r t) + \frac{1}{\omega_r}\tilde{\mathbf{X}}_r^T\mathbf{M}\dot{\mathbf{x}}(0)\sin(\omega_r t), \quad r = 1,2,\ldots,n. \tag{9.113}$$

Finally, applying Equation 9.102 gives the responses of n-degree-of-freedom systems to initial excitations

$$\mathbf{x}(t) = [\tilde{\mathbf{X}}_1 \ \cdots \ \tilde{\mathbf{X}}_n] \begin{Bmatrix} q_1(t) \\ \vdots \\ q_n(t) \end{Bmatrix} = \sum_{r=1}^{n} \tilde{\mathbf{X}}_r q_r(t) = \sum_{r=1}^{n} q_r(t)\tilde{\mathbf{X}}_r. \tag{9.114}$$

Equation 9.114 represents the so-called expansion theorem, by which the solution $\mathbf{x}(t)$ can be regarded as a superposition of the normal modes $\tilde{\mathbf{X}}_r(t)$ ($r = 1, 2, \ldots, n$). The modal responses $q_r(t)$ represent the contributions of the particular configuration $\tilde{\mathbf{X}}_r(t)$ to the total solution.

The approach using the orthogonality properties of the modal matrix to obtain a set of simultaneous independent equations is known as modal analysis. The basic steps to obtain the solutions using modal analysis are summarized as follows:

Step 1: Solve the eigenvalue problem and obtain the natural frequencies and modal vectors.
Step 2: Normalize the modal vectors to obtain orthonormality.
Step 3: Determine the modal responses and combine them into the response of the original system using the expansion theorem.

Example 9.11: Response to Initial Excitation by Modal Analysis

Consider the three-degree-of-freedom system discussed in Example 9.9. Assume m = 1 kg and k = 1 N/m. Using the natural frequencies obtained in Example 9.9 and the orthonormal modal vectors obtained in Example 9.10, determine the response of the system subjected to initial excitations $\mathbf{x}(0) = [0\ 0\ 0.01]^T$ and $\dot{\mathbf{x}}(0) = [0\ \ 0\ \ 0]^T$.

Solution

In Examples 9.9 and 9.10, we obtained the three natural frequencies

$$\omega_1 = 0.6448 \text{ rad/s}, \quad \omega_2 = 1.5147 \text{ rad/s}, \quad \omega_3 = 2.5080 \text{ rad/s}$$

and the three orthonormal modal vectors

$$\tilde{\mathbf{X}}_1 = \begin{bmatrix} 0.2150 \\ 0.4927 \\ 0.8433 \end{bmatrix}, \quad \tilde{\mathbf{X}}_2 = \begin{bmatrix} 0.5049 \\ 0.6831 \\ -0.5278 \end{bmatrix}, \quad \tilde{\mathbf{X}}_3 = \begin{bmatrix} 0.8360 \\ -0.5392 \\ 0.1019 \end{bmatrix}.$$

Application of Equation 9.113 gives the modal responses as

$$q_1(t) = [0.2150\ \ 0.4927\ \ 0.8433] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.01 \end{Bmatrix} \cos(0.6448t) = 0.0084\cos(0.6448t),$$

$$q_2(t) = [0.5049\ \ 0.6831\ -0.5278] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.01 \end{Bmatrix} \cos(1.5147t) = -0.0053\cos(1.5147t),$$

$$q_3(t) = [0.8360\ -0.5392\ \ 0.1019] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.01 \end{Bmatrix} \cos(2.5080t) = 0.0010\cos(2.5080t).$$

Inserting the modal responses into Equation 9.114, we obtain the response of the system to the given initial excitations as follows.

$$\mathbf{x}(t) = 0.0084\cos(0.6448t)\begin{bmatrix} 0.2150 \\ 0.4927 \\ 0.8433 \end{bmatrix} - 0.0053\cos(1.5147t)\begin{bmatrix} 0.5049 \\ 0.6831 \\ -0.5278 \end{bmatrix}$$

$$+0.0010\cos(2.5080t)\begin{bmatrix} 0.8360 \\ -0.5392 \\ 0.1019 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0018 \\ 0.0041 \\ 0.0071 \end{bmatrix}\cos(0.6448t) + \begin{bmatrix} -0.0027 \\ -0.0036 \\ 0.0028 \end{bmatrix}\cos(1.5147t) + \begin{bmatrix} 0.0084 \\ -0.0054 \\ 0.0010 \end{bmatrix}\cos(2.5080t).$$

9.4.4 Response to Harmonic Excitations

The essence of modal analysis is to determine the response of an n-degree-of-freedom system by decomposing it into n single-degree-of-freedom systems, determining the response of each single-degree-of-freedom system, and combining the individual responses to derive the response of the original system. The previous subsection showed how one can obtain the free response of an undamped system using modal analysis, which can also be used to find the response of an undamped system to harmonic excitations. In this case, the equations of motion in matrix form are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}, \tag{9.115}$$

where $\mathbf{f}$ is an $n \times 1$ force vector. Following the basic steps of modal analysis given in Subsection 9.4.3, we first need to solve the eigenvalue problem $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{X} = \mathbf{0}$ and obtain the solutions ω_r^2 and $\mathbf{X}_r$ ($r = 1, 2, \ldots, n$). The n modal vectors are then normalized and arranged to form the $n \times n$ modal matrix $\mathbf{\Phi} = [\tilde{\mathbf{X}}_1 \cdots \tilde{\mathbf{X}}_n]$, which satisfies the orthonormality relations $\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \mathbf{I}_n$ and $\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \mathbf{\Omega}$. Using the expansion theorem, the solution to Equation 9.115 is a linear combination of the modal vectors, $\mathbf{x} = \sum_{r=1}^{n} q_r(t)\tilde{\mathbf{X}}_r = \mathbf{\Phi}\mathbf{q}$. Substituting it into Equation 9.115 and premultiplying the result by $\mathbf{\Phi}^T$ gives

$$\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}\ddot{\mathbf{q}} + \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi}\mathbf{q} = \mathbf{\Phi}^T\mathbf{f}. \tag{9.116}$$

Combining the orthonormality relations, we have

$$\ddot{\mathbf{q}} + \mathbf{\Omega}\mathbf{q} = \mathbf{N}, \tag{9.117}$$

where $\mathbf{N} = \mathbf{\Phi}^T\mathbf{f}$. Note that

$$\mathbf{\Phi}^T\mathbf{f} = \begin{bmatrix} \tilde{\mathbf{X}}_1^T \\ \vdots \\ \tilde{\mathbf{X}}_n^T \end{bmatrix}\mathbf{f} = \begin{Bmatrix} \tilde{\mathbf{X}}_1^T\mathbf{f} \\ \vdots \\ \tilde{\mathbf{X}}_n^T\mathbf{f} \end{Bmatrix} = \begin{Bmatrix} N_1 \\ \vdots \\ N_n \end{Bmatrix}, \tag{9.118}$$

where the entries $\tilde{\mathbf{X}}_r^T \mathbf{f}(r = 1,2,\ldots,n)$ are the products of the $1 \times n$ row vectors $\tilde{\mathbf{X}}_r^T$ and the $n \times 1$ column vector $\mathbf{f}$, resulting in scalars N_r, known as modal forces. Equation 9.117 is equivalent to a set of n independent modal equations

$$\ddot{q}_r + \omega_r^2 q_r = N_r, \quad r = 1,2,\ldots,n. \tag{9.119}$$

We consider harmonic excitations, without loss of generality, in the form

$$\mathbf{f} = \mathbf{f}_0 \sin(\omega t). \tag{9.120}$$

Thus, the modal forces N_r can be written as $\tilde{\mathbf{X}}_r^T \mathbf{f}_0 \sin(\omega t)$ $(r = 1,2,\ldots,n)$. The solutions to Equation 9.119 can be obtained in a manner similar to the response of undamped single-degree-of-freedom systems to harmonic excitations. Applying Equations 9.31 through 9.33 with $\zeta = 0$, we have

$$q_r(t) = \frac{\tilde{\mathbf{X}}_r^T \mathbf{f}_0}{\omega_r^2 - \omega^2} \sin(\omega t), \quad r = 1,2,\ldots,n. \tag{9.121}$$

Thus, the steady-state response of the original system is

$$\mathbf{x} = \sum_{r=1}^{n} q_r(t) \tilde{\mathbf{X}}_r = \sum_{r=1}^{n} \frac{\tilde{\mathbf{X}}_r^T \mathbf{f}_0 \tilde{\mathbf{X}}_r}{\omega_r^2 - \omega^2} \sin(\omega t). \tag{9.122}$$

Example 9.12: Response to Harmonic Excitation by Modal Analysis

Consider the three-degree-of-freedom system discussed in Example 9.9. Assume $m = 1$ kg and $k = 1$ N/m. Using the natural frequencies obtained in Example 9.9 and the orthonormal modal vectors obtained in Example 9.10, determine the response of the system if a harmonic force $F(t) = 3\sin(2t)$ is applied to mass 3.

Solution

The force vector can be written as

$$\mathbf{f} = \begin{Bmatrix} 0 \\ 0 \\ F(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 3\sin(2t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix} \sin(2t).$$

From Examples 9.9 and 9.10, the three natural frequencies are

$$\omega_1 = 0.6448\text{rad/s}, \qquad \omega_2 = 1.5147\text{rad/s}, \qquad \omega_3 = 2.5080\text{rad/s}$$

and the three orthonormal modal vectors

$$\tilde{\mathbf{X}}_1 = \begin{bmatrix} 0.2150 \\ 0.4927 \\ 0.8433 \end{bmatrix}, \quad \tilde{\mathbf{X}}_2 = \begin{bmatrix} 0.5049 \\ 0.6831 \\ -0.5278 \end{bmatrix}, \quad \tilde{\mathbf{X}}_3 = \begin{bmatrix} 0.8360 \\ -0.5392 \\ 0.1019 \end{bmatrix}.$$

Equation 9.121 gives the modal responses as

$$q_1(t) = \frac{[0.2150\ 0.4927\ 0.8433]\begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix}}{0.6448^2 - 2^2}\sin(2t) = -0.7058\sin(2t),$$

$$q_2(t) = \frac{[0.5049\ 0.6831\ -0.5278]\begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix}}{1.5147^2 - 2^2}\sin(2t) = 0.9283\sin(2t),$$

$$q_3(t) = \frac{[0.8360\ -0.5392\ 0.1019]\begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix}}{2.5080^2 - 2^2}\sin(2t) = 0.1335\sin(2t).$$

Inserting the modal responses into Equation 9.122, we obtain the response of the system to the given harmonic excitation.

$$\mathbf{x}(t) = -0.7058\sin(2t)\begin{bmatrix} 0.2150 \\ 0.4927 \\ 0.8433 \end{bmatrix} + 0.9283\sin(2t)\begin{bmatrix} 0.5049 \\ 0.6831 \\ -0.5278 \end{bmatrix} + 0.1335\sin(2t)\begin{bmatrix} 0.8360 \\ -0.5392 \\ 0.1019 \end{bmatrix}$$

$$= \begin{Bmatrix} 0.4286 \\ 0.2144 \\ -1.0716 \end{Bmatrix}\sin(2t).$$

PROBLEM SET 9.4

1. Consider the two-degree-of-freedom mass–spring system shown in Figure 9.22. Assume that $m_1 = m_2 = m$, $k_1 = k$, and $k_2 = 2k$, where $m = 5$ kg and $k = 2000$ N/m.
   a. Derive the equations of motion and express them in matrix form.
   b. Solve the associated eigenvalue problem by hand. Plot the two modes and explain the nature of the mode shapes.
   c. Solve the associated eigenvalue problem using MATLAB.
2. A three-story building can be modeled as a three-degree-of-freedom system as shown in Figure 9.23, in which the horizontal members are rigid and the columns are massless beams acting as springs. Assume that $m_1 = 1500$ kg, $m_2 = 3000$ kg, $m_3 = 4500$ kg, $k_1 = 400$ kN/m, $k_2 = 800$ kN/m, and $k_3 = 1200$ kN/m.
   a. Derive the differential equations for the horizontal motion of the masses.

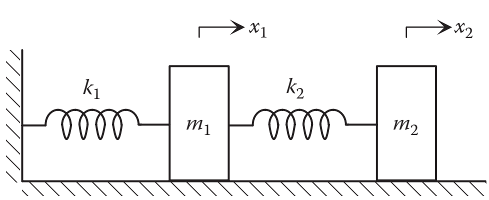


FIGURE 9.22 Problem 1.

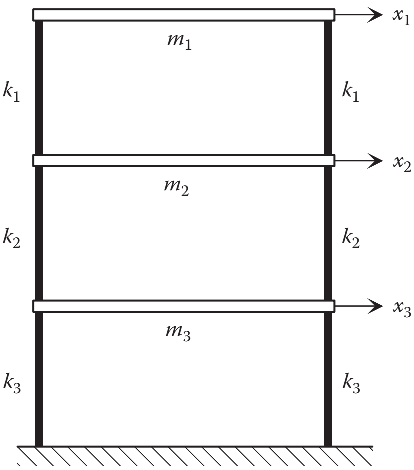


FIGURE 9.23 Problem 2.

b. Solve the associated eigenvalue problem by hand. Plot the three modes and explain the nature of the mode shapes.
c. Solve the associated eigenvalue problem using MATLAB.

3. Consider the two-degree-of-freedom mass–spring system in Figure 9.22. Normalize the modal vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ obtained in Part (b) of Problem 1 to a set of orthonormal modal vectors satisfying Equations 9.96 and 9.97.
4. Consider the three-story building system in Figure 9.23. Normalize the modal vectors $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ obtained in Part (b) of Problem 2 to a set of orthonormal modal vectors satisfying Equations 9.96 and 9.97.
5. Consider the two-degree-of-freedom mass–spring system in Figure 9.22. Determine the response of the system subjected to initial excitations $\mathbf{x}(0) = [0\ 0]^T$ and $\dot{\mathbf{x}}(0) = [1\ \ 0]^T$ by modal analysis. Note that the natural frequencies were found in Problem 1 and the modal vectors were normalized in Problem 3.
6. Consider the three-story building system in Figure 9.23. Determine the response of the system subjected to initial excitations $\mathbf{x}(0) = [0.01\ 0\ 0]^T$ and $\dot{\mathbf{x}}(0) = [0\ \ 0\ \ 0]^T$ by modal analysis. Note that the natural frequencies were calculated in Problem 2 and the modal vectors were normalized in Problem 4.
7. Consider the two-degree-of-freedom mass–spring system in Figure 9.22. Determine the response of the system if a harmonic force $F(t) = 2\cos t$ is applied to mass 2. Note that the natural frequencies were calculated in Problem 1 and the modal vectors were normalized in Problem 3.
8. Consider the three-story building system in Figure 9.23. Determine the response of the system if a harmonic force $F(t) = 0.15\sin(0.15t)$ is applied to the top story. Note that the natural frequencies were found in Problem 2 and the modal vectors were normalized in Problem 4.
9. Solve Problem 5 using MATLAB.
10. Solve Problem 6 using MATLAB.
11. Solve Problem 7 using MATLAB.
12. Solve Problem 8 using MATLAB.

9.5 VIBRATION MEASUREMENT AND ANALYSIS

Models are necessary for designing dynamic systems and understanding system dynamics. For many vibration systems, theoretical models are too difficult to develop. In these situations, we resort

to experimental models, which may be obtained using experimental modal techniques; specifically, the method known as frequency response function testing. There are two phases needed to obtain an experimental model of a vibration system. The first phase is the measurement phase, in which the frequency response functions of the system are measured. The second phase is the analysis phase, in which system parameters are estimated from the measured frequency response functions. In this section, we first introduce the measurement methods for acquiring frequency response data and then discuss the parameter estimation methods for extracting system properties.

9.5.1 Vibration Measurement

As presented in Section 8.3, a frequency response function is a transfer function, expressed in the frequency domain as opposed to the *s* domain. For a vibration system, a frequency response function describes the system response to an external excitation force as a function of frequency. The response may be displacement, velocity, or acceleration. To experimentally obtain the frequency response function of a vibration system, the input excitation and output response must be measured simultaneously. The excitation and response are both obtained in the frequency domain via fast Fourier transform, and the frequency response function is the ratio of the response to the excitation.

Figure 9.24 shows a diagram of the basic components for frequency response function measurement. An actuator, or so-called exciter, is used to apply force to the system under test. Sensors, or so-called transducers, are used to measure the force and responses. A data acquisition system is used to acquire and process the signals from the sensors. A computer with analysis software provides measurement functions such as windowing, averaging, and fast Fourier transform computation.

The first step in the measurement process involves selecting an excitation function along with an excitation system. The excitation function is the mathematical signal used for the input. The excitation system is the physical mechanism used to provide the signal. Two of the general categories of excitation functions are steady-state and transient. For example, a sine function is a steady-state signal and an impulse function is a transient signal. Two of the most common excitation mechanisms are a shaker and an impact hammer.

Figure 9.25 shows an electromagnetic shaker, which is one of the most common shakers for vibration testing. With the electromagnetic shaker, a force is generated by an alternating current that drives a magnetic coil. Such a shaker can generate a variety of time-varying forces, such as a sinusoidal force with constant frequency and a swept sine function with gradually increasing frequency but constant amplitude. The maximum frequency range and the maximum force level depend on the size of the shaker. Smaller shakers have a higher frequency range and a lower force level. When using a shaker for excitation, the shaker should be physically mounted on the system via a force transducer.

An impact hammer is another common excitation mechanism, which is used to apply impulse. As shown in Figure 9.26, the impact hammer has a transducer at the tip for measuring the impact force. If the system being tested is struck by the hammer quickly, the resulting excitation force resembles an impulse. Because the impact hammer does not have to be attached to the system under

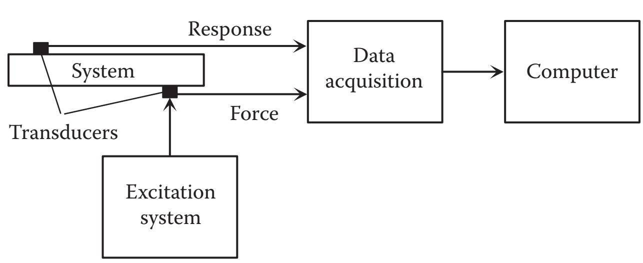


FIGURE 9.24 Basic components for frequency response function measurement.

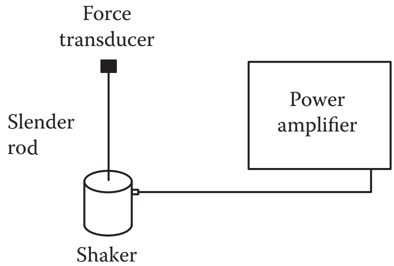


FIGURE 9.25 An electromagnetic shaker with power amplifier.

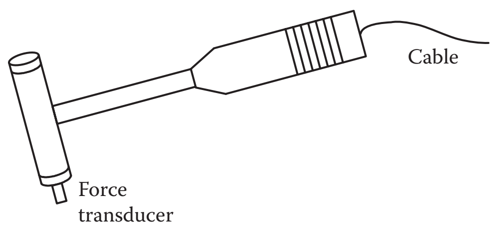


FIGURE 9.26 An impact hammer for vibration testing.

test, this technique is relatively easy to implement, but it is difficult to obtain consistent results. The frequency content and the pulse duration depend on the material of the tip. The harder the tip, the shorter the pulse duration included and thus the higher the frequency content measured.

The second step in the measurement process is to select the transducers for sensing force and response. The piezoelectric type is the most widely used for vibration testing. The piezoelectric transducer is an electromechanical sensor that generates an electrical output when subjected to vibration. The response measured is usually acceleration. This particular type of response transducer is called an accelerometer. Sometimes, the system is too small, or the working environment is too hot, to attach an accelerometer to the system under consideration. In those situations, a laser Doppler vibrometer can be used to make noncontact vibration measurements.

With the selected excitation mechanism and transducers, the frequency response function can be obtained in several different ways. For example, if a shaker is used, then harmonic excitation is applied to the system under test and the resulting harmonic response is measured. This type of test is referred to as sine wave testing. The frequency range is covered either by stepping from one frequency to the next or by slowly sweeping the frequency continuously. In both cases, the measurement time should be long enough to allow steady-state conditions to be attained. If an impact hammer is used, then impulsive excitation is applied to the system under test and the resulting transient response is measured.

9.5.2 System Identification

After having acquired frequency response data via vibration testing, the next major step is to identify the system parameters, more specifically, modal parameters. The basic information that can be determined from frequency response functions includes natural frequencies, the damping ratio associated with each mode, and mode shapes. The discussions in this subsection are only limited to the identification of natural frequencies and damping ratios.

To obtain an accurate estimation, it is important to understand the relationships between frequency response functions and their individual modal parameters. The basics of a single-degree-of-freedom

dynamic system form the basis for parameter estimation techniques. As discussed in Section 8.3, the frequency response function is a complex quantity, which can be presented in terms of magnitude and phase versus frequency. Another method of presenting the frequency response data is to plot the real part and the imaginary part versus frequency. Denote the Fourier transforms of displacement response and excitation force as $X(j\omega)$ and $F(j\omega)$, respectively. The expression of the frequency response function $X(j\omega)/F(j\omega)$ was given in Section 8.3. Figure 9.27 shows the frequency response presented in different forms, where the real and imaginary parts are

$$\mathrm{Re}\left(\frac{X(j\omega)}{F(j\omega)}\right)=\frac{k-m\omega^2}{\sqrt{(k-m\omega^2)^2+(b\omega)^2}}=\frac{1-(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2+(2\zeta\omega/\omega_n)^2}}, \tag{9.123}$$

$$\mathrm{Im}\left(\frac{X(j\omega)}{F(j\omega)}\right)=\frac{-b\omega}{\sqrt{(k-m\omega^2)^2+(b\omega)^2}}=\frac{-2\zeta\omega/\omega_n}{\sqrt{[1-(\omega/\omega_n)^2]^2+(2\zeta\omega/\omega_n)^2}}. \tag{9.124}$$

Assume that the system is lightly damped. As shown in Figure 9.27, when the excitation frequency approaches the natural frequency, $\omega = \omega_n$, the system resonates, and the magnitude of the frequency response function reaches its maximum. This conclusion was proved in Section 8.3. Also, at resonance, the real part is equal to zero, and the imaginary part reaches a peak. The former observation can be easily proved by setting $\omega = \omega_n$ in Equation 9.123, whereas the latter will be left as an exercise for the reader. It was discussed earlier in Section 9.2 that the damping ratio ζ can be estimated by the half-power bandwidth method.

In reality, most dynamic systems cannot be simplified as ideal single-degree-of-freedom systems. As presented in Section 9.4, an n-degree-of-freedom system has n modes. For systems with lightly damped and well-separated modes, as shown in Figure 9.28a, the natural frequencies and damping ratios can be estimated using the single-mode method. The basic assumption for single-mode approximation is that in the vicinity of a resonance, the response is due primarily to that single mode. Just like a single-degree-of-freedom system, the natural frequency associated with

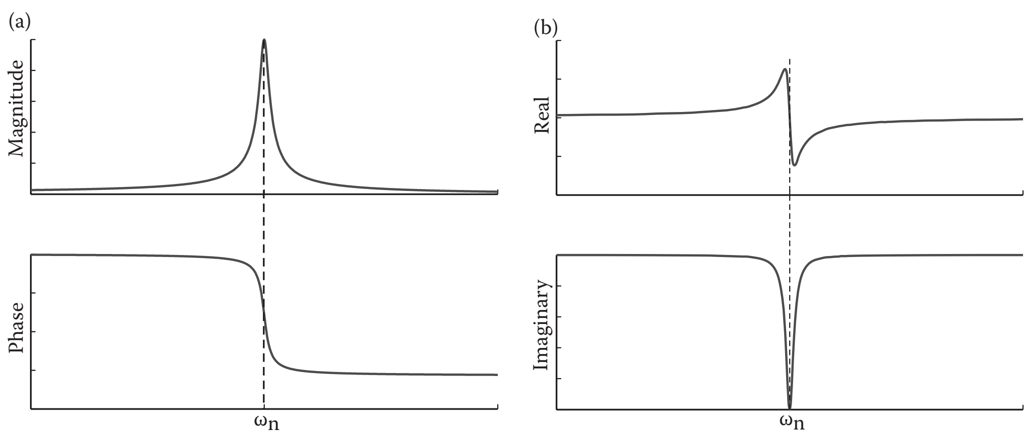


FIGURE 9.27 Frequency response of a single-degree-of-freedom system represented in terms of (a) magnitude and phase, and (b) real and imaginary.

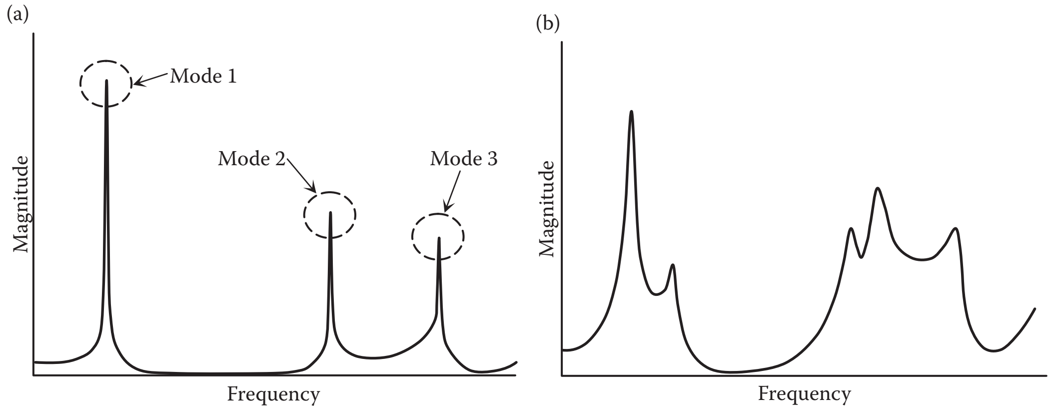


FIGURE 9.28 Frequency response of a multi-degree-of-freedom system with (a) light damping and (b) heavy damping.

that single mode can be estimated from the frequency response data by observing the frequency at which any of the following trends occur:

- The magnitude of the frequency response is a maximum.
- The real part of the frequency response is zero.
- The imaginary part of the frequency response is a maximum or minimum.

The damping ratio associated with that single mode can be estimated using the half-power bandwidth method. For systems with heavily damped and closely spaced modes, as shown in Figure 9.28b, the adjacent modes can affect each other significantly. In general, it will be necessary to implement a multiple-mode method to more accurately identify the modal parameters of these types of systems.

PROBLEM SET 9.5

1. For a single-degree-of-freedom system, denote the Fourier transforms of the displacement response and excitation force as $X(j\omega)$ and $F(j\omega)$, respectively. The expression of the imaginary part of the frequency response function $X(j\omega)/F(j\omega)$ is given by Equation 9.124. Prove that the imaginary part reaches a peak at resonance.
2. Accelerations are often measured in vibration testing. For a single-degree-of-freedom system, denote the Fourier transforms of the acceleration response and excitation force as $A(j\omega)$ and $F(j\omega)$, respectively. The expression of the frequency response function is given by

$$\frac{A(j\omega)}{F(j\omega)} = \frac{-\omega^2}{k - m\omega^2 + jb\omega}.$$

   Using MATLAB, write an m-file to plot the magnitude, phase, real part, and imaginary part of the frequency response versus ω/ω_n. Assume that $m = 50$ kg, $b = 30$ N·s/m, and $k = 2000$ N/m.
3. Rods, beams, plates, and so on are continuous systems, which have an infinite number of degrees of freedom and an infinite number of modes. For simplicity, assume that a cantilever beam is approximated as a single-degree-of-freedom mass–damper–spring system, for which the natural frequency is close to the first mode of the beam. The parameters of the cantilever beam are length $L = 0.5$ m, width $b = 0.025$ m, thickness $h = 0.005$ m, density $\rho = 7850$ kg/m^3, and Young's modulus $E = 210 \times 10^9$ N/m^2.

a. It is known that the equivalent mass for the beam is $m_{eq} = m/3$, where m is the actual mass of the beam. Determine the equivalent stiffness k_{eq} for the beam. Calculate the natural frequency of the equivalent single-degree-of-freedom system.
b. Figure 9.29 is the measured frequency response of the cantilever beam for the first mode. Determine the natural frequency and the damping ratio based on the given information in the plot.
c. Compare the frequencies obtained in Parts (a) and (b). What is the error if the cantilever beam is approximated as a single-degree-of-freedom system?

4. Figure 9.30 shows the magnitude of an experimentally determined frequency response. Estimate the number of degrees of freedom of the system and its natural frequencies.

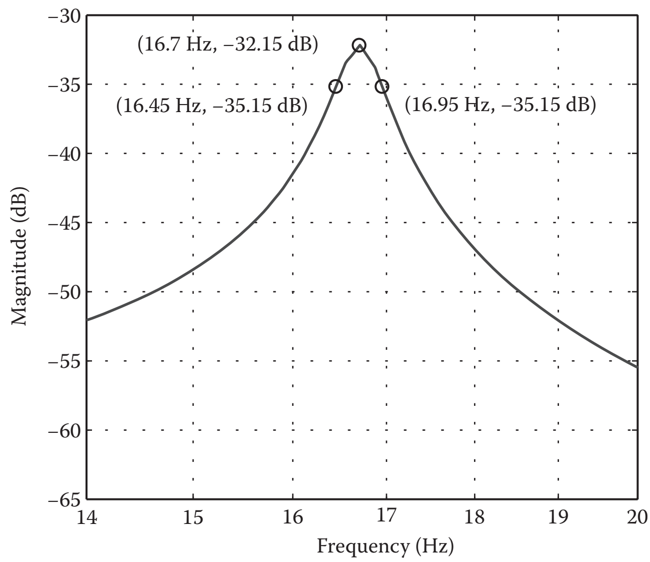


FIGURE 9.29 Problem 3.

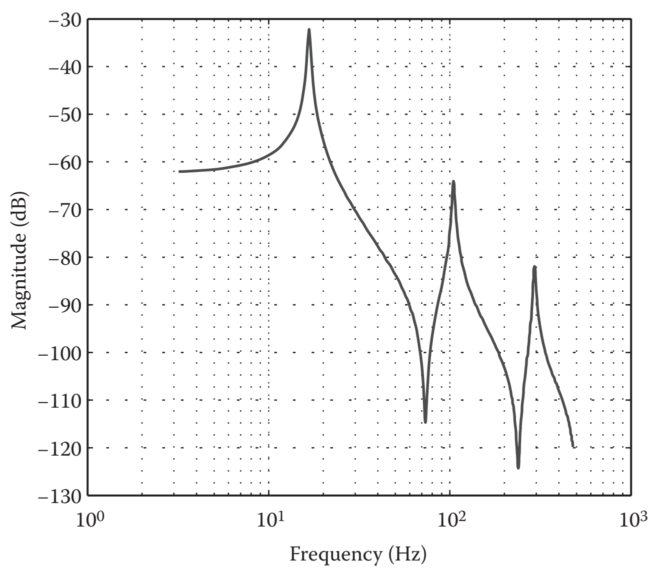


FIGURE 9.30 Problem 4.

9.6 SUMMARY

This chapter presented an introduction to vibrations. Two types of excitations cause a system to vibrate: initial excitation and external excitation. The vibration of a system caused by nonzero initial displacements or initial velocities (or both) is known as free vibration. The vibration of a system caused by externally applied forces is known as forced vibration.

The nature of system response to initial or external excitations depends on the system's damping, which is a very complex phenomenon. Logarithmic decrement and half-power bandwidth are two commonly used methods for measuring the damping of a vibration system. These two damping identification methods are only valid for the viscous damping model, which is widely used in vibration.

In the logarithmic decrement method, the free response to initial excitations is measured. The damping ratio ζ is estimated using the displacements of two consecutive peaks, x_1 and x_2. The value of the damping ratio ζ is given by

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}},$$

where δ is called the logarithmic decrement and $\delta = \ln(x_1/x_2)$. For a more accurate estimation, the damping ratio ζ can be determined by measuring the displacements of two peaks separated by a number of periods instead of two consecutive peaks.

In the half-power bandwidth method, the frequency response of the system is measured. The damping ratio ζ is estimated using a peak in the magnitude curve of the frequency response and two half-power points, which are 3 dB down from the peak. The value of the damping ratio ζ is given by

$$\zeta \approx \frac{\omega_2 - \omega_1}{2\omega_n},$$

where ω_1 and ω_2 are the frequencies at the two half-power points. For light damping, the natural frequency ω_n corresponds to the frequency at the peak or is approximated as $(\omega_1+\omega_2)/2$.

Frequency response is a very important concept in vibration. For harmonic excitations, more information on the steady-state response can be extracted using the frequency domain technique than the time domain technique. As presented in Chapter 8, the steady-state response of a single-degree-of-freedom mass–damper–spring system to harmonic excitation, for example, $f(t) = F_0\sin(\omega t)$, is still harmonic with a frequency that is the same as the excitation frequency. Denote the frequency response function of the system as $G(\mathrm{j}\omega)$; then, the steady-state response is $x(t) = X\sin(\omega t + \phi)$, where $X = F_0\left|G(\mathrm{j}\omega)\right|$ and $\phi = \angle G(\mathrm{j}\omega)$. The dimensionless ratio of the dynamic amplitude X and the static deflection x_{st} is given by

$$\frac{X}{x_{st}} = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}.$$

Rotating eccentric masses and harmonically moving supports are two common harmonic excitation sources in engineering applications. The steady-state responses to these two excitations can be obtained using the same approach as in the general case. The results are summarized in Table 9.1. The magnitude of the dimensionless ratios MX/me or X/Z_0 versus the driving frequency can be plotted using MATLAB. These plots provide significant information on harmonic responses.

Vibrations are undesirable in many systems. The reduction of vibration can be achieved through vibration isolators or vibration absorbers. A vibration isolation system attempts either to protect delicate equipment from vibration transmitted to it from its support system or to prevent the vibratory force

TABLE 9.1
Summarized Results for Rotating Unbalance and Harmonic Base Excitation

Rotating Unbalance	**Harmonic Base Excitation**
$M\ddot{x} + b\dot{x} + kx = me\omega^2\sin(\omega t)$, where the effect of a rotating unbalance mass is to exert a harmonic force $me\omega^2\sin(\omega t)$	$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 2\zeta\omega_n\dot{z}(t) + \omega_n^2 z(t)$, where the motion of the base is harmonic, for example, $z(t) = Z_0\sin(\omega t)$
$G(j\omega) = \dfrac{1/k}{1-(\omega/\omega_n)^2 + j2\zeta\omega/\omega_n}$,	$G(j\omega) = \dfrac{1 + j2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2 + j2\zeta\omega/\omega_n}$,
$x(t) = X\sin(\omega t + \phi)$, where	$x(t) = X\sin(\omega t + \phi)$, where
$X = \dfrac{me\omega^2/k}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}$	$X = Z_0\dfrac{\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}$
$\phi = -\tan^{-1}\dfrac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}$	$\phi = -\tan^{-1}\dfrac{2\zeta(\omega/\omega_n)^3}{1-(\omega/\omega_n)^2 + (2\zeta\omega/\omega_n)^2}$,
$\dfrac{MX}{me} = \dfrac{(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}$	$\dfrac{X}{Z_0} = \dfrac{\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}$.

generated by a machine from being transmitted to its surroundings. The essence of these two objectives is the same. The concept of displacement transmissibility X/Z_0 can be used for displacement isolation design, whereas force transmissibility F_T/F_0 can be used for force isolation design, where $X/Z_0 = F_T/F_0$.

To prevent harmonic resonance for a single-degree-of-freedom system, it is not always possible to prevent the natural frequency from being close to the driving frequency by changing either the mass or the spring stiffness. To address this issue, a vibration absorber consisting of a second mass and spring can be added to the system and protect the original single-degree-of-freedom system from harmonic excitation. The vibration of the original mass can be reduced to zero, provided that the natural frequency of the absorber is the same as the driving frequency. One disadvantage of the vibration absorber is that two new resonant frequencies are created.

For multi-degree-of-freedom systems, more advanced mathematical tools are needed to solve the equations of motion due to coordinate coupling. Modal analysis is an approach that utilizes the orthogonality of modal vectors to reduce the equations of motion to a set of independent second-order differential equations in modal coordinates. The basic steps in obtaining the response of an n-degree-of-freedom undamped system to initial excitations or external forces using modal analysis are summarized as follows.

Step 1: Solve the eigenvalue problem associated with the mass and stiffness matrices, that is,

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{X} = 0,$$

and obtain the natural frequencies, which are the square roots of eigenvalues $\omega_1^2, \omega_2^2, \ldots$, and ω_n^2, and the modal vectors, which are eigenvectors $\mathbf{X}_1$, $\mathbf{X}_2$, …, and $\mathbf{X}_n$. The natural frequencies are usually arranged in increasing order of magnitude, that is, $\omega_1 \le \omega_2 \le \cdots \le \omega_n$. The modal vectors represent the shape of the normal modes physically.

Step 2: Normalize each modal vector to satisfy the relations $\tilde{\mathbf{X}}_r^T\mathbf{M}\tilde{\mathbf{X}}_r = 1$ and $\tilde{\mathbf{X}}_r^T\mathbf{K}\tilde{\mathbf{X}}_r = \omega_r^2$, where $r = 1, 2, \ldots, n$. Then, the orthonormal modal matrix $\mathbf{\Phi} = [\tilde{\mathbf{X}}_1 \ \cdots \ \tilde{\mathbf{X}}_n]$ satisfies

$$\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \mathbf{I}_n, \quad \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \mathbf{\Omega},$$

TABLE 9.2
Responses to Initial Excitations or External Harmonic Excitations Using Modal Analysis

	Initial Excitations	External Harmonic Excitations
Equations of motion	$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$	$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}_0\sin(\omega t)$
Modal equations	$\ddot{q}_r + \omega_r^2 q_r = 0, \quad r = 1,2,\ldots,n$	$\ddot{q}_r + \omega_r^2 q_r = N_r, \quad r = 1,2,\ldots,n$ where the modal forces are $N_r = \tilde{\mathbf{X}}_r^T \mathbf{f}_0\sin(\omega t)$
Modal responses	$q_r(t) = q_r(0)\cos(\omega_r t) + \dfrac{\dot{q}_r(0)}{\omega_r}\sin(\omega_r t)$, where the modal initial conditions are $q_r(0) = \tilde{\mathbf{X}}_r^T \mathbf{M}\mathbf{x}(0), \dot{q}_r(0) = \tilde{\mathbf{X}}_r^T \mathbf{M}\dot{\mathbf{x}}(0)$	$q_r(t) = \dfrac{\tilde{\mathbf{X}}_r^T \mathbf{f}_0}{\omega_r^2 - \omega^2}\sin(\omega t)$
System responses	$\mathbf{x}(t) = \sum_{r=1}^{n} q_r(t)\tilde{\mathbf{X}}_r$	$\mathbf{x}(t) = \sum_{r=1}^{n} q_r(t)\tilde{\mathbf{X}}_r$

where $\mathbf{\Omega}$ is a diagonal matrix of the squares of the natural frequencies. Introducing the modal coordinate vector $\mathbf{q}$ and the coordinate transformation $\mathbf{x} = \mathbf{\Phi}\mathbf{q}$, we can decouple the original equations into n independent modal equations.

Step 3: Determine the modal responses and combine them to determine the response of the original system using the expansion theorem. Table 9.2 lists the responses to initial excitations or external harmonic excitations.

For many vibration systems, theoretical models are too difficult to develop. In these cases, experimental modal techniques can be used to obtain experimental models. There are two phases involved in obtaining an experimental model of a vibration system. The first phase is the measurement phase, in which the frequency response functions of the system are measured. Two of the most common excitation mechanisms are the shaker and the impact hammer. The second phase is the analysis phase, in which the system parameters are estimated from the measured frequency response functions. The basic information that can be determined from the frequency response functions includes natural frequencies, the damping ratio associated with each mode, and mode shapes.

REVIEW PROBLEMS

1. Consider the single-degree-of-freedom system shown in Figure 5.42.
   a. Determine the undamped natural frequency ω_n, the damping ratio ζ, and the damped natural frequency ω_d.
   b. Assume $f(t) = 0$. Find the free vibration response of the system subjected to the initial conditions $x(0) = 0.05$ m and $\dot{x}(0) = 0$ m/s.
   c. Write a MATLAB m-file to plot the system's response obtained in Part (b).
   d. Construct a Simulink® block diagram based on the differential equation of motion of the system and find the free vibration response.
   e. Build a Simscape model of the physical system and find the free vibration response.
2. Consider the single-degree-of-freedom system shown in Figure 5.42.
   a. Assume $f(t) = 500\sin(40t)$ and initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. Determine the dynamic amplitude X and the static deflection x_{st} of the system. Find the forced vibration response of the system.
   b. Write a MATLAB m-file to plot the system's response obtained in Part (a).

c. Construct a Simulink block diagram based on the differential equation of motion of the system and find the forced vibration response.

d. Build a Simscape model of the physical system and find the forced vibration response.

e. Assume that the driving frequency varies from 0 to 100 rad/s. Write a MATLAB m-file to plot the dimensionless ratio X/x_{st} versus the driving frequency ω.

3. A machine can be considered as a rigid mass with a rotating unbalanced mass. To reduce the vibration, the machine is mounted on a support system with a stiffness of 10 kN/m. Assume that the total mass is $M = 10$ kg and the unbalanced mass is $m = 0.5$ kg. Determine the range of the damping ratio of the support system so that the vibration amplitude will not exceed 10% of the rotating mass's eccentricity when the machine operates at a speed of 350 rpm.
4. As shown in Figure 9.31, a machine of a mass $M = 150$ kg is mounted on a fixed-fixed steel beam with negligible mass. The parameters of the beam are given as follows: Young's modulus of the beam $E = 210$ GPa, width $b = 0.75$ m, and thickness $h = 3$ cm. The rotating unbalance in the machine is $me = 0.015$ kg·m. The machine runs at a speed of 2400 rpm and the maximum allowable displacement is 4 mm. Determine the length of the beam. Assume damping to be negligible.
5. A 110-kg machine is placed on a floor that vibrates with a frequency of 20 Hz. The maximum acceleration of the floor is 15 cm/s^2. A vibration isolator consisting of four parallel-connected springs is designed to protect the machine from the vibration of the floor. Assume that the damping ratio of the isolator is 0.1 and the maximum allowable acceleration is 2.25 cm/s^2. Determine the stiffness of each spring.
6. Many vibration measuring instruments consist of a case containing a mass–damper–spring system, as shown in Figure 9.32. The displacement of the mass relative to the case is measured electronically. Denote the displacement of the mass, the displacement of the case, and the displacement of the mass relative to the case as $x(t)$, $z(t)$, and $y(t)$, respectively, where $y(t) = x(t) - z(t)$. Assume harmonic excitation, $z(t) = Z_0\sin(\omega t)$.

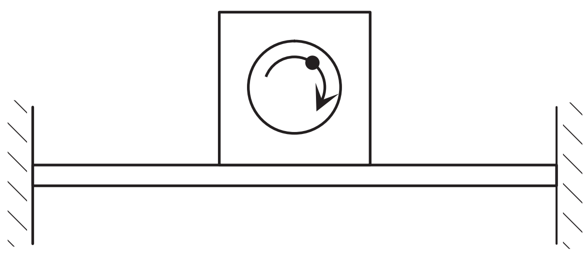


FIGURE 9.31 Problem 4.

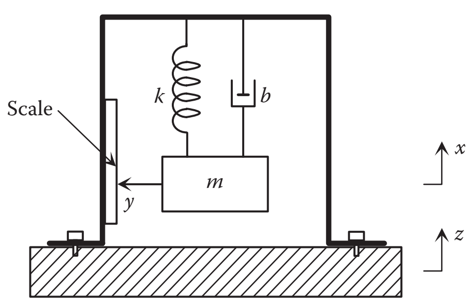


FIGURE 9.32 Problem 6.

a. Show that the amplitude Y of $y(t)$ is given by

$$Y = \frac{(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} Z_0$$

b. Using MATLAB, write an m-file to plot the frequency response Y/Z_0 versus ω/ω_n for different values of damping ratio: $\zeta = 0.1, 0.25, 0.5, 0.7$, and 1.0.

7. A 2000-kg rotating machine is mounted on a vibration isolator consisting of four parallel-connected springs. The machine operates at a speed of 1800 rpm. Determine the stiffness of each spring so that the force transmitted to the ground is reduced by 90%. Assume that the springs are identical and damping is negligible.
8. Consider the single-degree-of-freedom system shown in Figure 9.15a, where $m = 10$ kg. When the mass is in equilibrium, the static deformation of the spring is 2.5 mm. When the mass is allowed to vibrate freely, the maximum displacement amplitude during the fifth cycle is 20% of the first. Assume that a harmonic excitation force is applied to the system.
   a. Determine the minimum allowable driving frequency if the force transmitted to the ground is less than the excitation force.
   b. If the allowable force transmissibility is 20%, determine the stiffness of the spring.
9. Consider the vibration absorber shown in Figure 9.16. It is known that two new resonant frequencies in the neighborhood of the excitation frequency are created. Denote the two new natural frequencies as ω_{n1} and ω_{n2}. Assume $\nu = 1$. Determine ω_{n1}/ω_1 and ω_{n2}/ω_1 for the following cases: $\mu = 0.05, 0.1, 0.15, 0.2$, and 0.25.
10. A 75-kg table saw is driven by a motor that runs at a constant speed of 180 rpm and produces a 13-N force. Assume that the stiffness provided by the table legs is 2500 N/m and the damping is negligible.
    a. Determine the dynamic amplitude of the table.
    b. Design a vibration absorber to reduce the table oscillation to zero. Assume that the maximum allowable displacement of the absorber is 2 mm. What is the value of the mass ratio μ?
11. Consider the linearized model of the double pendulum system in Example 5.15. Assume $m = 1$ kg and $L = 1$ m.
    a. Use the modal analysis approach to determine the response of the system to the initial excitations $\theta_1(0) = 0.05$ rad, $\theta_2(0) = 0.1$ rad, $\dot{\theta}_1(0) = 0$ rad/s, and $\dot{\theta}_2(0) = 0$ rad/s.
    b. Write a MATLAB m-file to plot the response of the system.
    c. Construct a Simulink block diagram based on the linearized model and find the response of the system.
12. Consider the two-degree-of-freedom mass–spring system in Figure 5.118.
    a. Use modal analysis to determine the response of the system to the harmonic excitation $f = 40\sin(7\pi t)$ N.
    b. Write a MATLAB m-file, plot the response of the system, and compare the result with those obtained with Simulink and Simscape simulations.
13. A single-degree-of-freedom system undergoes free vibration. Figure 9.33 is the recorded displacement response of the first three cycles. The mass of the system is known to be 1750 kg. Determine the stiffness k and the damping b of the system.
14. The frequency response function of a single-degree-of-freedom system is shown in Figure 9.34. Determine the system's parameters including the mass m, the stiffness k, and the damping b.

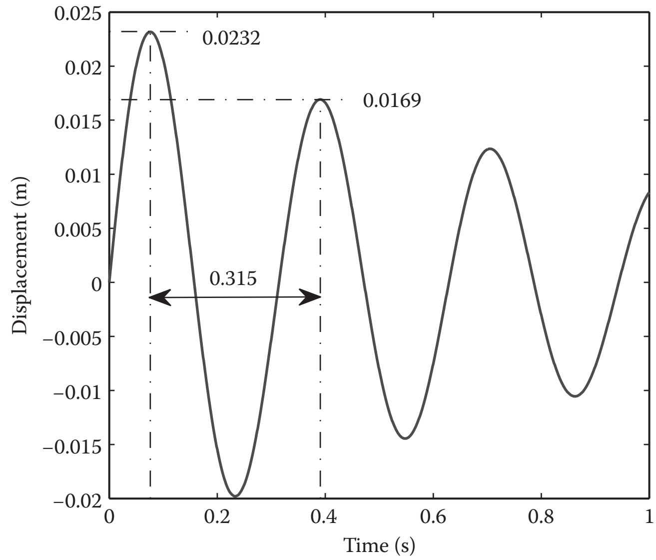


FIGURE 9.33 Problem 13.

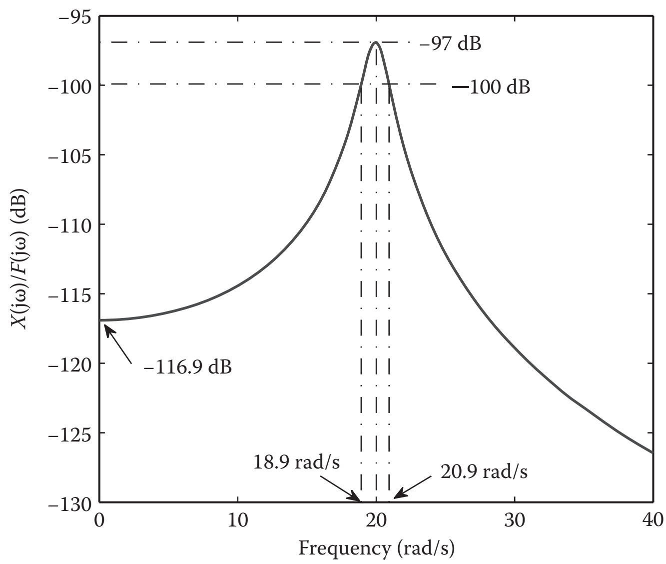


FIGURE 9.34 Problem 14.

10 Introduction to Feedback Control Systems

Control deals with the modeling of a variety of dynamic systems and the design of controllers that will ensure these systems behave in a desired manner. In Chapters 5 through 7, we discussed how to derive a mathematical model for a dynamic system, which can be mechanical, electrical, fluid, or thermal. In this chapter, we focus on how to design a controller for a dynamic system based on its mathematical model. Basic concepts such as feedback, open-loop control, closed-loop control, regulators, and servos, as well as basic terminologies in control are introduced in Section 10.1. In general, there are two main reasons why control is needed. One is to maintain the system stability, and the other is to improve the system performance. Section 10.2 covers how to determine the stability of a system and how to define the performance in either time domain or frequency domain. Section 10.3 discusses the advantages of feedback control, which is utilized in most situations. Following the overview of feedback in Section 10.3, the classical structure of proportional, integral, and derivative control is introduced in Section 10.4. Three different control design methods based on root locus, Bode plot, and state variable feedback are presented in Sections 10.5 through 10.7, respectively. The chapter concludes with controller design and implementation using MATLAB®, Simulink®, and Simscape™ computer tools.

10.1 BASIC CONCEPTS AND TERMINOLOGIES

Control is the process of manipulating, manually or automatically, the input of a dynamic system so that the system output will behave as desired. If the output signal is measured and fed back for use in computing the input signal, the system is called feedback control. A familiar example is the cruise control of an automobile. To maintain a constant vehicle speed set by the driver, the actual speed of the vehicle is measured by the speedometer and fed back to the controller, which adjusts the engine's throttle position. Then, the engine torque is changed accordingly, which influences the vehicle's actual speed.

To analyze and design a feedback control system, a block diagram is usually drawn to show the major components and their interconnections in graphical form. Figure 10.1 is a general block diagram of an elementary feedback control system. The essential components of this feedback control include a system we want to control, a controller we need to design, an actuator used to drive the controlled system, and a sensor used to measure the system output. The connecting lines in the block diagram carry signals. As shown in Figure 10.1, the important signals in this feedback control system include the output, the control signal, and the reference.

In the example of the automobile's cruise control, the controlled system is the auto-body, whose output is the speed. The speedometer, which acts as a sensor, measures the vehicle speed. The measured speed is fed back and compared with the desired speed, which is the reference signal. Based on the error between the measured and the reference signals, the controller computes the control signal, which is the engine's throttle position in our case. The engine is the actuator, and the torque provided by the engine is applied to the auto-body, which influences the vehicle speed.

Any control system must have these four essential components. Generally, the controlled system and the actuator are intimately connected, and they are combined as one component called the plant. There are two other signals shown in Figure 10.1, disturbance and sensor noise. Both of them are undesired system inputs that adversely affect the performance of a system.

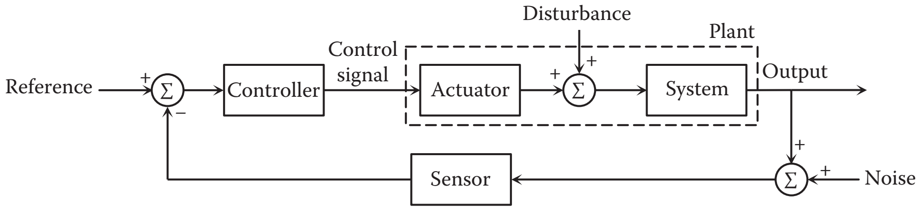


FIGURE 10.1 Block diagram of an elementary feedback control system.

Example 10.1: Block Diagram of a Feedback Control System

Consider the electromechanical system described in Problem 3 of Problem Set 6.4. It consists of a cart that moves along a linear track and a DC motor that drives the cart. An encoder is included to measure the position of the cart. Assume that a controller is designed to control the position of the cart. Draw a block diagram for this feedback control system. Clearly label essential components and signals.

Solution

Note that the essential components of this feedback control are the cart (the controlled system), the controller, the DC motor (the actuator), and the encoder (the sensor). The corresponding block diagram is shown in Figure 10.2. The actual position of the cart is the output, the desired position is the reference, and the voltage applied to the DC motor is the control signal.

Transfer functions are usually used to represent the mathematical model of each block in a block diagram. The input and output signals of each block are also expressed in the Laplace domain, and they are denoted by capital letters. If the disturbance and noise signals are negligible, then the general block diagram of a feedback control system given in Figure 10.1 can be redrawn as shown in Figure 10.3, in which $G(s)$ represents the dynamics of the plant, $C(s)$ is the controller, $H(s)$ is the sensor, $U(s)$ is the control signal, $Y(s)$ is the actual system output, $Y_m(s)$ is the measured output, and $R(s)$ is the reference. The difference between the reference and the feedback is defined as the error signal,

$$E(s) = R(s) - Y_m(s). \tag{10.1}$$

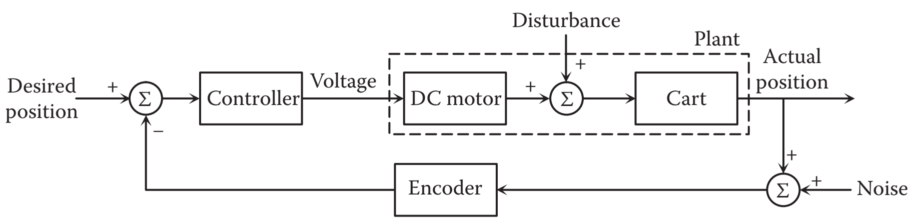


FIGURE 10.2 Block diagram of a cart position control system.

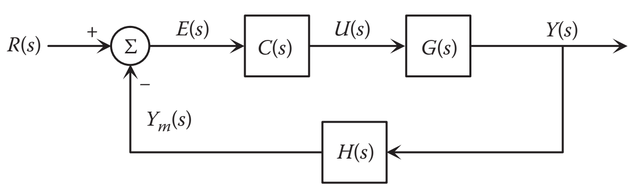


FIGURE 10.3 General block diagram of a feedback control system with transfer function representation.

For an ideal sensor, its output is exactly the same as its input, that is, $H(s) = 1$. Therefore, we have $Y_m(s) = Y(s)$ and

$$E(s) = R(s) - Y(s). \tag{10.2}$$

A control system with feedback is also called a closed-loop control. If the feedback is subtracted, it is called negative feedback, whereas feedback that is added is called positive feedback. Negative feedback is usually required for system stability, whereas positive feedback tends to make the system unstable. Unlike feedback control, open-loop control does not use the measured output to compute the control signal. The advantages of closed-loop control over open-loop control will be discussed in Section 10.3.

Example 10.2: Closed-Loop Transfer Function

Reconsider the cart position control system in Example 10.1. The transfer functions of the plant (combining the cart and the DC motor), the controller, and the sensor are

$$G(s) = \frac{3.778}{s^2 + 16.883s}, \quad C(s) = 85, \quad H(s) = 1.$$

Derive the closed-loop transfer functions $Y(s)/R(s)$ and $E(s)/R(s)$.

Solution

Using the result presented in Section 4.5, the equivalent transfer function for a negative feedback control system is

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)}.$$

Substituting the given transfer functions results in

$$\frac{Y(s)}{R(s)} = \frac{85(3.778/(s^2 + 16.883s))}{1 + 85(3.778/(s^2 + 16.883s))} = \frac{321.13}{s^2 + 16.883s + 321.13}.$$

Because $E(s) = R(s) - Y(s)$, we have

$$\frac{E(s)}{R(s)} = \frac{R(s) - Y(s)}{R(s)} = 1 - \frac{Y(s)}{R(s)}.$$

Inserting $Y(s)/R(s)$ gives

$$\frac{E(s)}{R(s)} = 1 - \frac{321.13}{s^2 + 16.883s + 321.13} = \frac{s^2 + 16.883s}{s^2 + 16.883s + 321.13}.$$

Note that the system in Example 10.1 is designed to track a reference signal. This type of control system is called a tracking or servo system, in which the reference signal usually varies with time. If the reference signal is constant, usually zero, and the system is designed to hold an output steady against unknown disturbances, then the control system is known as a regulator.

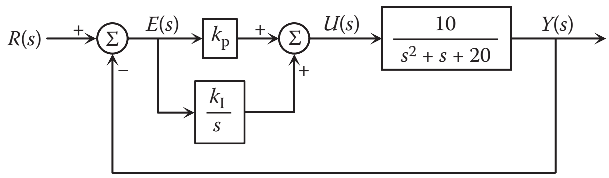


FIGURE 10.4 Problem 3.

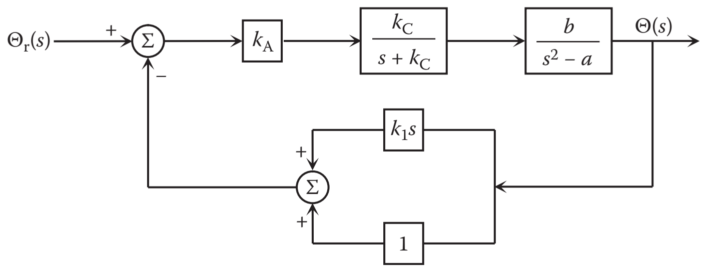


FIGURE 10.5 Problem 4.

PROBLEM SET 10.1

1. Draw a block diagram for the feedback control of a liquid-level system, which consists of a valve with a control knob (0%–100%) and a liquid-level sensor. Clearly label essential components and signals.
2. Draw a block diagram for the feedback control of a single-link robot arm system, which consists of a DC motor to produce the driving force and an encoder to measure the joint angle. Clearly label essential components and signals.
3. Determine the transfer functions $U(s)/E(s)$, $Y(s)/R(s)$, and $E(s)/R(s)$ in Figure 10.4.
4. The block diagram in Figure 10.5 represents a rocket attitude control system. Determine the transfer function $\Theta(s)/\Theta_r(s)$.
5. Consider the control system in Example 10.2. Build a Simulink block diagram to simulate reference tracking control, in which the signal $R(s)$ is a sine wave with a magnitude of 0.1 m and a frequency of 2 rad/s. Show the actual position response and the reference signal in the same scope.
6. Reconsider the control system in Example 10.2.
   a. Convert the transfer function $G(s) = Y(s)/U(s)$ to a differential equation of $y(t)$.
   b. Using the differential equation obtained in Part (a) to represent the plant, build a Simulink block diagram to simulate regulation control, in which the reference signal $R(s)$ is zero. Assume that the initial conditions are $y(0) = 0.1$ m and $\dot{y}(0) = 0$ m/s.

10.2 STABILITY AND PERFORMANCE

Stability and performance are two important subjects in control. Generally, before designing a controller for a dynamic system, control designers check the stability and performance of the uncontrolled system. Then they come up with reasonable control design objectives from the perspective of stability and performance. After a controller is designed, the stability and performance of the closed-loop control system are verified to meet the design objectives. In this section, we introduce

the stability condition, time-domain performance specifications, and frequency-domain performance specifications.

10.2.1 Stability of Linear Time-Invariant Systems

Intuitively, a system is stable if its transient response decays and is unstable if it diverges. Thus, we can determine the stability of a system by solving and plotting the transient response of the system. However, this is not the only way to determine the system stability.

Consider a linear time-invariant system, whose transfer function is given by

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}. \tag{10.3}$$

Assume that the numerator and denominator polynomials, $b(s)$ and $a(s)$, have no common factors. Setting the denominator of the transfer function equal to zero leads to the characteristic equation

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0. \tag{10.4}$$

The roots of the characteristic equation are called the poles (see Section 2.3), which are complex and can be defined in terms of real and imaginary parts, $p = \sigma + j\omega$. A complex s-plane is usually sketched to show pole locations.

The following is the condition for stability: a linear time-invariant system is said to be stable if all the poles of its transfer function have negative real parts and is unstable otherwise. In terms of the pole locations in the s-plane, a linear time-invariant system is stable if all the poles of the system are inside the left-half s-plane, that is, $\sigma < 0$. If the system has any poles, even only one pole, in the right-half s-plane, that is, $\sigma > 0$, then the system is unstable. Thus, the imaginary $j\omega$-axis is the stability boundary. If the system has nonrepeated poles on the $j\omega$-axis, that is, $\sigma = 0$, then the system is marginally stable. If the system has repeated poles on the $j\omega$-axis, then the system is unstable. The roots of the numerator of the transfer function are called zeros, which are not related to the stability of the system.

Essentially, the transient response of a linear time-invariant system is associated with its pole locations in the s-plane. For example, consider a first-order system whose transfer function is $Y(s)/U(s) = 1/(s - p)$. Assume that the input is an impulse function. We can obtain the response $y(t)$ by applying the inverse Laplace transform, $y(t) = e^{pt}$. The transient response e^{pt} approaches zero if and only if the real part of the pole p is negative. This simple example explains the reason why poles can be used to determine stability.

It is not an easy task to find the roots of a high-order characteristic equation by hand. Routh's stability criterion is a method for obtaining information about pole locations without solving for the poles. Consider the characteristic equation given in Equation 10.4. Routh's stability criterion consists of two conditions:

- A necessary (but not sufficient) condition for stability is that all the coefficients of the characteristic polynomial are positive.
- A necessary and sufficient condition for stability is that all the elements in the first column of the Routh array are positive.

If any of the coefficients in the characteristic polynomial is negative, we can conclude that the system is unstable by applying the first condition. If all of the coefficients in the characteristic

polynomial are positive, then we need to check the second condition by constructing a Routh array as shown in Equation 10.5. For an nth order characteristic polynomial, the Routh array has $n + 1$ rows. The first two rows are obtained by arranging the coefficients of the characteristic polynomial, beginning with the first and second coefficients and followed by the even-numbered and odd-numbered coefficients, respectively. Starting from the third row, the elements are formed from the two previous rows using determinants, with two elements in the first column and the other two elements from successive columns. Equation 10.6 shows how to compute the elements in the third and fourth rows. The rest of the rows can be obtained in the same manner as rows 3 and 4.

$$\begin{array}{cccccccc} s^n & : & 1 & a_2 & a_4 & a_6 & \cdots \\ s^{n-1} & : & a_1 & a_3 & a_5 & a_7 & \cdots \\ s^{n-2} & : & b_1 & b_2 & b_3 & \cdots & \\ s^{n-3} & : & c_1 & c_2 & c_3 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ s^2 & : & * & * & & & \\ s^1 & : & * & & & & \\ s^0 & : & * & & & & \end{array}, \tag{10.5}$$

$$\begin{aligned} b_1 &= \frac{-\det\begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} = \frac{a_1 a_2 - a_3}{a_1}, & c_1 &= \frac{-\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} = \frac{b_1 a_3 - a_1 b_2}{b_1}, \\ b_2 &= \frac{-\det\begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1} = \frac{a_1 a_4 - a_5}{a_1}, & c_2 &= \frac{-\det\begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1} = \frac{b_1 a_5 - a_1 b_3}{b_1}, \\ b_3 &= \frac{-\det\begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1} = \frac{a_1 a_6 - a_7}{a_1}, & c_3 &= \frac{-\det\begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}}{b_1} = \frac{b_1 a_7 - a_1 b_4}{b_1}. \end{aligned} \tag{10.6}$$

Example 10.3: Stability

The transfer function of a dynamic system is given by

$$G(s) = \frac{s+4}{s^5 + 2s^4 + 3s^3 + 8s^2 + 4s + 5}.$$

Determine the stability of the system

a. Using Routh's stability criterion without solving for the poles of the system.
b. Using MATLAB to solve for the poles.

Solution

a. The characteristic equation is $s^5 + 2s^4 + 3s^3 + 8s^2 + 4s + 5 = 0$. The Routh array can be formed as follows:

$$\begin{array}{llll}
s^5: & 1 & 3 & 4 \\
s^4: & 2 & 8 & 5 \\
s^3: & -1 = \dfrac{2(3)-8}{2} & 1.5 = \dfrac{2(4)-5}{2} & 0 \\
s^2: & 11 = \dfrac{(-1)(8)-2(1.5)}{-1} & 5 = \dfrac{(-1)(5)-0}{-1} & 0 \\
s^1: & \dfrac{43}{22} = \dfrac{11(1.5)-(-1)(5)}{11} & 0 & \\
s^0: & 5 & &
\end{array}$$

Because the elements in the first column of the Routh array are not all positive, we conclude that the system is unstable.

b. One of two MATLAB commands can be used to solve for the poles, `pole` or `roots`

```
>> num = [1 4];
>> den = [1 2 3 8 4 5];
>> sys = tf(num,den);
>> pole(sys)
```

The command `pole` returns the poles of the system: −2.16, 0.31 ± 1.65j, and −0.23 ± 0.88j. There are two poles that have positive real parts, and thus the system is unstable. We can also use the command `roots` to solve for the roots of the characteristic equation.

```
>> roots(den)
```

Routh's stability criterion was especially useful before the availability of mathematical and scientific computing software, such as MATLAB. However, it is still useful for determining the ranges of system parameters for stability (see Example 10.5 in Section 10.3). It should be pointed out that the study of stability discussed here is limited to only linear time-invariant systems. The study of stability for nonlinear and time-varying systems is very complex and is beyond the scope of this text.

10.2.2 Time-Domain Performance Specifications

Performance specifications are certain requirements associated with the response of the system. In the time domain, the requirements are usually given for the step response. Consider a second-order system whose transfer function is given by

$$\frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{10.7}$$

Figure 10.6 shows the unit-step response of this system, in which the vertical axis is normalized so that the steady-state value is equal to 1. Four quantities are defined to specify the performance of the system: rise time (t_r), overshoot (M_p), peak time (t_p), and settling time (t_s).

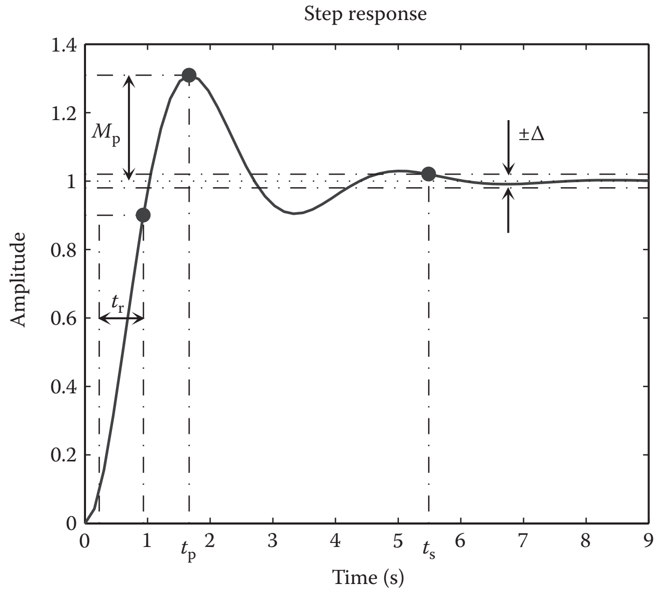


FIGURE 10.6 A unit-step response with time-domain performance specification indicated.

The rise time t_r is the time required for the response to rise from 10% to 90% of its steady-state value. The shorter the rise time, the faster the system reaches the vicinity of the steady-state value. The approximated relationship between t_r, the natural frequency ω_n, and the damping ratio ζ for a second-order system without zeros (see Equation 10.7) is given by

$$\omega_n t_r \approx 1.12 - 0.078\zeta + 2.30\zeta^2. \tag{10.8}$$

The overshoot M_p is the maximum amount of the response of the system exceeding the steady-state value divided by the steady-state value. It is usually expressed as a percentage value. The peak time t_p is the time it takes the response to reach its maximum value. As discussed in Section 8.2, assuming zero initial conditions, the unit-step response for the second-order system in Equation 10.7 is

$$y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right]. \tag{10.9}$$

Differentiating $y(t)$ with respect to t and setting it equal to zero yields the peak time

$$t_p = \frac{\pi}{\omega_d}. \tag{10.10}$$

Substituting Equation 10.10 into Equation 10.9 gives the value of y at the peak time. The overshoot M_p can be determined by computing $y(t_p) - 1$, as

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}. \tag{10.11}$$

The detailed derivation will be left to the reader as an exercise. As shown in Equation 10.11, the overshoot of the step response is related to the damping of the system. Figure 10.7 is the plot of M_p versus ζ. The larger the damping ratio, the smaller the overshoot.

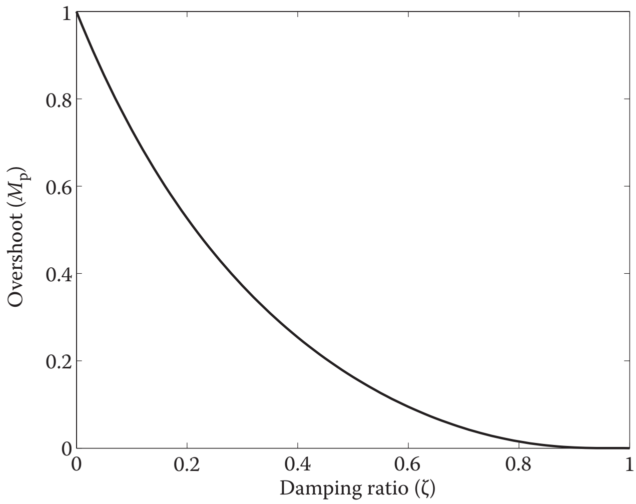


FIGURE 10.7 Plot of overshoot versus damping ratio for a second-order system.

The setting time t_s is the time required for the transient to decay to a small value so that $y(t)$ almost reaches the steady-state value. Note that Equation 10.9 can be rewritten as

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi), \tag{10.12}$$

where

$$\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}. \tag{10.13}$$

It is observed from Equation 10.12 that the value of $y(t)$ is within the bounds

$$1 - \Delta = 1 - \frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} \tag{10.14}$$

and

$$1 + \Delta = 1 + \frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}}, \tag{10.15}$$

where Δ is a small value, such as 1%, 2%, and 5%. Thus, we have

$$t_s = -\frac{\ln(\Delta\sqrt{1-\zeta^2})}{\zeta\omega_n}, \tag{10.16}$$

and for small damping,

$$t_s \approx -\frac{\ln \Delta}{\zeta \omega_n}. \tag{10.17}$$

As indicated by Equations 10.8, 10.10, and 10.11, as well as by Equations 10.16 and 10.17, the time-domain performance specifications are related to the system parameters, specifically the undamped natural frequency ω_n and the damping ratio ζ. From Equation 10.7, the poles of a second-order system are also related to ω_n and ζ,

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d, \tag{10.18}$$

which is a pair of complex conjugate poles for $0 \le \zeta < 1$. Figure 10.8 shows the plot of the poles in the s-plane. The two poles are equidistant from the origin with magnitude

$$\sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = \sqrt{(-\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \omega_n \tag{10.19}$$

and the angle between the pole and the imaginary axis satisfies

$$\sin\theta = \frac{|\mathrm{Re}|}{\sqrt{\mathrm{Re}^2 + \mathrm{Im}^2}} = \frac{\zeta\omega_n}{\omega_n} = \zeta. \tag{10.20}$$

Therefore, the time-domain performance specifications are associated with the pole locations. In control design, one or more of these requirements are often specified to determine the allowable region for the poles in the s-plane. The three plots in Figure 10.9 show the regions graphed based on the transient requirements t_r, M_p, and t_s, respectively.

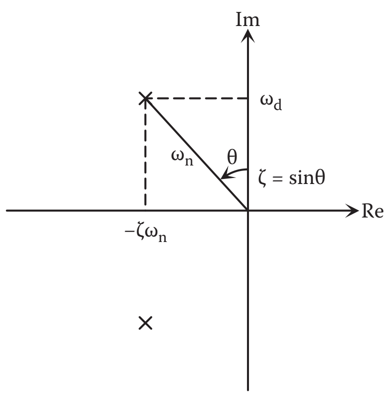


FIGURE 10.8 Relationship between a pair of complex conjugate poles and system parameters.

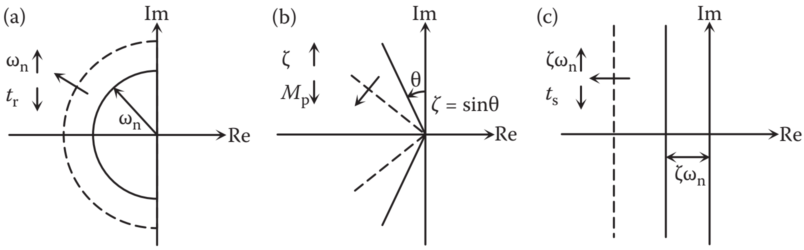


FIGURE 10.9 Regions graphed based on (a) rise time, (b) overshoot, and (c) settling time.

Example 10.4: Time-Domain Performance Specifications

Consider a second-order system whose transfer function is given by Equation 10.7. Assume that the requirements for the system unit-step response are overshoot $M_p \le 20\%$ and 2% settling time $t_s \le 2$ s. Sketch the allowable region for the poles in the *s*-plane.

Solution

To express the damping ratio in terms of the overshoot, we take the natural logarithm of both sides of Equation 10.11,

$$\ln M_p = -\frac{\pi\zeta}{\sqrt{1-\zeta^2}}.$$

Squaring both sides and solving for ζ,

$$\zeta = \frac{|\ln M_p|}{\sqrt{\pi^2 + (\ln M_p)^2}}.$$

As shown in both Figures 10.7 and 10.9b, ζ is mono decreasing with respect to M_p. That is, ζ will increase if M_p decreases and vice versa. Thus, for $M_p \le 20\%$, we have

$$\zeta \ge \frac{|\ln 0.2|}{\sqrt{\pi^2 + (\ln 0.2)^2}} = 0.46.$$

The corresponding angle θ is

$$\theta = \sin^{-1}\zeta \ge \sin^{-1}0.46 = 27°.$$

Equation 10.16 indicates that

$$\zeta\omega_n = -\frac{\ln(\Delta\sqrt{1-\zeta^2})}{t_s},$$

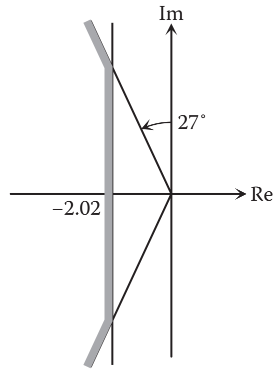


FIGURE 10.10 Allowable region of the poles in the s-plane.

and $\zeta\omega_n$ will increase when t_s decreases. Thus, for 2% settling time $t_s \le 2$ s and the smallest possible damping $\zeta = 0.46$, we have

$$\zeta\omega_n \ge -\frac{\ln\left(0.02\sqrt{1-0.46^2}\right)}{2} = 2.02.$$

The area to the left of the gray boundary shown in Figure 10.10 is the allowable region for the poles in the s-plane so that the two performance requirements are met.

Note that Equations 10.8 through 10.17 are derived based on the assumption that the system has no zeros and has two complex poles. Thus, they do not provide precise design formulas for all systems. However, they can be used as qualitative guides to provide a starting point for design. The transient time response of the system is often checked to verify the time-domain specifications after control design.

10.2.3 Frequency-Domain Performance Specifications

System performance can also be specified in terms of frequency response. Figure 10.11 illustrates the ideal frequency response magnitude of a closed-loop control system. Two frequency-domain specifications are defined: bandwidth ω_{BW} and resonant peak M_r.

Assume that the transfer function of the closed-loop system is $Y(s)/R(s)$, where $Y(s)$ is the system output and $R(s)$ is the reference input. The bandwidth is defined as the frequency at which the magnitude of the closed-loop transfer function crosses –3 dB (or 0.707). Recall that the steady-state response of a linear system to sinusoidal excitations is called the system's frequency response. As shown in Figure 10.11, if the excitation frequency is lower than ω_{BW}, the magnitude $|Y(s)/R(s)|$ is close to 0 dB (or 1). This indicates that the system output follows the reference input. If the excitation frequency is higher than ω_{BW}, the magnitude $|Y(s)/R(s)|$ is reduced to a small value, and the system output no longer follows the reference input. The higher the bandwidth, the faster the reference signal the system can follow. Thus, the bandwidth is a measure of the speed of the response.

The resonant peak M_r was introduced previously in Section 8.3. It is defined as the maximum value of the magnitude of the frequency response. As shown in Section 8.3, the resonant peak is related to the damping of the system. The smaller the damping, the higher the resonant peak. Compared with the time-domain performance specifications, the resonant peak M_r is similar to overshoot M_p, both of which are related to the damping ratio ζ, whereas the bandwidth ω_{BW} is similar to the rise time t_r, both related to the natural frequency ω_n.

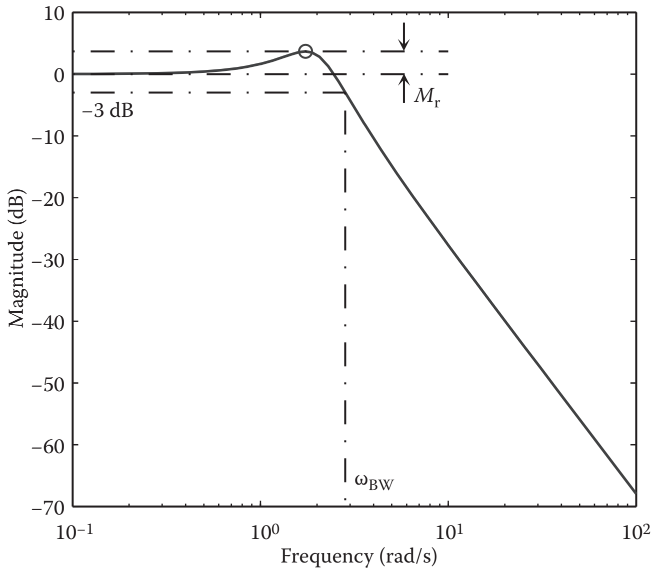


FIGURE 10.11 An ideal frequency response with performance specifications indicated.

PROBLEM SET 10.2

1. The transfer function of a dynamic system is given as

$$G(s) = \frac{s+1}{3s^5 + 6s^4 + 2s^3 + 5s^2 + s + 4}.$$

   a. Using Routh's stability criterion, determine the stability of the system.
   b. Using MATLAB, solve for the poles of the system and verify the result obtained in Part (a).

2. The transfer function of a dynamic system is given as

$$G(s) = \frac{20s+50}{s^3 + 10s^2 - 4s - 40}.$$

   a. Using Routh's stability criterion, determine the stability of the open-loop system.
   b. Suppose that a negative unity feedback is applied to this open-loop system. Using Routh's stability criterion, determine the stability of the resulting closed-loop system.
   c. Using MATLAB, solve for the poles of the open-loop and closed-loop systems and verify the results obtained in Parts (a) and (b).

3. The unit-step response for a second-order system $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$ is given by

$$y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right].$$

   Prove that the relationship between the overshoot M_p and the damping ratio is

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}.$$

4. Consider a second-order system $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, which has two poles at $-3 \pm 4j$.
   a. Determine the undamped natural frequency ω_n, damping ratio ζ, and damped natural frequency ω_d of the system.
   b. Estimate the rise time t_r, overshoot M_p, peak time t_p, and 2% settling time t_s in the unit-step response for the system.
5. Consider a second-order system $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$. Sketch the allowable region of the poles in the s-plane if the requirements for the system's unit-step response are $M_p \le 10\%$ and rise time $t_r \le 0.25$ s.
6. Consider a second-order system $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$. Sketch the allowable region of the poles in the s-plane if the requirements for the system's unit-step response are $M_p \le 15\%$, 2% settling time $t_s \le 2$ s, and rise time $t_r \le 0.4$ s.
7. Consider a second-order system $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$. Write a MATLAB m-file to plot the magnitude of the system's frequency response function for the following cases: $\omega_n = 2$ rad/s and $\zeta = 0.01$, 0.1, 0.5, and 1. Summarize the effects of the damping ratio on the frequency-domain performance.
8. Repeat Problem 7 for the following cases: $\zeta = 0.1$ and $\omega_n = 1$, 2, and 6 rad/s. Summarize the effects of the natural frequency on the frequency-domain performance.

10.3 BENEFITS OF FEEDBACK CONTROL

As introduced in Section 10.1, a closed-loop controller uses feedback to control the output of a plant. The input to a plant has an effect on its output, which is measured with a sensor and fed back to the controller, and then the computed control signal is used as the input to the plant, closing the loop. Feedback control has several advantages over open-loop control, such as stabilization, disturbance rejection, improved reference tracking performance, and reduced sensitivity to parameter variations. In this section, we show these benefits one by one using MATLAB or Simulink. The controller in each discussion and example is assumed to be a gain denoted by K.

10.3.1 Stabilization

Consider a plant represented by a transfer function $G(s) = b(s)/a(s)$, in which $a(s)$ and $b(s)$ are the denominator and numerator polynomials, respectively. Assume that $G(s)$ is unstable. This assumption implies that not all the poles, or the roots of the characteristic equation $a(s) = 0$, have negative real parts. If an open-loop control system is implemented as shown in Figure 10.12, in which the output is $Y(s)$ and the input is $R(s)$, then the transfer function is

$$\frac{Y(s)}{R(s)} = KG(s) = \frac{Kb(s)}{a(s)} . \tag{10.21}$$

It is obvious that the open-loop control system has the same poles as the plant $G(s)$, and thus it is still unstable.

Now, we use a negative feedback control system. Figure 10.13 shows the corresponding block diagram. The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1+KG(s)} = \frac{Kb(s)}{a(s)+Kb(s)} . \tag{10.22}$$

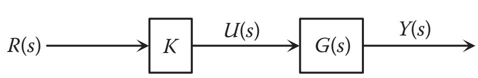


FIGURE 10.12 Open-loop control.

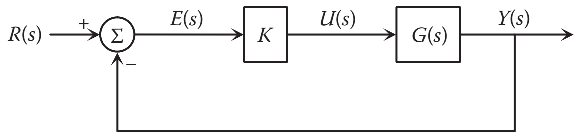


FIGURE 10.13 Closed-loop control.

The characteristic equation of the closed-loop system is therefore $a(s) + Kb(s) = 0$. Properly choosing the control gain K can possibly make the closed-loop system stable. Routh's stability criterion introduced in Section 10.2 is one way to determine the range of K such that the closed-loop control system is stable.

Example 10.5: Stabilization Using Feedback

Consider an unstable plant

$$G(s) = \frac{s+2}{s^3 + 4s^2 - 5s},$$

with feedback control as shown in Figure 10.13.

a. Using Routh's stability criterion, determine the range of the control gain K for which the closed-loop system is stable.
b. Use MATLAB commands to find the unit-step responses for open-loop and closed-loop control. Assuming that the control gain is $K = 20$, compare the open- and closed-loop responses.

Solution

a. Solving the characteristic equation of $G(s)$ gives the poles 0, 1, and −5. The positive real pole, 1, indicates that the plant $G(s)$ is unstable. The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K(s+2)/(s^3+4s^2-5s)}{1+K(s+2)/(s^3+4s^2-5s)} = \frac{Ks+2K}{s^3+4s^2+(K-5)s+2K}.$$

The closed-loop characteristic equation is

$$s^3 + 4s^2 + (K-5)s + 2K = 0,$$

for which we can construct the Routh array as follows:

$$\begin{array}{lll} s^3: & 1 & K-5 \\ s^2: & 4 & 2K \\ s^1: & \dfrac{K}{2} - 5 = \dfrac{4(K-5)-2K}{4} & 0 \\ s^0: & 2K & \end{array}.$$

To make the closed-loop system stable, all elements in the first column of the Routh array must be positive. Therefore,

$$\begin{cases} \dfrac{K}{2} - 5 > 0, \\ 2K > 0, \end{cases}$$

which leads to $K > 10$.

b. For the open-loop control system shown in Figure 10.12, the controller block is connected with the plant in series. The following is the MATLAB session:

```
>> G = tf([1 2],[1 4 -5 0]);
>> K = 20;
>> olp = K*G;
>> step(olp);
```

Figure 10.14 is the unit-step response of the open-loop control system, which is obviously unstable because the response diverges. Therefore, the unstable plant cannot be stabilized using open-loop control.

As introduced in Section 4.5, the MATLAB command `feedback` can be used to find the transfer function of the closed-loop system for the feedback control system in Figure 10.13.

```
>> clp = feedback(K*G,1);
>> step(clp);
```

As shown in Figure 10.15, the unit-step response of the closed-loop control system converges and the unstable plant is stabilized.

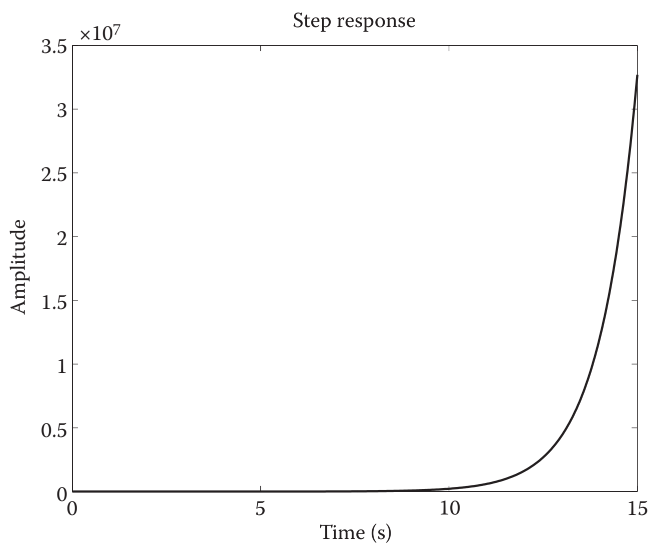


FIGURE 10.14 Unit-step response of the open-loop control system in Example 10.5.

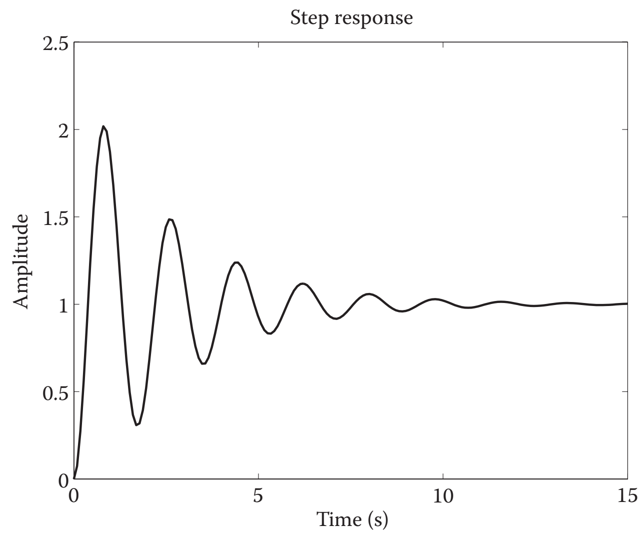


FIGURE 10.15 Unit-step response of the closed-loop control system in Example 10.5.

10.3.2 Disturbance Rejection

To compare the capabilities of the open-loop and the closed-loop control for disturbance rejection, assume that a disturbance is an input to the plant $G(s)$. Figure 10.16 is the block diagram for the open-loop control, where

$$Y(s) = G(s)U(s) + G(s)D(s) = KG(s)R(s) + G(s)D(s). \quad (10.23)$$

Equation 10.23 shows that the output $Y(s)$ depends on the reference input $R(s)$ and the disturbance input $D(s)$. Letting $R(s) = 0$ yields the transfer function relating the disturbance $D(s)$ and the output $Y(s)$,

$$\frac{Y(s)}{D(s)} = G(s). \quad (10.24)$$

Note that K does not appear in Equation 10.24 and it has no control over the disturbance in the open-loop case.

Using closed-loop control as shown in Figure 10.17, we have

$$Y(s) = G(s)U(s) + G(s)D(s) = KG(s)[R(s) - Y(s)] + G(s)D(s). \quad (10.25)$$

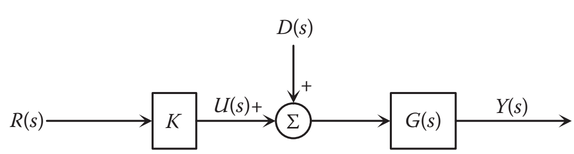


FIGURE 10.16 Open-loop control with disturbance input.

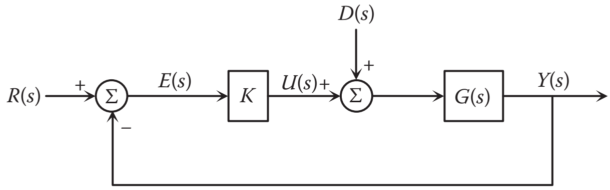


FIGURE 10.17 Closed-loop control with disturbance input.

Solving for $Y(s)$ gives

$$Y(s) = \frac{KG(s)}{1+KG(s)}R(s) + \frac{G(s)}{1+KG(s)}D(s). \tag{10.26}$$

The transfer function relating the disturbance $D(s)$ and the output $Y(s)$ is

$$\frac{Y(s)}{D(s)} = \frac{G(s)}{1+KG(s)}, \tag{10.27}$$

which indicates that K in the closed-loop case has control over the disturbance.

Example 10.6: Disturbance Rejection Using Feedback

Consider a plant whose transfer function is

$$G(s) = \frac{4}{s+10}.$$

a. Build a Simulink block diagram associated with Figure 10.16 to simulate open-loop control with disturbance input. Assume that the controller is $K = 2.5$, the disturbance is a constant of −1, and the reference is a unit-step function with the step time at $t = 0$ s. Compare the steady-state values of the responses without and with the disturbance.
b. Build a Simulink block diagram associated with Figure 10.17 to simulate closed-loop control with disturbance input. Assume that the controller is $K = 50$. The disturbance and the reference are the same as those in Part (a). Compare the steady-state values of the responses without and with the disturbance.

Solution

a. The Simulink block diagram of the open-loop control is shown in Figure 10.18, in which the `Constant` block is used to represent the disturbance signal. We first set the

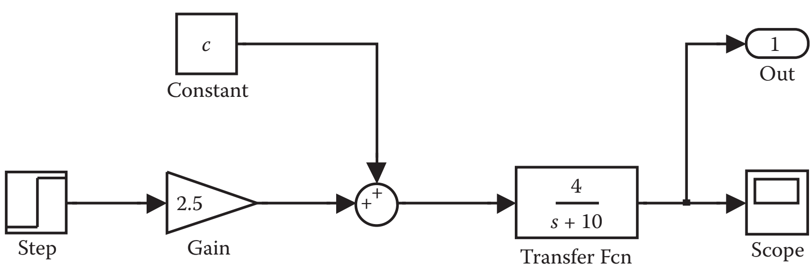


FIGURE 10.18 Simulink block diagram of open-loop control with disturbance input.

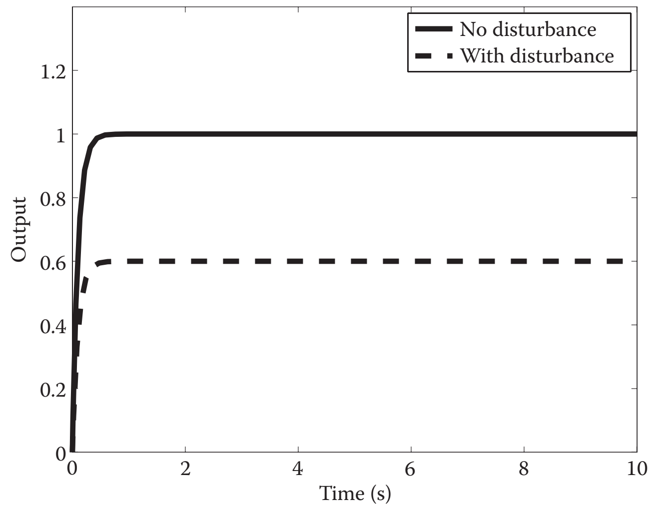


FIGURE 10.19 Unit-step responses of open-loop control without and with disturbance input.

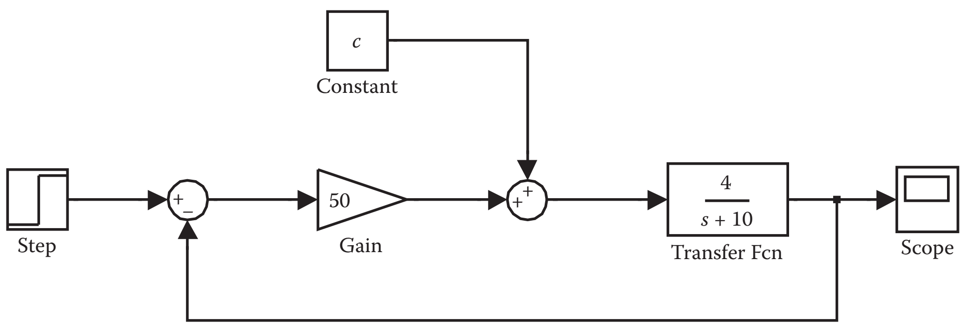


FIGURE 10.20 Simulink block diagram of closed-loop control with disturbance input.

`Constant` block to be 0 and run the simulation. Then, we change the `Constant` block to be −1 and rerun the simulation. The responses for those two cases are plotted in Figure 10.19. The steady-state value of the unit-step response is 1 without disturbance and 0.6 with disturbance.

b. The Simulink block diagram of the closed-loop control is shown in Figure 10.20. The responses are plotted in Figure 10.21. The steady-state value of the unit-step response is 0.95 without disturbance and 0.93 with disturbance.

Although the closed-loop control is not as good as the open-loop control in the absence of disturbance, the error resulting from a constant disturbance can be made smaller in a closed-loop feedback system compared with an open-loop system.

10.3.3 Reference Tracking

As stated in Section 10.1, there are two types of control systems: regulators and servos. The former are designed for disturbance rejection and the latter are designed for reference tracking. With

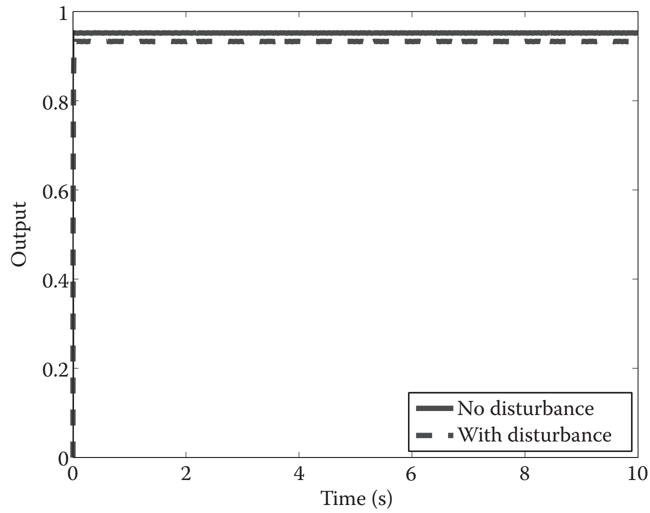


FIGURE 10.21 Unit-step responses of closed-loop control without and with disturbance input.

feedback, a control system can achieve improved reference tracking performance. To show this, let us consider the closed-loop control case in Figure 10.17, in which the disturbance $D(s)$ is now set as zero. As a result, the closed-loop transfer function of the system is

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)}. \tag{10.28}$$

A typical frequency response plot of the closed-loop system was given in Figure 10.11. As we discussed in Section 10.2, the output follows the reference input when $|Y(s)/R(s)| \approx 1$. Equation 10.28 implies that $Y(s)$ is approximately equal to $R(s)$ if the magnitude $|KG(s)| \gg 1$. In general, this can be achieved by increasing the value of the control gain K. Thus, a large control gain can reduce the steady-state error of the response.

Example 10.7: Reference Tracking Using Feedback

Consider the closed-loop control system shown in Figure 10.22.

a. Using MATLAB, plot the unit-step responses of the system for the following values of K: 5, 50, and 500.
b. Compute the steady-state errors for the different values of K.

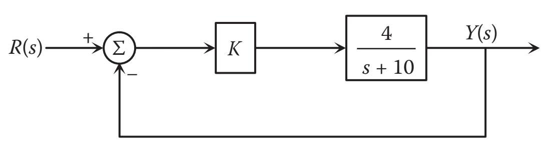


FIGURE 10.22 A reference tracking control system with feedback.

Solution

a. The following is the MATLAB script. The unit-step responses of the system for the three different values of the control gain are shown in Figure 10.23. It is observed that the steady-state error becomes smaller as the control gain increases.

```
K = [5 50 500];
G = tf([4],[1 10]);
for i = 1:length(K)
    clp = feedback(K(i)*G,1);
    step(clp);
    hold on;
end
```

b. The value of the steady-state error can be computed by applying the final-value theorem. The closed-loop transfer function is

$$\frac{E(s)}{R(s)} = \frac{1}{1+KG(s)}.$$

For a unit-step input, we have

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)\frac{E(s)}{R(s)} = \lim_{s\to 0} s\frac{1}{s}\frac{1}{1+KG(s)} = \lim_{s\to 0}\frac{s+10}{s+10+4K} = \frac{10}{10+4K}.$$

The steady-state errors for $K = 5$, 50, and 500 are 0.3333, 0.0476, and 0.005, respectively. It should be pointed out that a large control gain may also result in unsatisfactory transient response and even destabilize the system. Details will be discussed in Section 10.4.

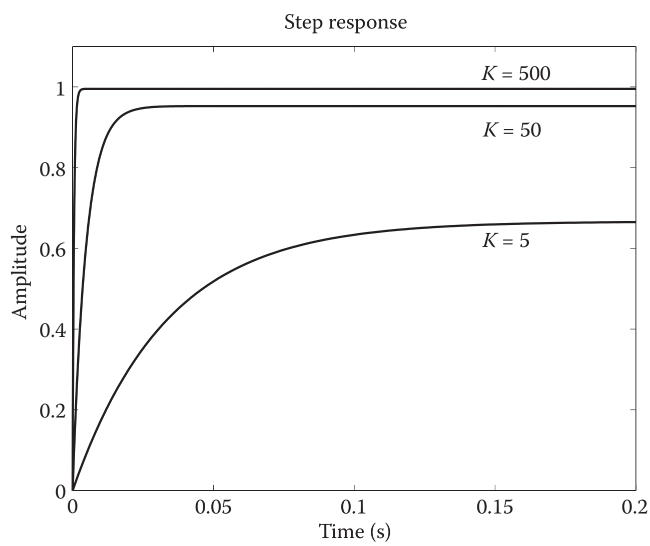


FIGURE 10.23 Unit-step responses for the system in Figure 10.22 with different values of K.

10.3.4 Sensitivity to Parameter Variations

In model-based control, a controller implemented for a practical system is designed based on a mathematical model of the system. Therefore, it is important to obtain a precise model of the plant for control design. However, modeling errors do exist due to uncertainties in system parameters. For example, the dynamic behavior of a mass–spring system depends on the values of the mass and stiffness. The values used for modeling might be different from the actual values due to inevitable measurement errors from the start or slight parameter changes caused by external effects. To maintain control performance, a controller should be insensitive to parameter changes.

Consider a function f that depends on a parameter a. Denote the change in the parameter as δa. If the change in f because of the parameter change is δf, then the sensitivity of the function f with respect to the parameter a is defined as

$$S_a^f = \frac{\delta f/f}{\delta a/a}, \tag{10.29}$$

where the first-order variation δf is proportional to the derivative $\mathrm{d}f/\mathrm{d}a$ and is given by

$$\delta f = \frac{\mathrm{d}f}{\mathrm{d}a}\delta a. \tag{10.30}$$

Thus, the sensitivity S_a^f can be written as

$$S_a^f = \frac{a}{f}\frac{\mathrm{d}f}{\mathrm{d}a}. \tag{10.31}$$

To compare the sensitivity of open-loop control with that of closed-loop control, we assume that one or more parameters in the plant change. Without loss of generality, the disturbance is assumed to be zero. For the open-loop control shown in Figure 10.16, denote the transfer function from the reference signal to the system output as $T_o(s)$, which is $KG(s)$. By Equation 10.31, the sensitivity of the open-loop control system $T_o(s)$ to the plant $G(s)$ is

$$S_G^{T_o} = \frac{G}{T_o}\frac{\mathrm{d}T_o}{\mathrm{d}G} = \frac{G}{KG}\frac{\mathrm{d}(KG)}{\mathrm{d}G} = \frac{G}{KG}\frac{K\,\mathrm{d}G}{\mathrm{d}G} = 1, \tag{10.32}$$

which implies that open-loop control is very sensitive to the parameter variations in the plant. For example, a 5% error in the plant would yield the same percentage error in the open-loop transfer function.

For the closed-loop control case as shown in Figure 10.17, in which the disturbance is still set as zero, denote the closed-loop transfer function as $T_c(s)$. Applying Equation 10.31 gives

$$S_G^{T_c} = \frac{G}{T_c}\frac{\mathrm{d}T_c}{\mathrm{d}G}. \tag{10.33}$$

The closed-loop transfer function $T_c(s)$ is given by Equation 10.28. Thus, we have

$$\frac{\mathrm{d}T_c}{\mathrm{d}G} = \frac{\mathrm{d}}{\mathrm{d}G}\left(\frac{KG}{1+KG}\right) = \frac{K(1+KG)-KG(K)}{(1+KG)^2} = \frac{K}{(1+KG)^2}. \tag{10.34}$$

Substituting it into Equation 10.33 yields

$$S_G^{T_c} = \frac{G(1+KG)}{KG}\frac{K}{(1+KG)^2} = \frac{1}{1+KG}, \tag{10.35}$$

which can be made much less than 1 by adjusting the controller gain K. Thus, the overall transfer function in feedback control is less sensitive to variations in the plant gain compared with the one in open-loop control.

Example 10.8: Sensitivity to Parameter Variations

Consider a mass–damper–spring system $G(s) = Y(s)/U(s) = 1/(ms^2 + bs + k)$, where $m = 1$ kg, $b = 8$ N · s/m, and $k = 40$ N/m.

a. Assume that the system is controlled in an open-loop control system with a controller $K = 40$. Determine the steady-state value of the response to a unit-step input. If the spring stiffness is actually 50 N/m, recalculate the steady-state value of the response and determine the fractional change in the steady-state value.
b. Repeat Part (a) assuming that the mass–damper–spring system is controlled in a feedback control system with a controller $K = 2000$.

Solution

a. In the open-loop case, the steady-state value of the response to a unit-step input is

$$y_{ss} = \lim_{s\to 0} sR(s)\frac{Y(s)}{R(s)} = \lim_{s\to 0} s\frac{1}{s}KG(s) = \lim_{s\to 0}\frac{40}{s^2+8s+40} = 1.$$

If the spring stiffness is 50 N/m, then the steady-state value becomes

$$y_{ss} = \lim_{s\to 0} sR(s)\frac{Y(s)}{R(s)} = \lim_{s\to 0} s\frac{1}{s}KG(s) = \lim_{s\to 0}\frac{40}{s^2+8s+50} = 0.8.$$

Note that y_{ss} is reduced by 20%, which is the same as the relative error in k.
b. In the closed-loop case, the steady-state value of the response to a unit-step input is

$$y_{ss} = \lim_{s\to 0} sR(s)\frac{Y(s)}{R(s)} = \lim_{s\to 0} s\frac{1}{s}\frac{KG(s)}{1+KG(s)} = \lim_{s\to 0}\frac{2000}{s^2+8s+2040} = 0.9804.$$

If the spring stiffness is 50 N/m, then the steady-state value becomes

$$y_{ss} = \lim_{s\to 0} sR(s)\frac{Y(s)}{R(s)} = \lim_{s\to 0} s\frac{1}{s}\frac{KG(s)}{1+KG(s)} = \lim_{s\to 0}\frac{2000}{s^2+8s+2050} = 0.9756.$$

Note that y_{ss} is reduced by 0.49% only, which is much less than the relative error in k. Although the closed-loop control is not as good as the open-loop control in the absence of modeling error, the tracking error resulting from parameter variations can be made smaller in a closed-loop feedback system compared with an open-loop system.

PROBLEM SET 10.3

1. Consider the feedback system shown in Figure 10.24.
   a. Using Routh's stability criterion, determine the range of the control gain K for which the closed-loop system is stable.
   b. Use MATLAB commands to find the unit-step responses for open-loop and closed-loop control. Assume that the control gain is $K = 50$. Compare the open-loop and closed-loop responses.
2. Consider the feedback system shown in Figure 10.25. Using Routh's stability criterion, determine the range of the control gain K for which the closed-loop system is stable.
3. Reconsider Example 10.6. Using the final-value theorem, verify the steady-state errors to a unit-step input for open-loop and closed-loop control without and with disturbance.
4. A stable system can be classified by a system type, which is defined as the degree of the polynomial for which the steady-state error is a nonzero finite constant. For instance, if the error to a step input, which is a polynomial of zero degree, is a nonzero finite constant, then such a system is called type 0, and so on. Consider the system in Figure 10.26.
   a. Compute the steady-state errors for: (1) a unit-step reference input, (2) a unit-ramp reference input $u = t$, and (3) a parabolic reference input $u = 0.5t^2$.
   b. Determine the system type.
5. Reconsider Example 10.8. Build Simulink block diagrams to simulate open-loop and closed-loop controls with parameter variations. Verify the steady-state response values y_{ss} obtained in Example 10.8.
6. Consider the feedback control system shown in Figure 10.27.
   a. Compute the sensitivity of the closed-loop transfer function to changes in the parameter τ.

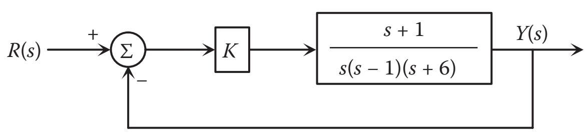


FIGURE 10.24 Problem 1.

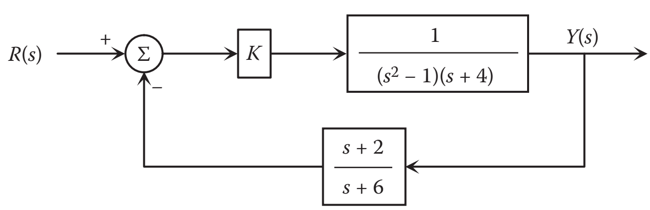


FIGURE 10.25 Problem 2.

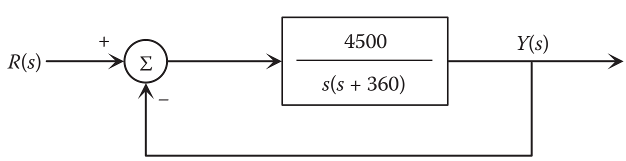


FIGURE 10.26 Problem 4.

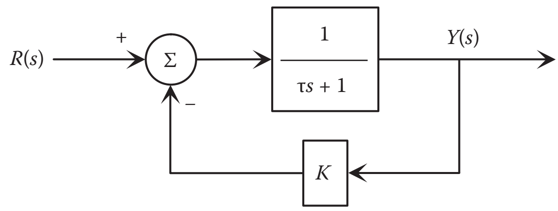


FIGURE 10.27 Problem 6.

b. Compute the sensitivity of the closed-loop transfer function to changes in the parameter K.
c. Assuming $\tau = 1$ and $K = 1$, use MATLAB to plot the magnitude of each of the sensitivity functions for $s = j\omega$. Use the logarithmic scale for the y-axis. Comment on the effect of parameter variations in τ and K for different driving frequencies ω.

10.4 PROPORTIONAL–INTEGRAL–DERIVATIVE CONTROL

A proportional–integral–derivative (PID) controller is a generic feedback control structure widely used in industries. The PID controller involves three terms. The proportional term determines the control signal based on the current error, the integral term determines the control signal based on the integral of error, and the derivative term determines the control signal based on the derivative of error. The expression of the PID controller in the time domain is given by

$$u(t) = k_\mathrm{p} e(t) + k_\mathrm{I} \int_0^t e(\tau)\,\mathrm{d}\tau + k_\mathrm{D} \frac{\mathrm{d}e(t)}{\mathrm{d}t}, \tag{10.36}$$

where k_p is the proportional control gain, k_I is the integral control gain, and k_D is the derivative control gain. Taking the Laplace transform of Equation 10.36 yields the transfer function of the PID controller

$$\frac{U(s)}{E(s)} = k_\mathrm{p} + \frac{k_\mathrm{I}}{s} + k_\mathrm{D} s. \tag{10.37}$$

Figure 10.28 shows a block diagram of feedback control using a PID controller denoted by $C(s)$. In this section, we discuss the advantages and disadvantages of PID control.

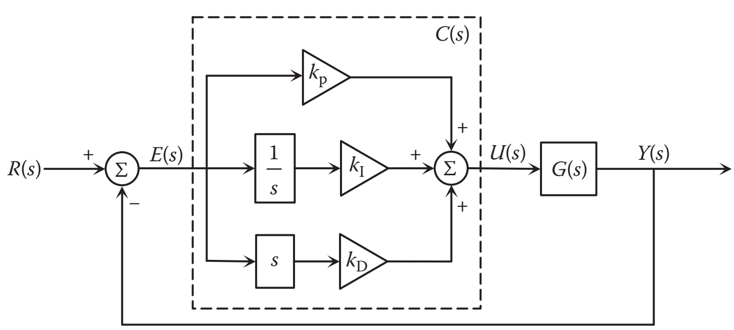


FIGURE 10.28 A block diagram of PID feedback control.

10.4.1 Proportional Control

For proportional feedback control, the controller transfer function is

$$\frac{U(s)}{E(s)} = k_p. \tag{10.38}$$

Note that the controller structures in the previous sections are the most basic proportional feedback control. As discussed in Section 10.3, a high proportional control gain can result in smaller steady-state error. However, if k_p is made too large, the closed-loop system may experience damping reduction and even become unstable.

Example 10.9: Proportional Control

Consider the mass–damper–spring system in Example 10.8,

$$G(s) = \frac{1}{s^2 + 8s + 40}.$$

Use Simulink to build a block diagram for proportional feedback control. Find the unit-step responses for k_p = 25, 250, and 2000. Discuss the effects of the proportional feedback on the unit-step response.

Solution

The Simulink block diagram is shown in Figure 10.29. Note that Figure 10.29 also includes integral and derivative control loops, which will be used later in this section. Set k_I and k_D to be zero. Run the simulation for k_p = 25, 250, and 2000.

Figure 10.30 illustrates the effects of proportional feedback control. On the one hand, as k_p increases, the steady-state error to the unit-step input becomes smaller. Applying the final-value theorem gives

$$e_{ss} = \lim_{s\to 0} sR(s)\frac{E(s)}{R(s)} = \lim_{s\to 0} s\frac{1}{s}\frac{1}{1+k_pG(s)} = \frac{40}{40+k_p}.$$

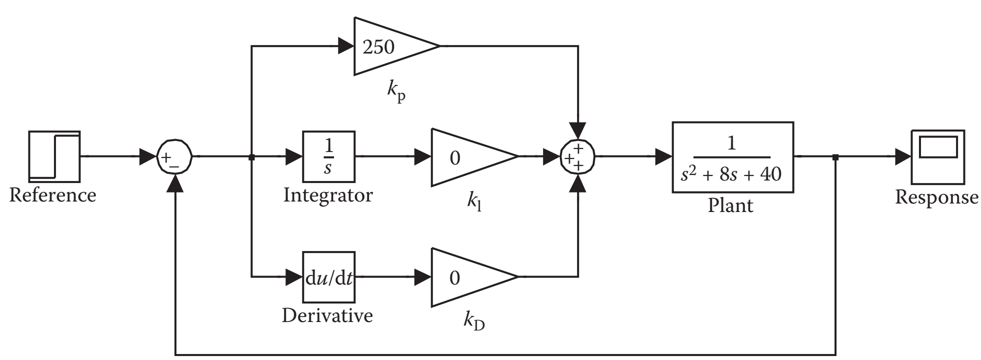


FIGURE 10.29 Simulink block diagram for PID feedback control.

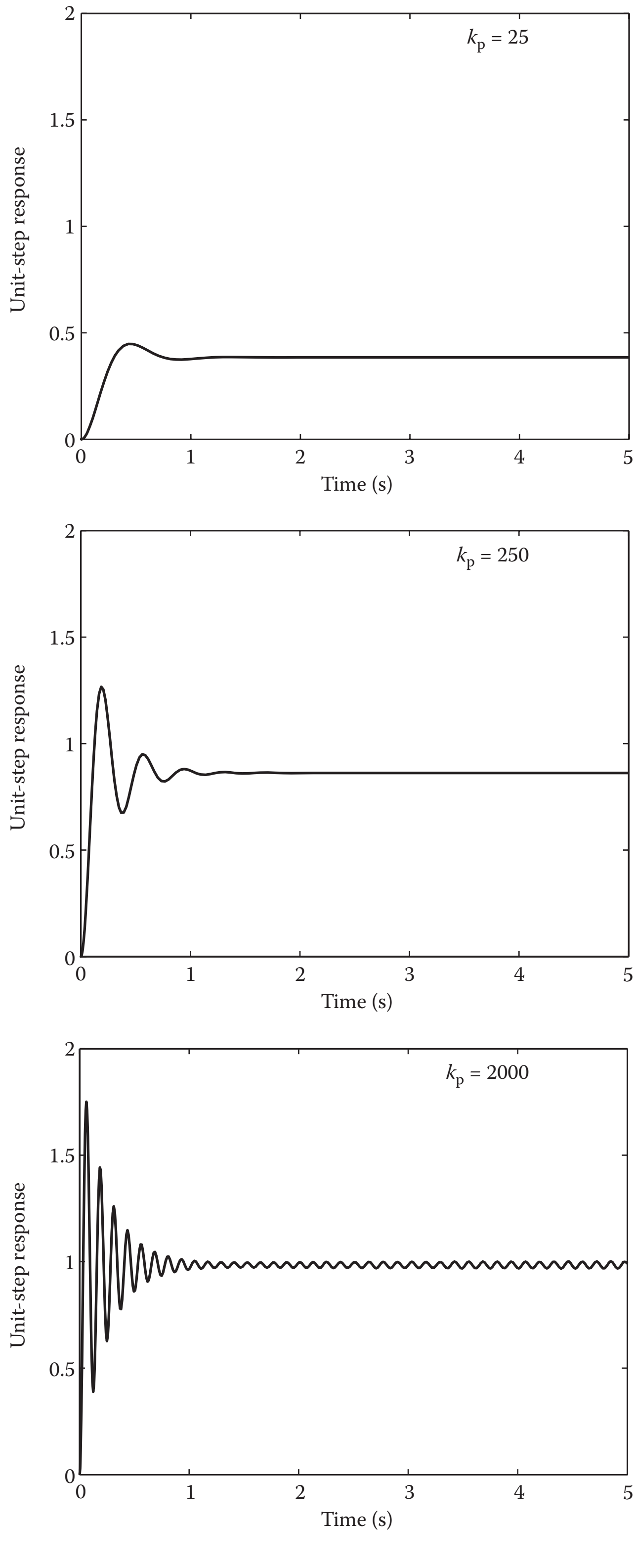


FIGURE 10.30 Responses of proportional control.

Substituting the three different proportional control gains yields e_{ss} = 0.62, 0.14, and 0.02, and k_p = 25, 250, and 2000, respectively. For this particular system, the steady-state error will approach zero as k_p increases. However, it will never be zero.

On the other hand, the larger proportional gain results in less satisfactory oscillatory response. This is caused by reduced damping. Note that the characteristic equation of the closed-loop system with proportional control is

$$s^2 + 8s + 40 + k_p = 0.$$

For a second-order system, the coefficients in the characteristic equation are related to the natural frequency and the damping ratio of the system, that is,

$$8 = 2\zeta\omega_n, \quad 40 + k_p = \omega_n^2.$$

Obviously, if k_p is made large, the natural frequency is increased. However, the damping ratio becomes smaller. This leads to a faster response, but with a bigger overshoot and much more severe oscillations.

10.4.2 Proportional–Integral Control

As seen in Example 10.9, increasing the proportional gain k_p can reduce steady-state error, but cannot achieve zero steady-state error. Adding an integral term to the controller in Equation 10.38 results in a proportional–integral (PI) controller

$$\frac{U(s)}{E(s)} = k_p + \frac{k_I}{s}. \tag{10.39}$$

If PI control is used in the previous example, the steady-state error becomes

$$e_{ss} = \lim_{s\to 0} s \frac{1}{s} \frac{1}{1 + (k_p + k_I/s)G(s)} = \lim_{s\to 0} \frac{s}{s + (k_p s + k_I)G(s)}. \tag{10.40}$$

Substituting the transfer function $G(s)$ given in Example 10.9, we can obtain the result by evaluating the limit in Equation 10.40. The system with PI control has a zero steady-state error. Thus, the primary reason to introduce the integral control is to reduce, or possibly eliminate, the steady-state error.

Example 10.10: PI Control

Use the Simulink block diagram built in Example 10.9 to find the unit-step responses for k_I = 50, 500, and 1550. Set k_p = 250 and k_D = 0. Discuss the effects of the integral term on the unit-step response.

Solution

Figure 10.31 shows the unit-step responses of PI feedback control for k_I = 50, 500, and 1550. The system response will achieve zero steady-state error for each case if the simulation time is long enough. However, a large integral control gain results in lightly damped oscillations.

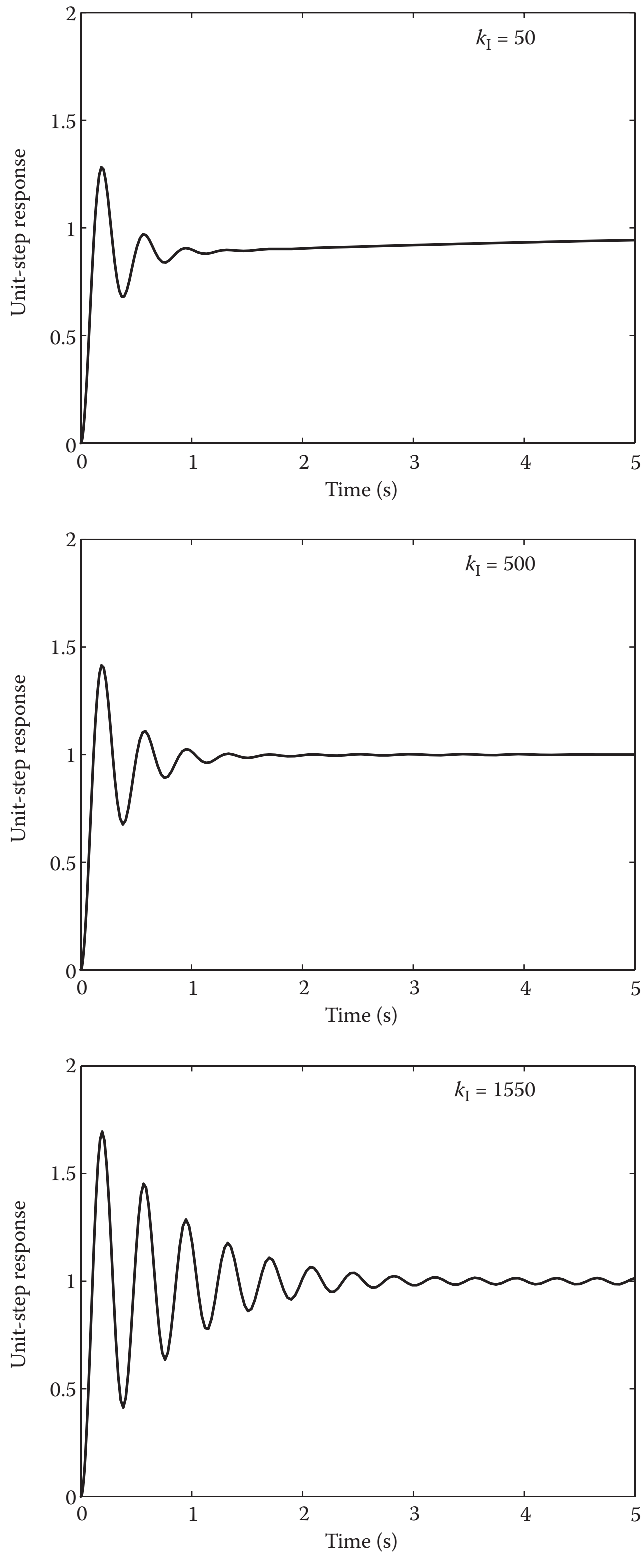


FIGURE 10.31 Responses of PI control.

10.4.3 PID Control

The final term in a PID controller represents the derivative control. The complete three-term controller is given by Equation 10.37. The main reason to introduce the derivative control is to increase the damping and thus to improve the stability of the system.

Example 10.11: PID Control

Use the Simulink block diagram built in Example 10.9 to find the unit-step responses for k_D = 1, 10, and 50. Set k_p = 250 and k_I = 500. Discuss the effects of the derivative term on the unit-step response.

Solution

Figure 10.32 shows the unit-step responses of PID feedback control for k_D = 1, 10, and 50. As k_D increases, the overshoot of the unit-step response becomes smaller. This implies that the damping of the feedback control system becomes larger. However, a large derivative control gain k_D leads to a slower response.

In summary, a larger proportional gain k_p results in a faster response and a smaller steady-state error. However, an excessively large proportional gain k_p leads to lightly damped oscillations and even instability. A larger integral gain k_I eliminates steady-state errors more quickly, but reduces damping and leads to a larger overshoot. A larger derivative control decreases the overshoot, but slows down the speed of the response.

If a PID feedback control system is second-order, then any two free control gains (among k_p, k_I, and k_D) can be determined based on the system stability and performance requirements.

Example 10.12: Proportional–Derivative Control of a DC Motor–Driven Cart

Consider the feedback control system shown in Figure 10.33, in which the plant is the DC motor–driven cart given in Example 10.2. The input to the plant is the voltage applied to the DC motor, and the output is the position of the cart. Design a proportional–derivative (PD) controller such that the maximum overshoot in the response to a unit-step reference input is less than 10%, and the rise time is less than 0.15 s.

Solution

The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1+C(s)G(s)} = \frac{(k_p + k_D s)(3.778/(s^2+16.883s))}{1+(k_p+k_D s)(3.778/(s^2+16.883s))} = \frac{3.778k_D s + 3.778k_p}{s^2+(16.883+3.778k_D)s+3.778k_p},$$

which is a second-order system. The closed-loop characteristic equation is

$$s^2 + (16.883 + 3.778k_D)s + 3.778k_p = 0,$$

where the coefficients are related to the natural frequency and the damping ratio of the closed-loop system via

$$16.883 + 3.778k_D = 2\zeta\omega_n,$$

$$3.778k_p = \omega_n^2.$$

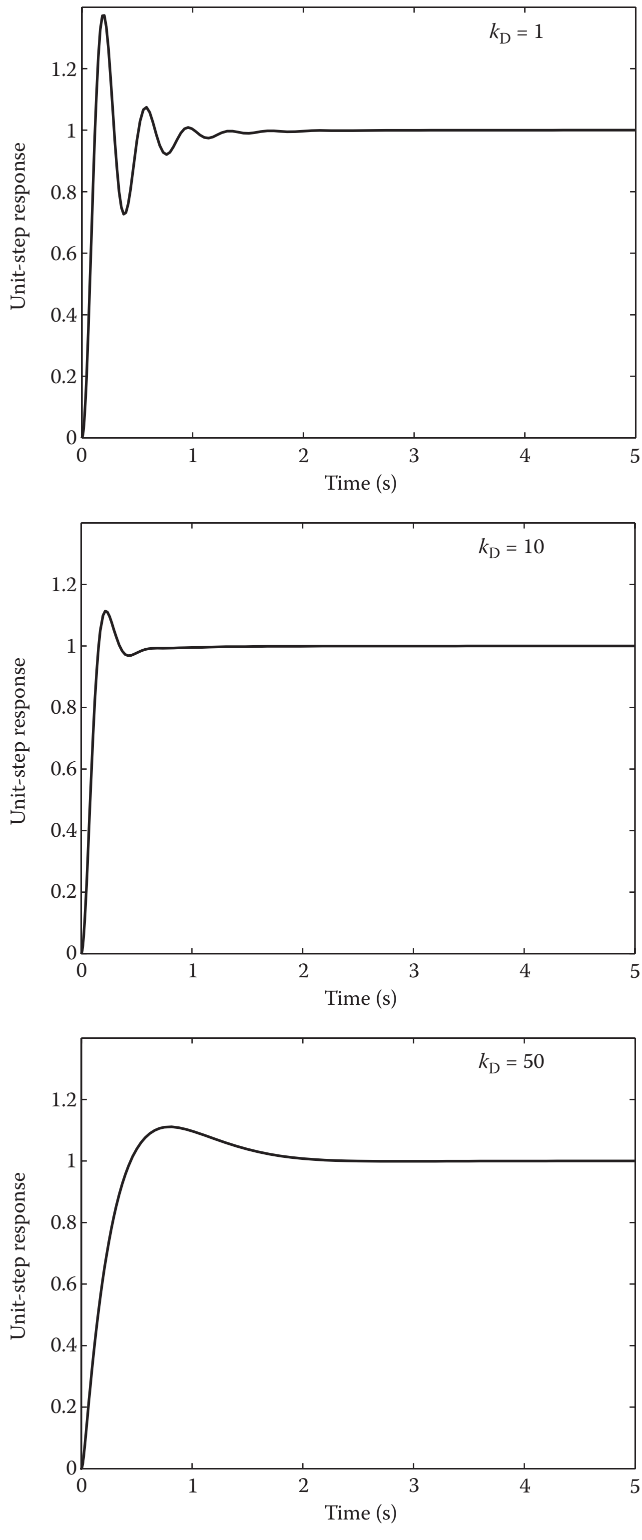


FIGURE 10.32 Responses of PID control.

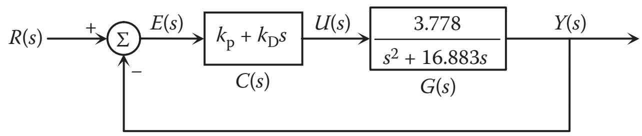


FIGURE 10.33 Block diagram of a feedback control system.

Note that there are two requirements for the transient response of the closed-loop system, that is, $M_P < 10\%$ and $t_r < 0.15$ s. To satisfy these two requirements, a set of reasonable values of ω_n and ζ can be approximated using the relationships given in Section 10.2.

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} < 10\%,$$

$$t_r \approx \frac{1.12 - 0.078\zeta + 2.230\zeta^2}{\omega_n} < 0.15\text{ s}.$$

The requirement for overshoot yields

$$\zeta > 0.59.$$

Letting $\zeta = 0.59$ and substituting it into the requirement for rise time gives

$$\omega_n > 12.33\text{ rad/s}.$$

This region is shown in Figure 10.34, which can be used as a starting point for control design. Note that damping slows the motion of the system. Thus, for a damping ratio higher than 0.59, the critical value of natural frequency should be higher than 12.33 rad/s to speed up the motion of the system. If the closed-loop poles are located to the left of the gray boundary in Figure 10.34, then the closed-loop response to a unit-step reference input will very likely meet the desired requirements.

Choosing $\zeta = 0.65$ and $\omega_n = 13.5$ rad/s yields $k_p = 48.24$ and $k_D = 0.18$. Figure 10.35 shows the response of the closed-loop system to a unit-step reference input.

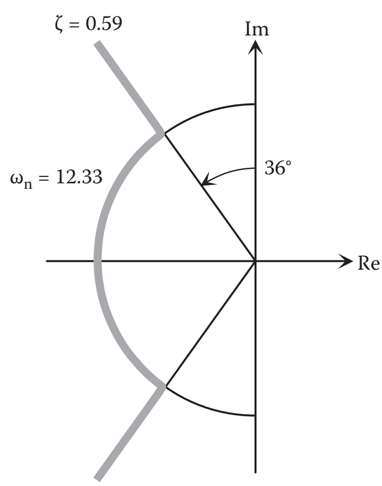


FIGURE 10.34 Allowable region of the closed-loop poles in the s-plane.

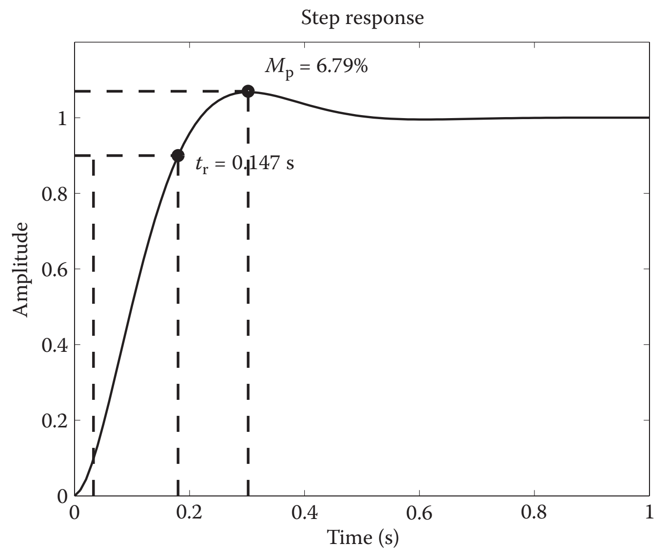


FIGURE 10.35 Unit-step response of the cart system with PD control $C(s) = 48.24 + 0.18s$.

10.4.4 Ziegler–Nichols Tuning of PID Controllers

As mentioned earlier, all the methods considered in this chapter are model-based control, which requires that a dynamic model of the plant be available. To avoid this requirement, in the early 1940s, Ziegler and Nichols conducted numerous experiments and proposed two useful tuning methods for determining the PID control gains. The form of the PID controller used by Ziegler and Nichols is given by

$$C(s) = k_p\left(1 + \frac{1}{T_I s} + T_D s\right), \tag{10.41}$$

where T_I is the integral time and T_D is the derivative time. The gain parameters T_I and T_D are related to the parameters k_p, k_I, and k_D through $k_I = k_p/T_I$ and $k_D = k_pT_D$.

For the first method, known as the reaction curve method, a step response of the plant is measured. As shown in Figure 10.36, the *S*-shaped curve is characterized by two constants, lag time *L*

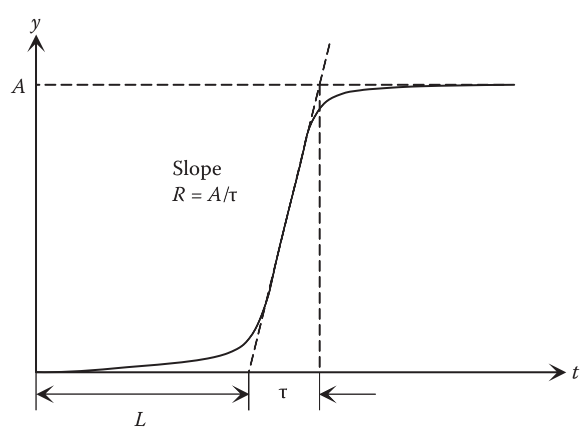


FIGURE 10.36 Reaction curve method.

and reaction rate R, which are determined by drawing a line tangent to the curve and finding the intersections of the tangent line with the time axis and the steady-state level line. The values of k_p, T_I, and T_D can be set using the parameters L and R according to the rules in Table 10.1.

For the second method, known as the ultimate sensitivity method, the frequency of the oscillations of the plant at the limit of stability is measured. To use this method, a proportional feedback control is applied to the plant and the proportional gain is increased until the closed-loop system becomes marginally stable. The corresponding proportional gain is defined as K_u, also called the ultimate gain. Figure 10.37 shows the response of a marginally stable system. It is a pure harmonic response, in which the period of oscillation P_u can be measured, also known as the ultimate period. The values of k_p, T_I, and T_D can then be set using the parameters K_u and P_u according to the rules in Table 10.2.

TABLE 10.1
Ziegler–Nichols Tuning Based on Reaction Curve

Type of Controller	Optimum Gains
P	$k_p = 1/(RL)$
PI	$k_p = 0.9/(RL)$, $T_I = L/0.3$
PID	$k_p = 1.2/(RL)$, $T_I = 2L$, $T_D = 0.5L$

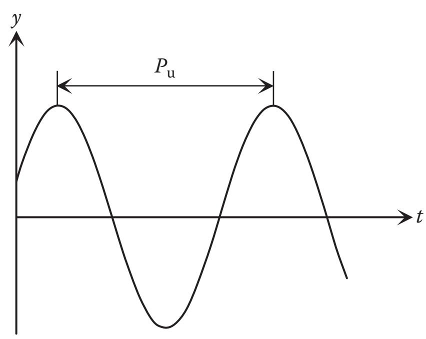


FIGURE 10.37 Response of a marginally stable system.

TABLE 10.2
Ziegler–Nichols Tuning Based on Ultimate Gain and Ultimate Period

Type of Controller	Optimum Gains
P	$k_p = 0.5K_u$
PI	$k_p = 0.45K_u$, $T_I = P_u/1.2$
PID	$k_p = 0.6K_u$, $T_I = P_u/2$, $T_D = P_u/8$

PROBLEM SET 10.4

1. Figure 10.38 shows a negative feedback control system.
   a. Design a P controller such that the damping ratio of the closed-loop system is 0.7.
   b. Estimate the rise time, overshoot, and 1% settling time in the unit-step response for the closed-loop system.
2. Consider the negative feedback control system shown in Figure 10.39.
   a. Design a P controller such that the maximum overshoot in the response to a unit-step reference input is less than 15%, the 2% settling time is less than 1 s, and the rise time is less than 0.2 s.
   b. Use MATLAB to plot the unit-step response of the closed-loop system. Find the overshoot, 2% settling time, and rise time. If the time-domain specifications exceed the requirements, fine-tune and reduce them to be approximately the specified values or less.
3. Consider the feedback control system shown in Figure 10.40.
   a. Design a PD controller such that the closed-loop poles are at $p_{1,2} = -3 \pm 3j$.
   b. Use MATLAB to plot the unit-step response of the closed-loop system. Find the rise time, overshoot, peak time, and 1% settling time.
4. Consider the feedback control system shown in Figure 10.41.
   a. Find the values for k_p and T_D such that the maximum overshoot in the response to a unit-step reference input is less than 10% and the 1% settling time is less than 0.5 s.

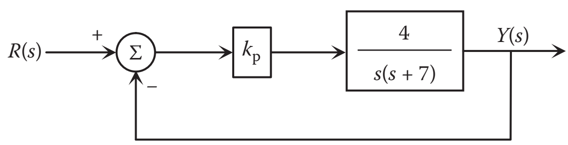


FIGURE 10.38 Problem 1.

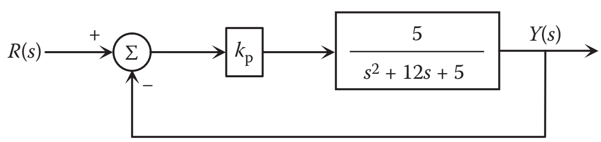


FIGURE 10.39 Problem 2.

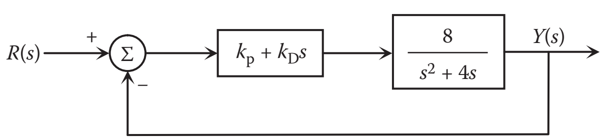


FIGURE 10.40 Problem 3.

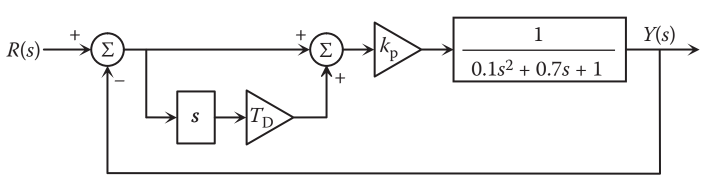


FIGURE 10.41 Problem 4.

b. Compute the steady-state error of the closed-loop system to a unit-step reference input.
c. Verify the results of Parts (a) and (b) using MATLAB by plotting the unit-step response of the closed-loop system. If the maximum overshoot and the settling time exceed the requirements, fine-tune and reduce them to be approximately the specified values or less.

5. Consider the feedback control system shown in Figure 10.42a.
   a. If the desired closed-loop poles are located within the shaded regions shown in Figure 10.42b, determine the corresponding ranges of ω_n and ζ of the closed-loop system.
   b. Design a PI controller such that the closed-loop poles are at $p_{1,2} = -9 \pm 12j$.
   c. Compute the steady-state errors of the plant and the closed-loop system to a unit-step reference input.
   d. Verify the results of Part (c) using MATLAB by plotting the unit-step responses of the plant and the closed-loop system.
6. Consider the feedback control system shown in Figure 10.43.
   a. Find the values for k_p and T_I such that the maximum overshoot in the response to a unit-step reference input is less than 15% and the peak time is less than 0.25 s.
   b. Verify the results in Part (a) using MATLAB by plotting the unit-step response of the closed-loop system. If the maximum overshoot and the peak time exceed the requirements, fine-tune and reduce them to be approximately the specified values or less.
7. The unit-step response of a plant is shown in Figure 10.44.
   a. The lag time L and the reaction rate R can be determined from the figure. Find the P, PI, and PID controller parameters using the Ziegler–Nichols reaction curve method.
   b. Assume that the transfer function of the plant is $3/(10s^2 + 8s + 1)$. Use MATLAB to plot the unit-step response of the closed-loop system with P, PI, or PID control.
8. Consider a proportional feedback control system. As shown in Figure 10.45, the closed-loop system becomes marginally stable when the proportional gain is 0.75. Find the P, PI, and PID controller parameters using the Ziegler–Nichols ultimate sensitivity method.

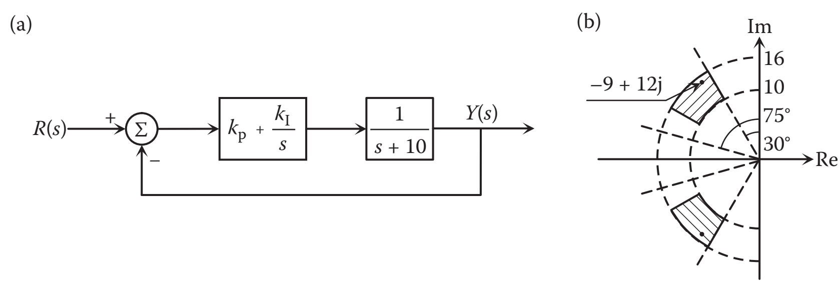


FIGURE 10.42 Problem 5 (a) A feedback control system, (b) closed-loop pole locations.

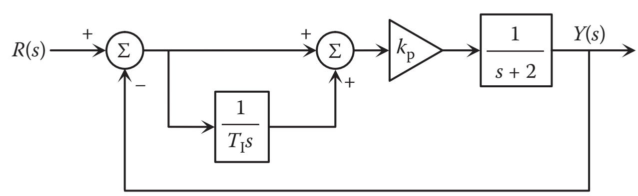


FIGURE 10.43 Problem 6.

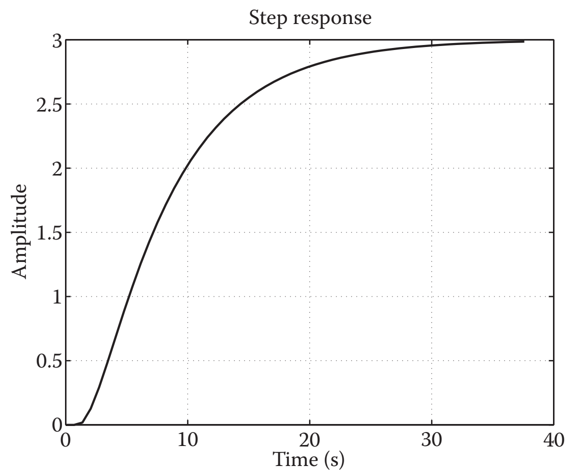


FIGURE 10.44 Problem 7.

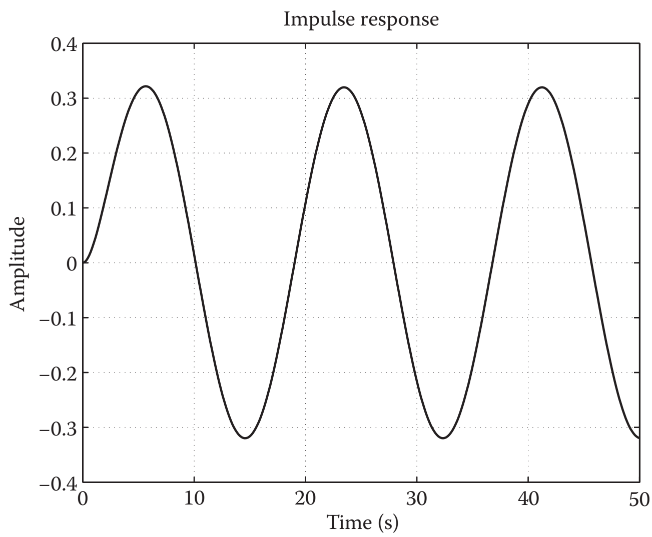


FIGURE 10.45 Problem 8.

10.5 ROOT LOCUS

Beginning with this section, and for the remainder of this chapter, the discussion focuses on how to design a feedback controller to meet stability and performance requirements. Root locus, Bode plot, and state-space techniques are three control design methods that are introduced in Sections 10.5 through 10.7, respectively. To simplify the discussion, all controllers in those three sections are limited to proportional feedback control.

As presented in Section 10.2, the time-domain specifications, such as rise time t_r, overshoot M_p, peak time t_p, and settling time t_s, are related to the natural frequency ω_n and the damping ratio ζ, both of which can be used to express the pole locations of a second-order system in the s-plane. Thus, the dynamic response of a system can be influenced by changing the system's pole locations. Root locus, developed by W.R. Evans in the late 1940s, is a graphical design technique that shows how changes in one of the system parameters will modify the roots of the closed-loop characteristic

equation, or the closed-loop poles, and thus change the dynamic response of the system. In this section, we first introduce the procedure to sketch the root locus of a feedback control system. Then, we discuss ways to analyze the stability and performance of the closed-loop system based on the root locus. Finally, we will learn how to design a proportional feedback controller using the root locus technique.

10.5.1 Root Locus of a Basic Feedback System

Consider a basic feedback control system as shown in Figure 10.46, in which $G(s)$ is the transfer function of the plant. The controller is assumed to be a proportional gain, $C(s) = K$. The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1+C(s)G(s)} = \frac{KG(s)}{1+KG(s)} \tag{10.42}$$

and the characteristic equation is

$$1 + KG(s) = 0. \tag{10.43}$$

Note that the roots of the closed-loop characteristic equation are the poles of the closed-loop system. As observed from Equation 10.43, the closed-loop poles are affected by the value of K. When K varies from 0 to ∞, the closed-loop poles will move around the s-plane and create a trajectory, or a locus of poles.

Intuitively, a root locus can be constructed by changing the value of K from 0 to ∞, solving the characteristic equation for the roots, and plotting the poles in the s-plane. However, it was difficult to obtain the poles for high-order systems before the availability of mathematical and scientific computing software. In the late 1940s, Evans developed rules to plot a locus without actually solving for the roots of the characteristic equation. The following example shows the step-by-step procedure to manually construct a root locus.

Example 10.13: Root Locus Sketching

For the system in Figure 10.47, find the locus of closed-loop poles with respect to K.

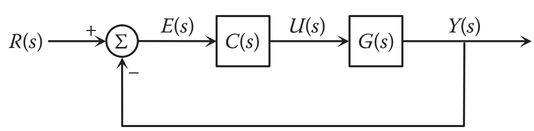


FIGURE 10.46 Block diagram of a basic feedback control system.

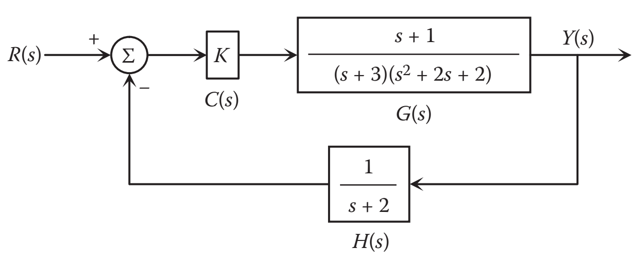


FIGURE 10.47 Block diagram for the feedback control in Example 10.13.

Solution

Step 1: Express the closed-loop characteristic equation in the form of

$$1+K\frac{b(s)}{a(s)}=0. \tag{10.44}$$

From Figure 10.47, the closed-loop transfer function is

$$\frac{Y(s)}{R(s)}=\frac{C(s)G(s)}{1+C(s)G(s)H(s)}.$$

Thus, the closed-loop characteristic equation is

$$1+C(s)G(s)H(s)=1+K\frac{s+1}{(s+3)(s^2+2s+2)}\frac{1}{s+2}=0,$$

which is in the form of $1 + Kb(s)/a(s) = 0$ with

$$a(s)=(s+3)(s^2+2s+2)(s+2),$$
$$b(s)=s+1.$$

Denote $L(s) = b(s)/a(s)$, where $L(s)$ is the loop gain. Note that the roots of $a(s)$ are the poles of $L(s)$ and the roots of $b(s)$ are the zeros of $L(s)$. The number of poles determines the number of branches of the root locus.

Step 2: Draw the axes of the s-plane to a suitable scale and mark a cross symbol "×" for each pole of $L(s)$ and a circle symbol "o" for each zero of $L(s)$.

RULE 1: Assume that L(s) *has* n *poles and* m *zeros. The* n *branches of the locus start at the poles of* L(s) *and the* m *of these branches end at the zeros of* L(s).

Solving $a(s) = 0$ for the poles gives

$$p_1 = -3, \quad p_{2,3} = -1 \pm j, \quad p_4 = -2$$

Solving $b(s) = 0$ for the zero gives

$$z_1 = -1.$$

The locations of the four poles and one zero are shown in Figure 10.48.

Step 3: Find the real axis portions of the locus

RULE 2: The portions of the root locus on the real axis are to the left of an odd number of poles and zeros.

This implies that a point on the real axis is part of the root locus if there is an odd number of poles and zeros to its right. As shown in Figure 10.48, there are two poles, –2 and –3, and one zero, –1, located on the real axis, which is divided into four segments, (–∞, –3), (–3, –2), (–2, –1), and (–1, +∞). To demonstrate RULE 2, consider a point within (–2, –1). There are two complex poles and one zero (three in total), to its right. Because three is an odd number, this portion of the real axis is part of the root locus. Similarly, the segment (–∞, –3) is also part of the root locus because there are four poles and one zero (a total of five), to its right. The thick solid lines in Figure 10.48 represent the portions of the root locus on the real axis.

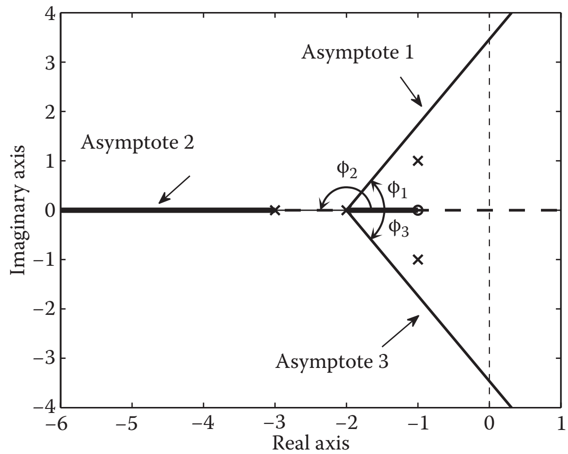


FIGURE 10.48 Real axis portions of the root locus and asymptotes for Example 10.13.

Step 4: Draw the asymptotes for large values of K.

RULE 3: For large s *and* K*,* n − m *branches of the locus are asymptotic to lines radiating out from the point* s = α *on the real axis at an angle* ϕ_l, where

$$\alpha = \frac{\sum \mathrm{Re}(p_i) - \sum \mathrm{Re}(z_i)}{n-m}, \tag{10.45}$$

$$\phi_l = \frac{180° + 360°(l-1)}{n-m}, \quad l = 1, 2, \ldots, n-m. \tag{10.46}$$

In our case, n = 4 and m = 1. Thus, there are three asymptotes, which radiate from a centroid α on the real axis

$$\alpha = \frac{(-1-1-2-3)-(-1)}{3} = -2,$$

with angles of

$$\phi_1 = \frac{180° + 360°(1-1)}{3} = 60°,$$

$$\phi_2 = \frac{180° + 360°(2-1)}{3} = 180°,$$

$$\phi_3 = \frac{180° + 360°(3-1)}{3} = 300° \quad \text{or} \quad -60°.$$

The centroid and asymptotes are shown in Figure 10.48.

Step 5: Compute the departure and arrival angles.

RULE 4: The angle of departure of a branch of the locus from a pole is given by

$$_{\text{dep}} = \sum \psi_i - \sum \ {}_i - 180°, \tag{10.47}$$

where $\sum {}_i$ *is the sum of the angles from this pole to the remaining poles and* $\sum \psi_i$ *is the sum of the angles from this pole to all the zeros. The angle of arrival of a branch at a zero is given by*

$$\psi_{\text{arr}} = \sum {}_i - \sum \psi_i + 180°, \tag{10.48}$$

where $\sum \psi_i$ *is the sum of the angles from this zero to the remaining zeros and* $\sum {}_i$ *is the sum of the angles from this zero to all the poles.*

Note that Equations 10.47 and 10.48 are valid for nonrepeated poles and zeros. The formula used for computing the departure or arrival angles from a repeated pole or to a repeated zero can be found in control books.

In Figure 10.48, two branches of the locus on the real axis have been completely drawn. One of them starts from the pole at –2 and ends at the zero at –1, and the other starts from the pole at –3 and ends at $-\infty$ by approaching the second asymptote. There is no need to compute the departure or arrival angles for the poles and the zero on the real axis. Note that there is a pair of complex conjugate poles at –1 ± j, from each of which one branch of the locus starts. Selecting the pole at –1 + j and applying Equation 10.47 gives

$$_{\text{dep}} = \sum \psi_i - \sum {}_i - 180° = \psi_1 - ({}_1 + {}_2 + {}_3) - 180°.$$

As sketched in Figure 10.49, $\psi_1 = 90°$, which is the angle of the line connecting the complex pole at –1 + j and the zero with respect to the positive real axis. Similarly, we can determine the angles φ_i, as $\varphi_1 = 90°$, $\varphi_2 = 45°$, and ${}_3 = \tan^{-1}\left(\frac{1}{2}\right) = 26.57°$. Thus,

$$\varphi_{\text{dep}} = 90° - (90° + 45° + 26.57°) - 180° = -251.57° = 108.43°.$$

The departure angle from the pole at –1 – j is –108.43° because the root locus is symmetric about the real axis.

Step 6: Determine the points where the root locus crosses the imaginary axis.

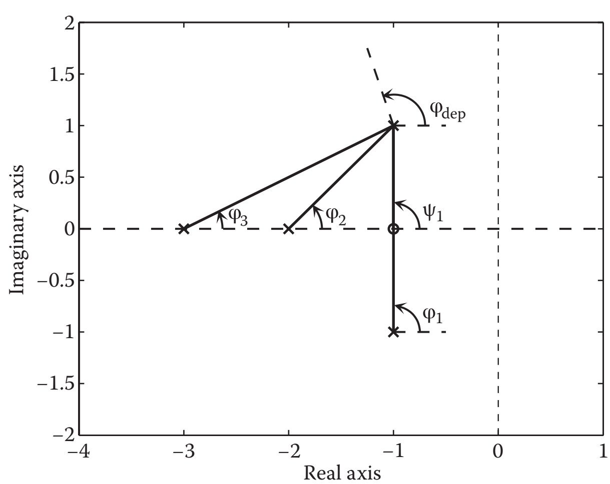


FIGURE 10.49 Departure angles for Example 10.13.

RULE 5: The root locus crosses the imaginary axis at points where the characteristic equation satisfies

$$1 + K\frac{b(j\omega)}{a(j\omega)} = 0 \tag{10.49}$$

or

$$a(j\omega) + Kb(j\omega) = 0. \tag{10.50}$$

Note that if a point on the imaginary axis is part of the root locus, then it must be a root of the characteristic equation given by Equation 10.44. The point on the imaginary axis can be expressed as $s = j\omega$ and substituting it into Equation 10.44 leads to Equation 10.49 or 10.50. Solving either one of them yields the $j\omega$-axis crossing points and the corresponding values of K.

In this example, we have

$$(s + 3)(s^2 + 2s + 2)(s + 2) + K(s + 1) = 0.$$

Substituting $s = j\omega$ and rearranging the equation gives

$$(j\omega)^4 + 7(j\omega)^3 + 18(j\omega)^2 + 22j\omega + 12 + Kj\omega + K = 0.$$

Separating the real and imaginary terms results in

$$(\omega^4 - 18\omega^2 + 12 + K) + j(-7\omega^3 + 22\omega + K\omega) = 0,$$

which is equivalent to two equations

$$\begin{cases} \omega^4 - 18\omega^2 + 12 + K = 0, \\ -7\omega^3 + 22\omega + K\omega = 0. \end{cases}$$

Solving for ω, we obtain

$$\omega_1 = 0, \quad \omega_{2,3} = \pm 3.44, \quad \omega_{4,5} = \pm 0.92j,$$

where the last two roots are invalid because ω represents frequency and is real. The corresponding values of K are

$$K_1 = -12, \quad K_{2,3} = 60.91,$$

where the first value is rejected because the gain K varies from 0 to $+\infty$. Thus, the root locus crosses the $j\omega$-axis at $\pm 3.44j$ when $K = 60.91$.

Step 7: Complete the sketch.

As shown in Figure 10.50, two complex branches of the locus depart from the complex poles, cross the imaginary axis, enter the right-half s-plane, and end at infinity by approaching the asymptotes. Combining the complex branches with the real axis portions of the locus, we have the sketch completed.

Rules 1 through 5 can be used to roughly sketch a root locus. One more rule is available for computing the so-called break-in or break-away points, but this will not be covered in this text. With the availability of MATLAB, these rules are not necessary to plot a root locus. However,

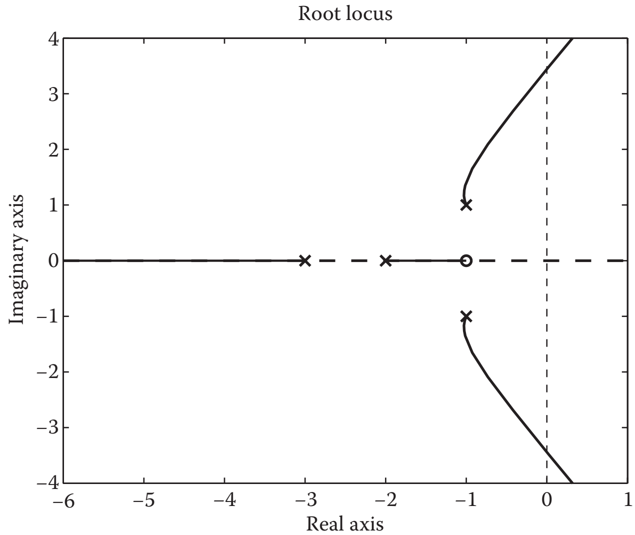


FIGURE 10.50 Root locus for Example 10.13.

learning these rules can help understand classical control design techniques and evaluate the correctness of a computer-generated root locus.

The MATLAB command used to sketch a root locus is `rlocus`. The following is the MATLAB session to generate the root locus shown in Figure 10.48:

```
>> num = [1 1];
>> den = conv(conv([1 3],[1 2 2]),[1 2]);
>> sysL = tf(num,den);
>> rlocus(sysL);
```

10.5.2 Analysis Using Root Locus

Using the root locus technique, it is very easy to determine the stability of a closed-loop system when the proportional gain K varies from 0 to ∞. For a particular value of K, if any of the poles are in the right-half s-plane, then the corresponding closed-loop system is unstable. If all of the poles are in the left-half s-plane, then the closed-loop system is stable. If any of the poles are on the imaginary axis and they are nonrepeated, then the closed-loop system is marginally stable.

From the root locus, we can also obtain information about the performance of a closed-loop system using the concept of a dominant pole. For a system with multiple poles, the pole closest to the origin is called the dominant pole. If the dominant poles are a pair of complex conjugates, the distance between either one of them and the origin is associated with the natural frequency if the system is approximated as a second-order system. If the dominant pole is a real pole, the distance between the pole and the origin is associated with the time constant if the system is approximated as a first-order system. Both the natural frequency and the time constant determine the speed of transient response. The dominant pole has the slowest speed of response, and it dominates the effect of all other poles with higher frequencies or lower time constants.

Example 10.14: Analysis Using Root Locus

Refer to the root locus obtained in Example 10.13. Comment on the stability and performance of the closed-loop system when K varies from 0 to ∞.

Solution

When $K = 0$, which corresponds to having no control, the root locus starts from the poles of the loop gain $L(s)$. As shown in Figure 10.51, all four open-loop poles are located in the left s-plane. This implies that the open-loop system is stable. As K increases, the four closed-loop poles move along four different branches of the root locus. For $0 \leq K < 60.91$, all of the poles are in the left-half s-plane, and thus the closed-loop system is stable. When $K = 60.91$, two complex poles, $\pm 3.44j$, appear on the imaginary axis, and thus the closed-loop system becomes marginally stable. For $K > 60.91$, the two complex branches of the root locus cross the imaginary axis and enter the right-half s-plane and the closed-loop system becomes unstable.

When K varies between 0 and 60.91, the pair of complex poles is always closer to the origin than the other poles. Consequently, they dominate the effects of all other poles and thus the closed-loop system exhibits underdamped oscillations. For example, the closed-loop poles are −3.29, −1.66, and

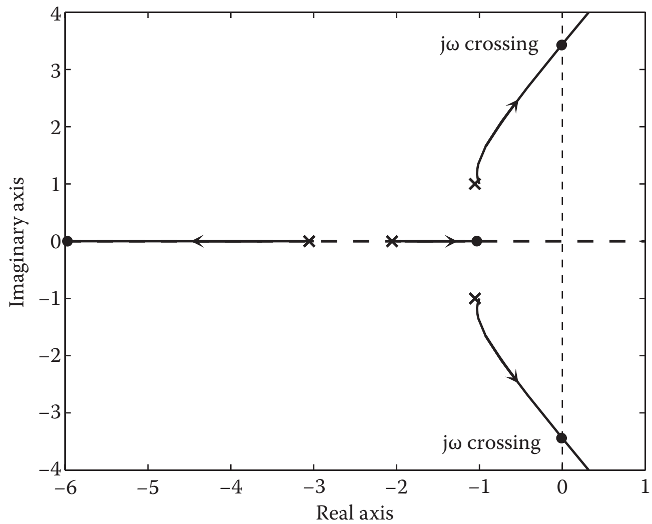


FIGURE 10.51 Root locus for Example 10.13 with poles highlighted when $K = 60.91$.

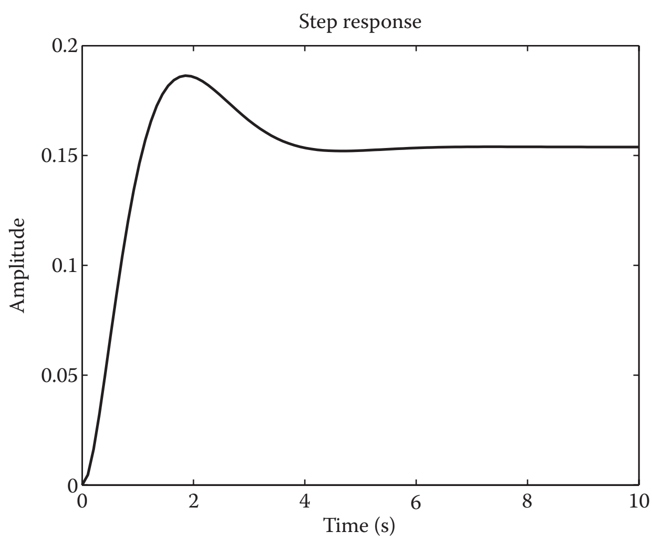


FIGURE 10.52 Unit-step response of the closed-loop system when $K = 1$.

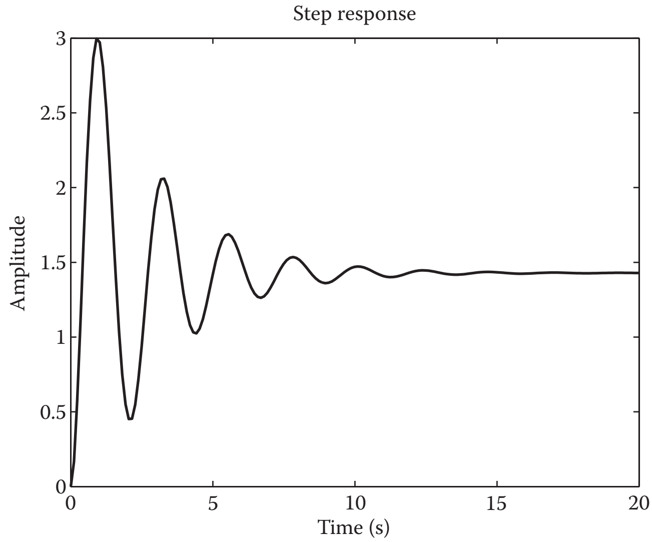


FIGURE 10.53 Unit-step response of the closed-loop system when $K = 30$.

$-1.03 \pm 1.15j$ when $K = 1$. Figure 10.52 is the corresponding closed-loop unit-step response, which has an overshoot of 21.1%. As K increases, the closed-loop system will exhibit severe oscillations because the dominant complex poles move toward the imaginary axis and the damping decreases. For example, the closed-loop poles are -5.16, -1.06, and $-0.39 \pm 2.74j$ when $K = 30$. Figure 10.53 is the corresponding closed-loop unit-step response, in which the overshoot is as high as 110%.

Note that although the root locus is constructed based on the loop gain $L(s)$, it also gives information on the stability and performance of the closed-loop system varying with respect to the proportional gain K. This is what makes the root locus technique attractive.

10.5.3 Control Design Using Root Locus

As we learned in Subsection 10.5.1, the root locus is a plot of all possible roots of the closed-loop characteristic equation $1 + KL(s) = 0$ for real positive values of K, which is generally the gain of a proportional controller. It is very easy to select K from a particular root locus so that the closed-loop system meets the performance specifications.

Example 10.15: Proportional Control Design Using Root Locus

Design a proportional controller for the cart system in Example 10.12 using the root locus technique.

Solution

For proportional feedback control of the DC motor–driven cart, the loop gain $L(s)$ is equal to the transfer function of the plant $G(s)$,

$$L(s) = G(s) = \frac{3.778}{s^2 + 16.883s}.$$

The root locus is plotted in Figure 10.54 using the MATLAB command `rlocus`. Note that the closed-loop system is a second-order system. Two closed-loop poles that are real appear for small values of K, specifically $K \leq 18.9$. When $K > 18.9$, the closed-loop system has a pair of complex

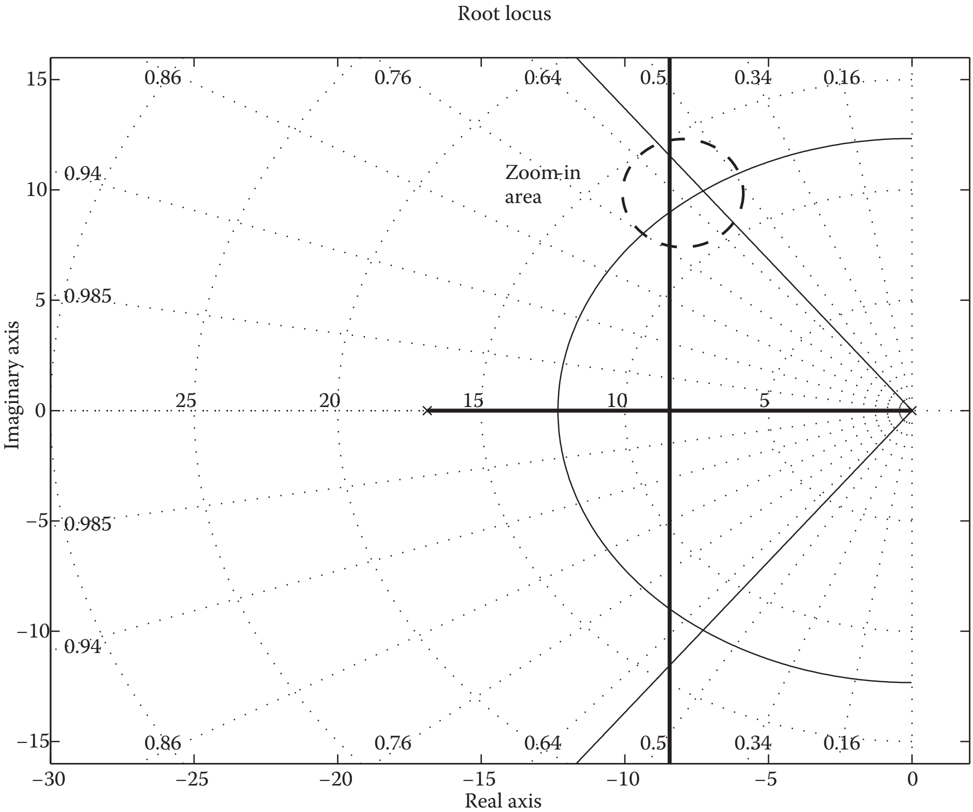


FIGURE 10.54 Root locus for Example 10.15 with grid lines.

conjugate poles, which move along the complex portion of the root locus as K varies between 18.9 and ∞.

To select the value of the proportional control gain that will meet the performance specifications, we can turn on the grid lines to the root locus using the command

```
>> grid on
```

As discussed in Subsection 10.2.2, the semicircles in Figure 10.54 indicate lines of constant natural frequencies ω_n and the diagonal lines indicate constant damping ratios ζ. In this example, we need an overshoot that is less than 10% (which implies $\zeta > 0.59$) and a rise time that is less than 0.15 s (which implies $\omega_n > 12.33$ rad/s). In Figure 10.54, the solid diagonal lines indicate pole locations with a damping ratio of approximately 0.59. In between these lines, the poles have $\zeta > 0.59$ and outside of the lines $\zeta < 0.59$. The solid semicircle is the locus of all poles with $\omega_n = 12.33$ rad/s, whereas those inside the semicircle have $\omega_n < 12.33$ rad/s, and those outside correspond to $\omega_n > 12.33$ rad/s. Thus, only the part of the root locus in between the two diagonal lines and outside of the semicircle is acceptable.

Figure 10.55 zooms in on the desired region, in which the vertical line is a part of the root locus. Left-clicking on the root locus, you will see the values of the pole and the gain that are required to place one of the closed-loop poles at that particular location. Holding down the left mouse button and moving the mouse along the root locus, you can see the values of the pole and the gain varying correspondingly.

Let us select $K = 50$. Figure 10.56 is the corresponding unit-step response of the closed-loop system with proportional feedback control. The closed-loop system meets the given specifications.

Note that the closed-loop poles cannot be placed arbitrarily in the s-plane with only a static proportional controller because the shape of the root locus is fixed for a given plant. A more useful

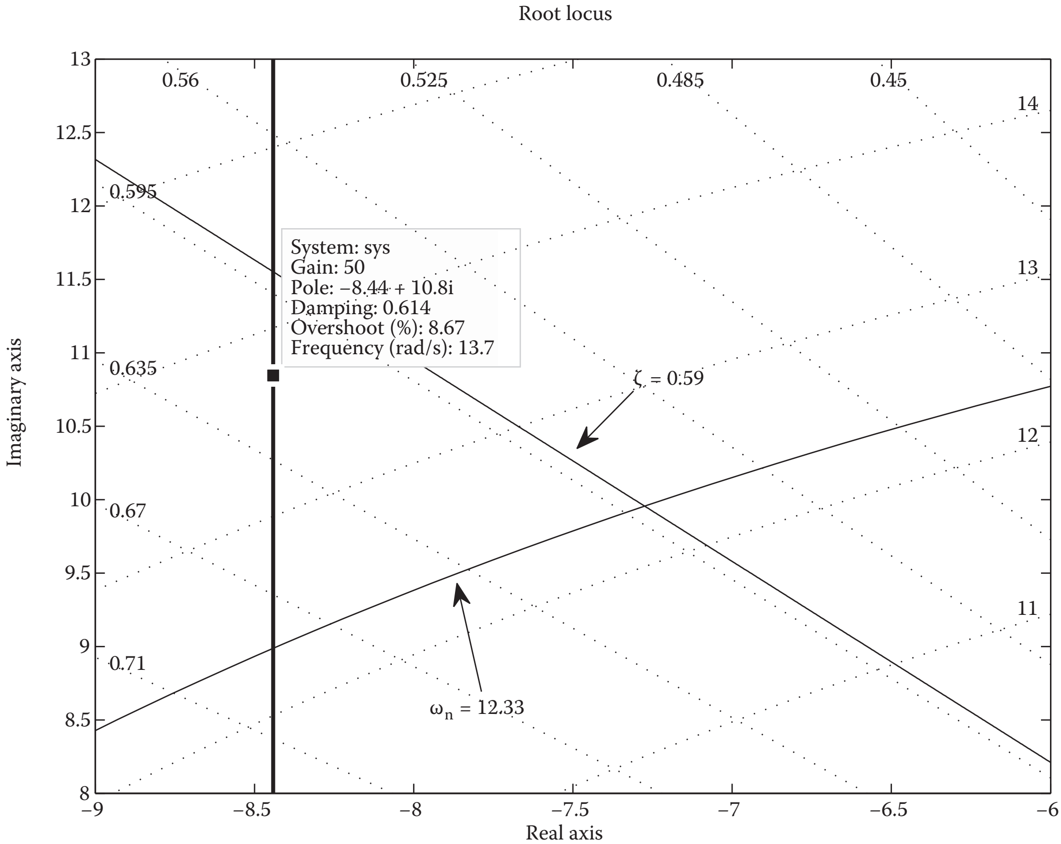


FIGURE 10.55 Zoom-in of the designed region in Example 10.15.

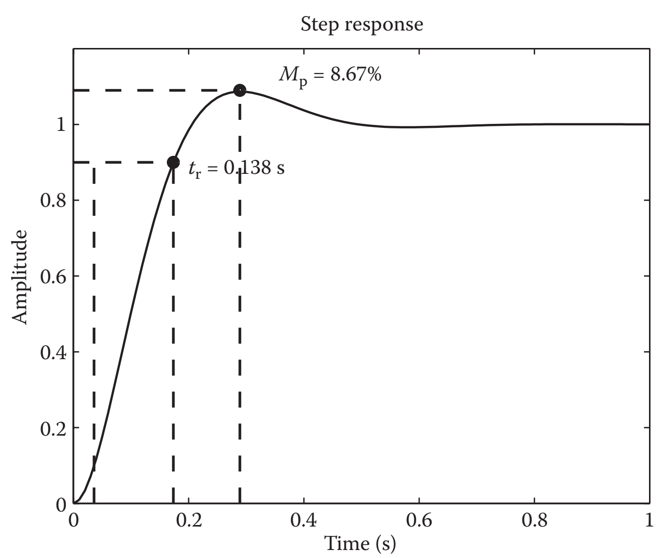


FIGURE 10.56 Unit-step response of the cart system with proportional feedback control $K = 50$.

design can be obtained by adding a pole or zero to the controller and making it a dynamic controller. This results in so-called lead or lag compensators, $C(s) = K(s + z)/(s + p)$. The reader can refer to control systems books for more details.

PROBLEM SET 10.5

1. Roughly sketch the root locus with respect to K for the equation of $1 + KL(s) = 0$ and the following choices for $L(s)$. Make sure to give the asymptotes, arrival or departure angles, and points crossing the imaginary axis. Verify your results using MATLAB.
   a. $L(s) = \dfrac{1}{(s+1)(s+3)}$
   b. $L(s) = \dfrac{1}{(s+1)(s+3)(s+11)}$
   c. $L(s) = \dfrac{s+5}{(s+1)(s+3)(s+11)}$
   d. $L(s) = \dfrac{s(s+5)}{(s+1)(s+3)(s+11)}$
2. Repeat Problem 1 for the following choices for $L(s)$.
   a. $L(s) = \dfrac{1}{s^2+2s+5}$
   b. $L(s) = \dfrac{1}{(s+4)(s^2+2s+5)}$
   c. $L(s) = \dfrac{s^2+4s+5}{(s+4)(s^2+2s+5)}$
   d. $L(s) = \dfrac{(s+1)(s^2+4s+5)}{(s+4)(s^2+2s+5)}$
3. A control system is represented using the block diagram shown in Figure 10.57. Sketch the root locus with respect to the proportional control gain K. Determine all the values of K for which the closed-loop system is stable.
4. A control system is represented using the block diagram shown in Figure 10.58, where the parameter a is subject to variations. Sketch the root locus with respect to the parameter a. Determine all values of a for which the closed-loop system is stable.
5. Figure 10.59 shows the root locus of a unity negative feedback control system, where K is the proportional control gain.
   a. Determine the transfer function of the plant. Use MATLAB to plot the root locus based on your choice of the plant, compare it with the root locus shown in Figure 10.59, and check the accuracy of your plant transfer function.
   b. Give comments on the stability of the closed-loop system when K varies from 0 to ∞.

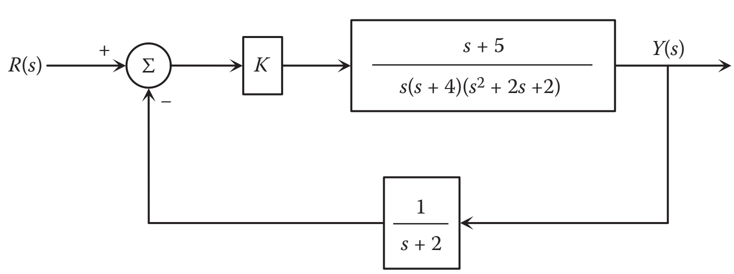


FIGURE 10.57 Problem 3.

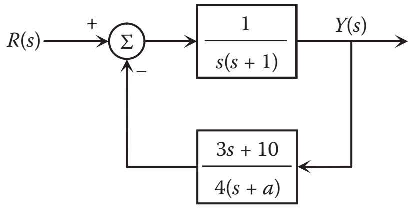


FIGURE 10.58 Problem 4.

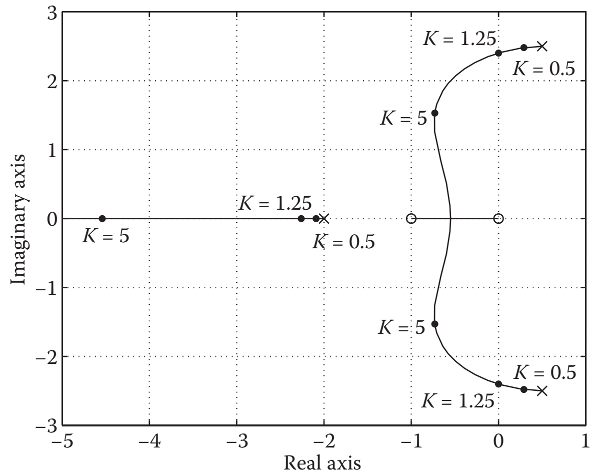


FIGURE 10.59 Problem 5.

c. Give comments on the transient performance of the closed-loop system when $K = 0.5$, 1.25, and 5. Use MATLAB to plot the corresponding unit-step responses and verify your comments.

6. Figure 10.60 shows the root locus of a unity negative feedback control system, where K is the proportional control gain.
   a. Determine the transfer function of the plant. Use MATLAB to plot the root locus based on your choice of the plant, compare it with the root locus shown in Figure 10.60, and check the accuracy of your plant.
   b. Find the range of values of K for which the system has damped oscillatory response. What is the largest value of K that can be used before pure harmonic oscillations occur? Also, what is the frequency of pure harmonic oscillations? Use MATLAB to plot the corresponding unit-step response and verify the accuracy of your computed frequency.
7. Consider the feedback system shown in Figure 10.61.
   a. Find the locus of the closed-loop poles with respect to K.
   b. Find a value of K such that the maximum overshoot in the response to a unit-step reference input is less than 10%. What is the corresponding steady-state error of the closed-loop system?
   c. Plot the unit-step response of the closed-loop system to verify the result for Part (b).
8. Consider the feedback system shown in Figure 10.62.
   a. Find the locus of the closed-loop poles with respect to K.
   b. Find a value of K such that the maximum overshoot in the response to a unit-step reference input is less than 20% and the 2% settling time is less than 1.1 s.
   c. Plot the unit-step response of the closed-loop system to verify the result in Part (b).

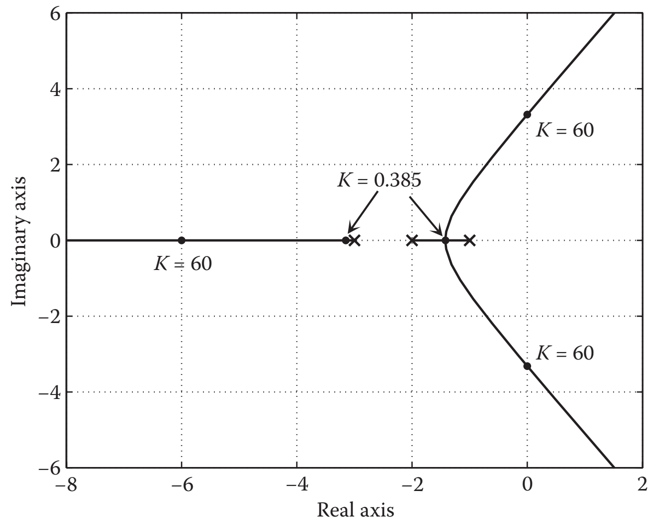


FIGURE 10.60 Problem 6.

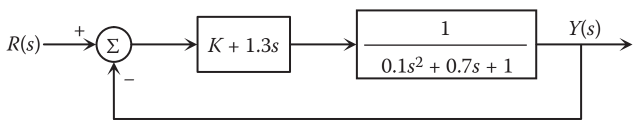


FIGURE 10.61 Problem 7.

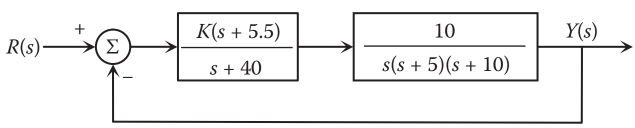


FIGURE 10.62 Problem 8.

10.6 BODE PLOT

The Bode plot technique is widely used to display a frequency response function. It also gives useful information for analyzing and designing control systems. Stability criteria can be interpreted using the Bode plot and numerous control design techniques are based on the Bode plot. In Section 8.3, we introduced the concept of the Bode plot and presented the Bode plot of the frequency response function for two fundamental systems: first-order and second-order. In this section, we first show how to use the Bode plot to display the frequency response function for a general dynamic system. Subsequently, we will learn how the Bode plot is utilized to determine stability. Finally, we will see how the Bode plot technique is used to design a proportional feedback controller.

10.6.1 Bode Plot of a Basic Feedback System

Consider the basic feedback control system shown in Figure 10.63. The open-loop transfer function is $KG(s)$, which can be written in the form

$$KG(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_\mathrm{m})}{(s-p_1)(s-p_2)\cdots(s-p_\mathrm{n})}, \tag{10.51}$$

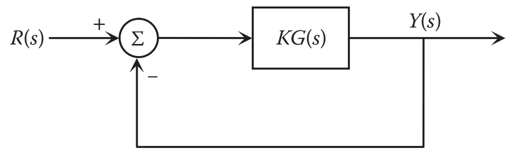


FIGURE 10.63 Simplified feedback control system.

explicitly showing the poles and zeros. Replacing s with jω yields the frequency response function

$$KG(j\omega) = K\frac{(j\omega - z_1)(j\omega - z_2)\cdots(j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2)\cdots(j\omega - p_n)}. \tag{10.52}$$

The frequency response function can be displayed using two curves: the Bode magnitude plot and the Bode phase plot. By definition, the magnitude of $KG(j\omega)$ in decibels is

$$\begin{aligned} \left|KG(j\omega)\right|_{dB} &= 20\log_{10}\left|KG(j\omega)\right| \\ &= 20\log_{10}\left|K\right| + 20\log_{10}\left|j\omega - z_1\right| + 20\log_{10}\left|j\omega - z_2\right| + \cdots \\ &\quad - 20\log_{10}\left|j\omega - p_1\right| - 20\log_{10}\left|j\omega - p_2\right| - \cdots \end{aligned} \tag{10.53}$$

and the phase of $KG(j\omega)$ is

$$\angle[KG(j\omega)] = \angle K + \angle(j\omega - z_1) + \angle(j\omega - z_2) + \cdots - \angle(j\omega - p_1) - \angle(j\omega - p_2) - \cdots. \tag{10.54}$$

Equations 10.53 and 10.54 show that the magnitude (in decibels) and phase of the frequency response function each is the sum of the magnitudes and phases of simple terms, which are similar to each other. If we know how to draw the Bode plot for each individual term, then the composite curve can be obtained by combining all the terms involved.

Depending on the locations of poles or zeros, there are four classes of basic terms:

1. Constant terms K (no pole or zero)
2. Integral or derivative terms $(j\omega)^{\pm n}$ (with pole(s) or zero(s) at the origin)
3. First-order terms $(j\omega\tau + 1)^{\pm 1}$ (with a real pole or zero at $-1/\tau$)
4. Second-order terms $[(j\omega/\omega_n)^2 + 2\zeta j\omega/\omega_n + 1]^{\pm 1}$ (with a pair of complex conjugate poles or zeros at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$)

Figure 10.64 is an example of a plot with a constant term $K = 10$. Because K is independent of the frequency, both magnitude and phase in the entire frequency region are horizontal lines. The magnitude in decibels is

$$\left|K\right|_{dB} = 20\log_{10}\left|K\right| \tag{10.55}$$

and the phase is

$$\angle K = 0°. \tag{10.56}$$

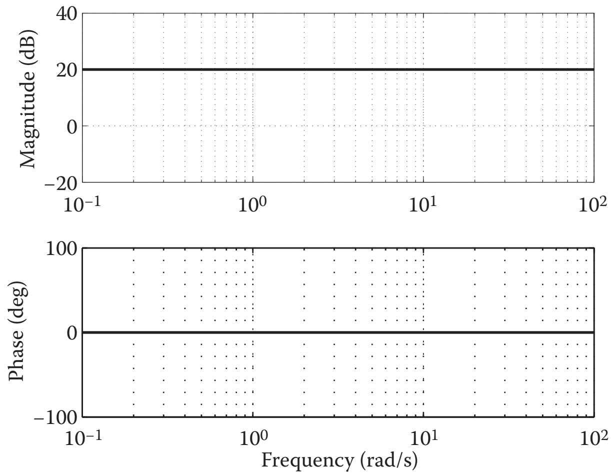


FIGURE 10.64 Bode plot for $K = 10$.

For integral or derivative terms, the magnitude of $(j\omega)^{\pm n}$ in decibels is

$$\left|(j\omega)^{\pm n}\right|_{dB} = 20\log_{10}\left|(j\omega)^{\pm n}\right| = \pm n \times 20\log_{10}\left|j\omega\right| = \pm 20n\log_{10}\omega. \tag{10.57}$$

Note that the magnitude plot is drawn using the logarithmic scale for the frequency, that is, $\log_{10}\omega$. Thus, the magnitude plot of an integral or derivative term is a straight line with a slope $\pm 20n$ dB/decade, which means that the magnitude will change by $\pm 20n$ dB as the frequency increases by a factor of 10. Geometrically, a straight line is uniquely determined by its slope and one point that it goes through. By Equation 10.57, this line always crosses (1 rad/s, 0 dB) regardless of the value of n. Figure 10.65 shows Bode plots for an integral term $1/j\omega$ and a derivative term $j\omega$. Their magnitude

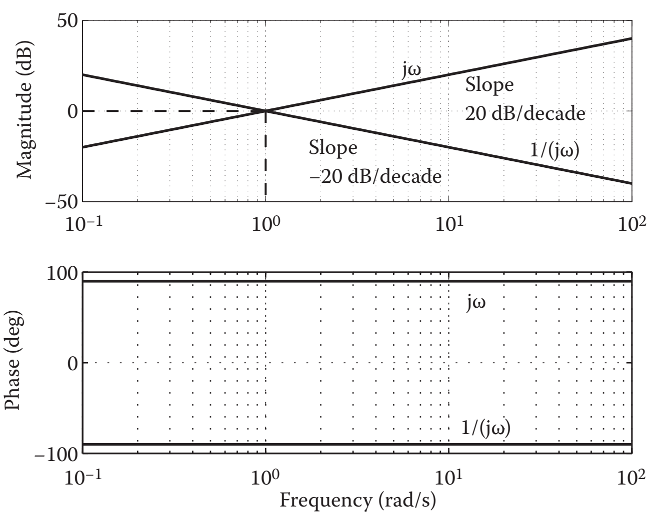


FIGURE 10.65 Bode plot for $1/j\omega$ and $j\omega$.

plots intersect at (1 rad/s, 0 dB) and the slope is −20 dB/decade for 1/jω and +20 dB/decade for jω. The phase of $(j\omega)^{\pm n}$ is

$$\angle(j\omega)^{\pm n} = \pm n\angle(j\omega) = \pm n \times 90°, \tag{10.58}$$

which is independent of frequency. As shown in Figure 10.65, the phase is a horizontal line and it is −90° for 1/jω and +90° for jω.

The Bode plots of the frequency response function for a first-order and a second-order system are shown in Figures 8.23 and 8.25. In this section, we discuss the plotting of asymptotes for more general cases as indicated in the third and fourth classes.

For the first-order terms $(j\omega\tau + 1)^{\pm 1}$, Figure 10.66 shows the magnitude plots with the asymptotes. Let us take $j\omega\tau + 1$ as an example. At low frequencies, $\omega\tau \ll 1$, we have $j\omega\tau + 1 \approx 1$. Thus, the magnitude approaches a horizontal line crossing 0 dB. At high frequencies, $\omega\tau >> 1$, we have $j\omega\tau + 1 \approx j\omega\tau$, for which the magnitude is

$$20\log_{10}\left|j\omega\tau\right| = 20\log_{10}(\omega\tau) = 20\log_{10}\omega + 20\log_{10}\tau \approx 20\log_{10}\omega. \tag{10.59}$$

Note that τ is a finite number for a given frequency response function, and its effect on the magnitude can be ignored at very high frequencies. Thus, the magnitude approaches a straight line with a slope of 20 dB/decade. When $\omega\tau = 1$ (or $\omega = 1/\tau$), the magnitude is

$$20\log_{10}\left|j+1\right| = 20\log_{10}\sqrt{2} = 3\,\text{dB}. \tag{10.60}$$

The corresponding frequency 1/τ is the corner frequency, in which the slope of the asymptote changes from 0 to 20 dB/decade. The magnitude plot in the entire frequency region can be obtained by drawing a smooth curve following the asymptotes with 3 dB above the line at the corner frequency.

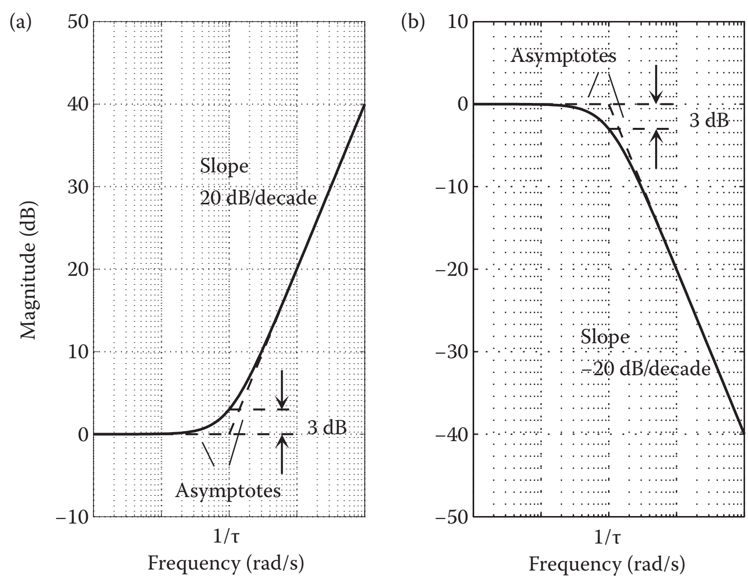


FIGURE 10.66 Magnitude plots for (a) $(j\omega\tau + 1)^{+1}$ and (b) $(j\omega\tau + 1)^{-1}$.

Figure 10.67 shows the asymptotes of the phase plots for the terms $(j\omega\tau + 1)^{\pm 1}$. Again, let us take $j\omega\tau + 1$ as an example. The phase can be approximated as $\angle 1 = 0°$ at low frequencies and $\angle(j\omega\tau) = 90°$ at high frequencies. The phase at the corner frequency $1/\tau$ is $\angle(j + 1) = 45°$.

The Bode plot of the second-order terms $[(j\omega/\omega_n)^2 + 2\zeta j\omega/\omega_n + 1]^{\pm 1}$ can be drawn in a similar manner as the first-order terms. The asymptotes for magnitude and phase plots are shown in Figures 10.68 and 10.69, respectively. Unlike the first-order terms, the corner frequency is $\omega = \omega_n$, at which the magnitude changes slope from 0 to +40 dB/decade if the term is in the numerator or to −40 dB/decade if the term is in the denominator. Correspondingly, the phase changes from 0° to +180° or −180°.

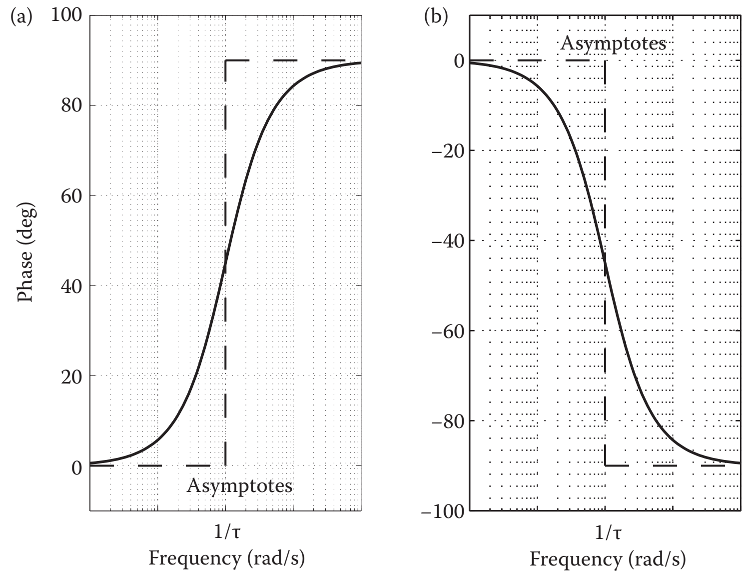


FIGURE 10.67 Phase plots for (a) $(j\omega\tau + 1)^{+1}$ and (b) $(j\omega\tau + 1)^{-1}$.

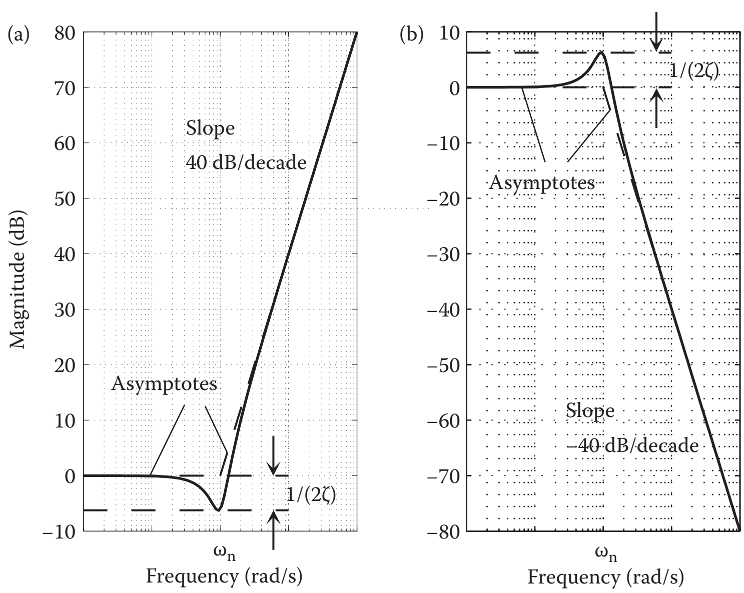


FIGURE 10.68 Magnitude plot for a second-order term in the (a) numerator and (b) denominator.

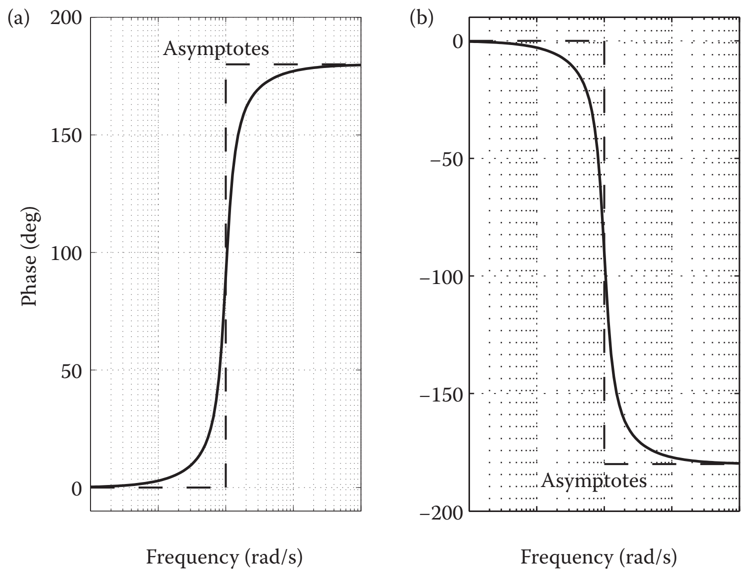


FIGURE 10.69 Phase plot for a second-order term in the (a) numerator and (b) denominator.

The magnitude at the corner frequency ω_n greatly depends on the damping ratio ζ. As shown in Equation 8.35, a rough sketch can be made by noting whether the peak is either $1/(2\zeta)$ below or above the asymptotes.

For a general dynamic system, the frequency response function can always be written as the product of several basic terms. Equations 10.53 and 10.54 suggest that the composite magnitude curve and the phase curve are the sum of their respective individual curves. The following example shows how to obtain a quick sketch of the composite curve using the asymptotes of the basic terms.

Example 10.16: Bode Plot Sketching

Plot the Bode magnitude and phase for the system with the transfer function

$$KG(s) = K\frac{s+2}{s(s^2+8s+400)},$$

where $K = 1$.

Solution

Step 1: Convert the transfer function to the frequency response function

$$KG(j\omega) = \frac{j\omega+2}{j\omega[(j\omega)^2+8j\omega+400]} = \frac{0.005((j\omega/2)+1)}{j\omega[(j\omega/20)^2+2(0.2)(j\omega/20)+1]}.$$

Note that first-order and second-order terms are expressed in their corresponding basic forms, $j\omega\tau + 1$ and $(j\omega/\omega_n)^2 + 2\zeta j\omega/\omega_n + 1$.

Step 2: Identify the basic terms and the corner frequencies associated with first-order and second-order terms.

The basic terms in this example are listed as follows:

1. One constant term 0.005
2. One integral term $1/j\omega$
3. One first-order term $j\omega/2 + 1$ in the numerator with $1/\tau = 2$ rad/s
4. One second-order term in the denominator with $\omega_n = 20$ rad/s and $\zeta = 0.2$

Step 3: Draw the asymptotes for the magnitude curve.

We start with indicating the corner frequencies on the frequency axis. Then, the asymptote for the derivative term is drawn through the point (1 rad/s, 0 dB) with a slope of –20 dB/decade. The asymptote is extended until the first corner frequency 2 rad/s is met, which is associated with the first-order term in the numerator. At the first corner frequency, the slope increases by 20 dB/decade and changes to 0 dB/decade. We then continue extending the asymptote until the second corner frequency 20 rad/s is met, which is associated with the second-order term in the denominator. At the second corner frequency, the slope decreases by −40 dB/decade and changes from 0 dB/decade to −40 dB/decade. Finally, we consider the effect of the constant term 0.005 by sliding the asymptotes downward by 46 dB (i.e., $20\log_{10}0.005 = -46$ dB). This completes the composite magnitude asymptotes.

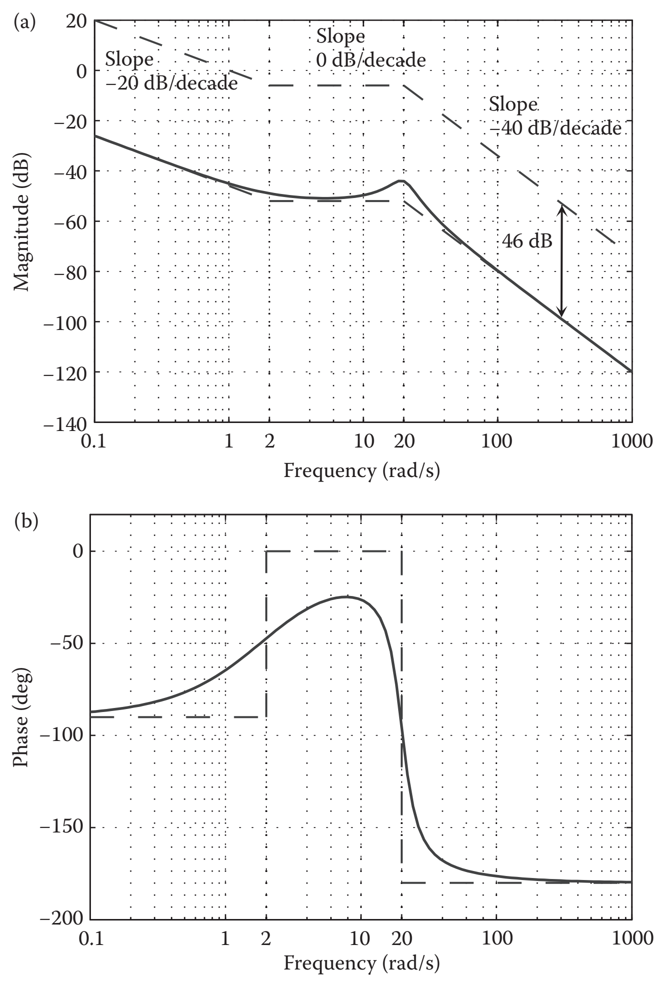


FIGURE 10.70 Bode plot for the system in Example 10.16: (a) magnitude plot with asymptotes and (b) phase plot with asymptotes.

The approximate Bode magnitude plot can be obtained by drawing a smooth curve following the asymptotes. The magnitude is 3 dB above the asymptote at the first-order numerator corner frequency. A resonant peak is sketched at the second-order denominator corner frequency. Note that the associated damping ratio ζ is 0.2. Thus, the magnitude above the asymptote is approximately 1/0.4 = 2.5 or 7.9588 dB. The accurate value of the peak above the asymptote is $1/(20 \log_{10}(2\zeta)) = 8$ dB. Figure 10.70a shows the magnitude plot and the asymptotes.

Step 4: Draw the asymptotes for the phase curve.

Following the same procedure as in Step 3, we can sketch the asymptotes for the composite phase plot. We start with sketching the asymptote for the derivative term with a horizontal line at −90°. The phase changes by 90° at the first-order numerator corner frequency 2 rad/s, and −180° at the second-order denominator corner frequency 20 rad/s. Because the phase of a constant is 0° for any frequency, the constant is not considered when sketching the phase plot. Figure 10.70b shows the phase plot and the asymptotes.

In summary, the asymptote for a composite magnitude or phase curve is plotted by starting with the $(j\omega)^{\pm n}$ term, and changing the slope or the phase at each corner frequency depending on whether the corner frequency is associated with a first-order or second-order term in the numerator or denominator. For first-order terms, the changes of slope and phase are +20 dB/decade and +90°, respectively, when in the numerator, and −20 dB/decade and −90°, respectively, when in the denominator. For second-order terms, the changes of slope and phase are +40 dB/decade and +180°, respectively, when in the numerator, and −40 dB/decade and −180°, respectively, when in the denominator. The asymptote for the magnitude is completed by shifting it up or down depending on the value of the constant term.

10.6.2 Analysis Using Bode Plot

As with the root locus technique, a Bode plot can be used to determine the stability of a closed-loop system without solving for the poles. Consider the proportional feedback control system shown in Figure 10.63. Often, the Bode plot for $K = 1$ is first drawn. Two margins can be read from the magnitude and phase plots. As shown in Figure 10.71, the gain margin (GM) is the amount of the gain that can be added before the magnitude curve reaches 0 dB at the frequency where the phase plot crosses −180°. The phase margin (PM) is the amount of the phase that can be subtracted before the phase curve reaches −180° at the frequency where the magnitude plot crosses 0 dB.

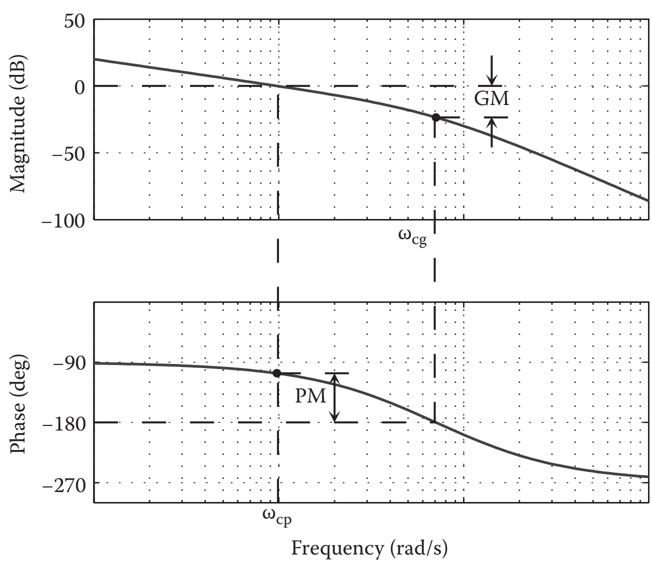


FIGURE 10.71 GM and PM.

The stability criteria are given by

$$\begin{cases} \mathrm{GM} > 0\,\mathrm{dB} & \text{stable} \\ \mathrm{GM} = 0\,\mathrm{dB} & \text{marginally stable} \\ \mathrm{GM} < 0\,\mathrm{dB} & \text{unstable} \end{cases} \tag{10.61}$$

or

$$\begin{cases} \mathrm{PM} > 0^\circ & \text{stable} \\ \mathrm{PM} = 0^\circ & \text{marginally stable} \\ \mathrm{PM} < 0^\circ & \text{unstable} \end{cases} \tag{10.62}$$

Example 10.17: Stability Analysis Using a Bode Plot

Consider the feedback control system in Figure 10.63, where

$$KG(s) = K\frac{100}{s(s+5)(s+20)},$$

and K is assumed to be 1. Plot the Bode magnitude and phase curves using MATLAB. Give comments on the stability of the closed-loop system.

Solution

Note that the Bode plot is drawn based on the loop gain

$$L(s) = K\frac{100}{s(s+5)(s+20)}.$$

The MATLAB command used to sketch the Bode plot is `bode`. Assuming $K = 1$, the following is the MATLAB session:

```
>> num = [100];
>> den = conv(conv([1 0],[1 5]),[1 20]);
>> sysL = tf(num,den);
>> bode(sysL);
```

As observed in Figure 10.72, the phase curve crosses –180° at frequency 10 rad/s, and the corresponding magnitude is approximately –28 dB. This implies that a gain of 28 dB can be added before the magnitude plot reaches 0 dB at that frequency. Thus, the GM is 28 dB. The magnitude curve crosses 0 dB at frequency 1 rad/s, and the corresponding phase is –104°. This implies that a phase of 76° can be subtracted before the phase plot reaches –180°. Thus, the PM is 76°.

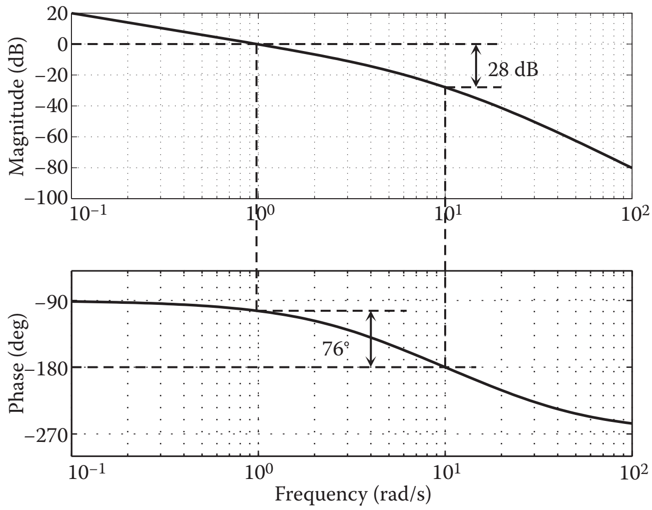


FIGURE 10.72 Bode plot for the system in Example 10.17.

According to Equations 10.61 and 10.62, the closed-loop system with the proportional controller $K = 1$ is stable.

More information on stability can be extracted from the GM. Specifically, the GM when $K = 1$ gives the stability range of K for which the proportional feedback control system is stable. In our case, GM = 28 dB or 25, and this indicates that the closed-loop system is stable for $0 < K < 25$. We can also use the MATLAB command `margin` as follows:

```
>> [gm,pm,wcg,wcp] = margin(sysL)
```

which returns GM, PM, and the associated frequencies as defined in Figure 10.72. In our case, the GM returned by the command `margin` is 25. Note that the stability range of K can be determined in this way only for systems that change from being stable to unstable as K increases.

10.6.3 Control Design Using Bode Plot

Unlike the root locus technique, which uses time-domain performance specifications, the Bode plot technique deals with control design in the frequency domain. The requirements are defined in terms of GM, PM, bandwidth, resonant peak, and so on. If a time-domain specification is given, it will usually be converted to one in the frequency domain.

Example 10.18: Proportional Control Design Using a Bode Plot

Design a proportional controller for the cart system in Example 10.12 using the Bode plot technique.

Solution

The Bode plot for the open-loop transfer function $KG(s)$, where

$$G(s) = \frac{3.778}{s^2 + 16.883s} \quad \text{and} \quad K = 1$$

is shown in Figure 10.73.

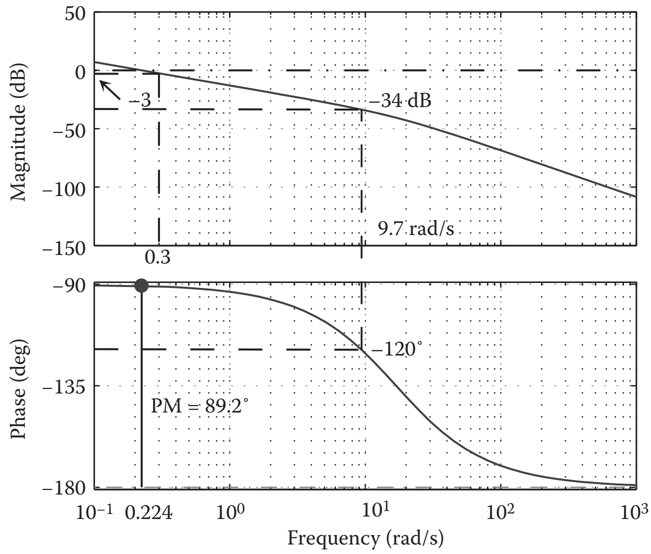


FIGURE 10.73 Bode plot for the system in Example 10.18 with $K = 1$.

Note that the requirements are given as overshoot $M_p < 10\%$ and rise time $t_r < 0.15$ s. These conditions correspond to $\zeta > 0.59$ and $\omega_n > 12.33$ rad/s. It can be shown that the relationship between the damping ratio and PM is

$$\text{PM} \approx 100\zeta, \tag{10.63}$$

which, for the current example, yields the requirement PM > 59°. In addition, the closed-loop natural frequency ω_n is related to the closed-loop bandwidth, which is somewhat greater than the frequency when the Bode magnitude plot of $KG(s)$ crosses –3 dB. Denote this crossover frequency as ω_c, and we have

$$\omega_c \le \omega_{BW} \le 2\omega_c. \tag{10.64}$$

The higher the crossover frequency, the higher the bandwidth and the natural frequency.

As shown in Figure 10.73, PM = 89.2°, which meets the requirement. However, the crossover frequency ω_c is only approximately 0.3 rad/s, which is too small. We must adjust the value of the proportional control gain K to meet both requirements. Because the current PM is way above the requirement, let us decrease it and pick PM = 60°. Based on the definition of PM, this implies that the frequency at which the magnitude plot crosses 0 dB should be –120°. It is observed from Figure 10.73 that the frequency corresponding to –120° is 9.7 rad/s, at which the magnitude is –34 dB. To make the magnitude 0 dB, the magnitude plot should slide upward by 34 dB. This is the effect of multiplying a constant term of

$$10^{34/20} = 50,$$

which is the value of the proportional control gain K.

Let us set K to be 50, which is also what was found in Example 10.15 using the root locus design method. The Bode plot of the open-loop transfer function $KG(s)$ with the new value of K is shown in Figure 10.74. The PM is 60.1° and the crossover frequency ω_c is 12.6 rad/s. The Bode plot with $K = 1$ is also shown in Figure 10.74. Comparing the two magnitude plots, we find the magnitude plot corresponding to $K = 50$ to be 34 dB above the one corresponding to $K = 1$, as designed.

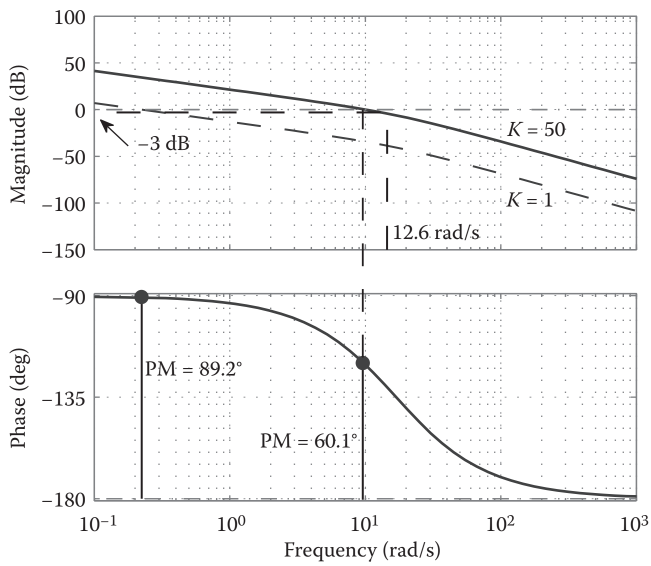


FIGURE 10.74 Bode plots for the system in Example 10.18 with $K = 50$ and $K = 1$.

PROBLEM SET 10.6

1. Sketch the asymptotes of the Bode plot magnitude and phase for the following open-loop transfer functions. Make sure to give the corner frequencies, slopes of the magnitude plot, and phase angles. Verify the results using MATLAB.

   a. $G(s) = \dfrac{s+10}{s+100}$

   b. $G(s) = \dfrac{s+10}{(s+100)(s+5000)}$

   c. $G(s) = \dfrac{s+10}{s(s+100)(s+5000)}$

   d. $G(s) = \dfrac{(s+10)^2}{s(s+100)(s+5000)}$

2. Repeat Problem 1 for the following open-loop transfer functions.

   a. $G(s) = \dfrac{1}{s^2+4s+100}$

   b. $G(s) = \dfrac{s+0.5}{s^2+s+25}$

   c. $G(s) = \dfrac{s^2+0.1s+25}{s^2+0.24s+144}$

   d. $G(s) = \dfrac{100(s^2+7s+49)}{(s+1)(s+500)}$

3. For each of the following open-loop transfer functions, construct a Bode plot for $K = 1$ using the MATLAB command `bode`. Estimate the GM, PM, and their associated crossover frequencies from the plot. Verify the results using the MATLAB command `margin`. Determine the stability of the corresponding closed-loop system.

   a. $KG(s) = \dfrac{K}{s(s^2+2s+25)}$

b. $KG(s) = \dfrac{100K}{(s+4)(s^2+2s+2)}$

c. $KG(s) = K\dfrac{s+0.5}{s(s+2)^2}$

d. $KG(s) = K\dfrac{1}{(s+10)(s+1)^2}$

4. Reconsider Problem 3. Each plant $G(s)$ is controlled by a proportional controller K via unity negative feedback. Determine the stability range of K by sliding the magnitude plot up or down until instability occurs. Verify the results by sketching a root locus.
5. Figure 10.75 shows the Bode plot for an open-loop transfer function $KG(s)$ with $K = 500$.
   a. Determine the stability of the closed-loop system.
   b. Determine the value of K that would yield a PM of 45°.

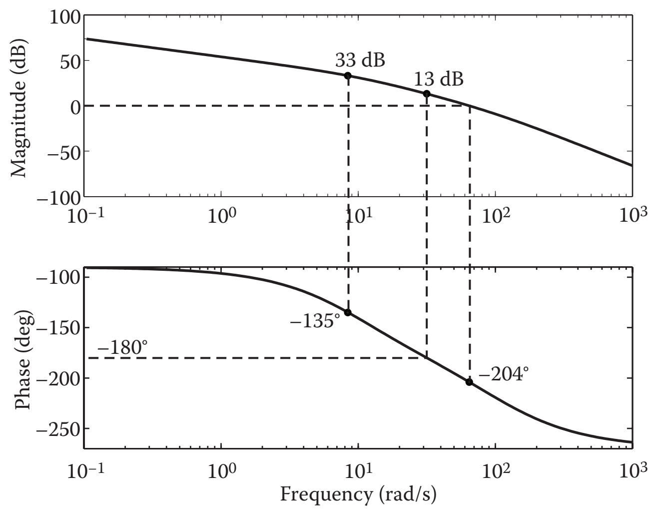


FIGURE 10.75 Problem 5.

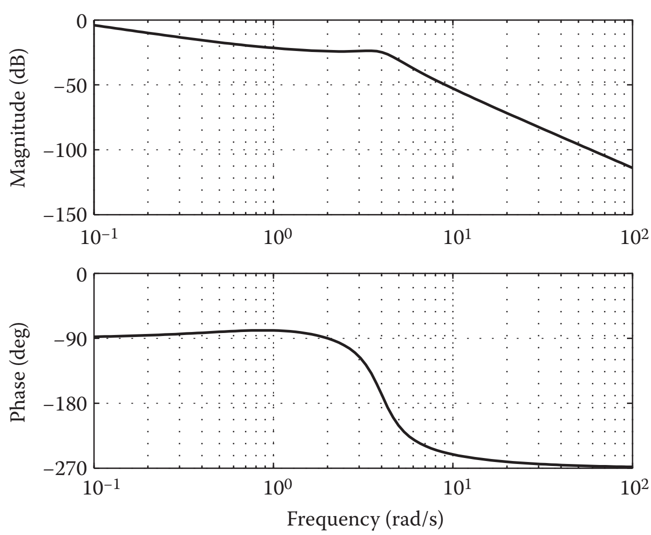


FIGURE 10.76 Problem 6.

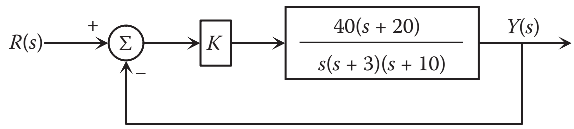


FIGURE 10.77 Problem 7.

6. The Bode plot for an open-loop transfer function $KG(s)$ is shown in Figure 10.76.
   a. Determine the stability of the closed-loop system.
   b. Assume that the proportional control gain K is increased by a factor of 100. Will the closed-loop system still be stable with the new value of K?
7. Consider the unity negative feedback system shown in Figure 10.77.
   a. Use MATLAB to obtain the Bode plot for $KG(s)$ when $K = 2$.
   b. Determine the stability of the closed-loop system when $K = 2$ using the stability margins.
   c. Determine the value of K that would yield a PM of 30°.
   d. Verify the result obtained in Part (c) by using the MATLAB command `margin`.
8. Reconsider the feedback system in Figure 10.62. Using the Bode plot technique, find a value of K such that the maximum overshoot in the response to a unit-step reference input is less than 20% and the 2% settling time is less than 1.1 s. Plot the unit-step response of the closed-loop system to verify the result.

10.7 FULL-STATE FEEDBACK

Unlike the root locus and Bode plot techniques, the state-space method works directly with mathematical models in state-space form instead of transfer function form. Often, use of the state-space method is referred to as modern control design, and use of transfer function–based methods, such as the root locus and Bode plot, is referred to as classical control design. Compared with the techniques based on transfer functions, the state-space method provides a convenient and compact way to model and analyze systems with multiple inputs and multiple outputs. This is one of the advantages of state-space design because most practical systems have more than one control input or more than one measured output. In this section, we only discuss single-input/single-output systems to show the basic ideas of state-space design. We first show how to analyze stability of a system whose model is given in state-space form. Two other important properties of control systems, known as controllability and observability, are also briefly introduced. Then, we will learn how to design a full-state feedback controller using the pole placement method.

10.7.1 Analysis of State-Space Equations

Consider a linear dynamic system with single input, single output, and n states. The state-space representation is written in the form (see more details in Section 4.2)

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u, \\ y &= \mathbf{C}\mathbf{x} + Du, \end{aligned} \tag{10.65}$$

where u, y, and D are scalars. The stability characteristics of a dynamic system in state-space form can be determined by the eigenvalues of matrix $\mathbf{A}$, which are the roots of

$$\left| s\mathbf{I} - \mathbf{A} \right| = 0 \tag{10.66}$$

known as the characteristic equation (see more details in Section 3.3).

Example 10.19: Stability Analysis in State Space

a. Compute the poles of the system described by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -16.883 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 3.778 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}.$$

b. Verify the results by converting the state-space representation to a transfer function and then identifying the poles of the transfer function.

Solution

a. The characteristic equation is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 \\ 0 & s + 16.883 \end{vmatrix} = s^2 + 16.883s = 0$$

which yields the poles $s_1 = 0$ and $s_2 = -16.883$.

b. As presented in Section 4.4, state-space equations for a single-input/single-output system can be converted to a transfer function using

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D.$$

Substituting the system matrices **A**, **B**, **C**, and D, which is 0 in this example, gives

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s + 16.883 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 3.778 \end{bmatrix} + 0 = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s + 16.883 & 1 \\ 0 & s \end{bmatrix}}{s^2 + 16.883s} \begin{bmatrix} 0 \\ 3.778 \end{bmatrix}$$

$$= \frac{3.778}{s^2 + 16.883s}.$$

The characteristic equation is $s^2 + 16.883s = 0$, which yields the poles at 0 and -16.883. The results agree with the poles obtained in Part (a).

Two other important properties for a control system are controllability and observability. Before we introduce their definitions, let us consider the following example.

Example 10.20: Controllability and Observability

Consider a dynamic system described by $G(s) = 2/(s + 4)$, which can be converted to state-space form as

$$\dot{x}_1 = -4x_1 + 2u,$$

$$y = x_1.$$

a. A new state is added and the resulting state-space equation is

$$\dot{x}_1 = -4x_1 + 2u,$$

$$\dot{x}_2 = -x_2,$$

$$y = x_1 + 3x_2.$$

Determine the transfer function for this new model.

b. Determine the transfer function for another model with state-space form

$$\dot{x}_1 = -4x_1 + 2u,$$

$$\dot{x}_2 = -x_2 + u,$$

$$y = x_1.$$

Solution

a. The system matrices **A**, **B**, **C**, and D are

$$\mathbf{A} = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 3],\ D = 0.$$

The transfer function is

$$G(s) = [1 \quad 3] \begin{bmatrix} s+4 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0 = \frac{[1 \quad 3] \begin{bmatrix} s+1 & 0 \\ 0 & s+4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}}{(s+1)(s+4)} = \frac{2}{s+4}.$$

b. Similarly, we have

$$\mathbf{A} = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0], \quad D = 0$$

and

$$G(s) = [1 \quad 0] \begin{bmatrix} s+4 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 = \frac{[1 \quad 0] \begin{bmatrix} s+1 & 0 \\ 0 & s+4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}{(s+1)(s+4)} = \frac{2}{s+4}.$$

Note that the dynamic models in Parts (a) and (b) are second-order systems, with state-space forms that differ from the original first-order system. However, they both end up with the same transfer function as the given first-order system due to pole–zero cancellation. As seen in Part (a), the second state cannot be affected by the input matrix **B**,

$$G(s) = \frac{[s+1 \quad 3(s+4)]}{(s+1)(s+4)} \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

This implies that the second state is uncontrollable by the actuator defined by matrix **B**. Similarly, in Part (b), the second state cannot be observed by the output matrix **C**,

$$G(s) = [1 \quad 0]\frac{\begin{bmatrix} 2(s+1) \\ s+4 \end{bmatrix}}{(s+1)(s+4)}.$$

This implies that the second state is unobservable by the sensor defined by matrix **C**.

Now we introduce more rigorous definitions of controllability and observability. A system is controllable if there exists a control signal $u(t)$ that will take the state of the system from any initial state $\mathbf{x}_0$ to any desired final state $\mathbf{x}_f$ in a finite amount of time. A system is observable if for any initial state $\mathbf{x}_0$ there is a finite time τ such that $\mathbf{x}_0$ can be determined from $u(t)$ and $y(t)$ for $0 \le t \le \tau$.

An nth-order single-input/single-output system is controllable if and only if the square matrix given by

$$\mathbf{P} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}] \tag{10.67}$$

is nonsingular, where **P** is called the controllability matrix. Similarly, the system is observable if and only if the square matrix given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{10.68}$$

is nonsingular, where **Q** is called the observability matrix. Because a nonsingular square matrix is of full rank, we can also check the rank of matrix **P** or **Q** to determine the controllability or observability.

Example 10.21: Controllability and Observability

Determine the controllability and observability for the second-order systems given in Example 10.20.

Solution

For the system in Part (a), the controllability matrix **P** is

$$\mathbf{P} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 2 & -8 \\ 0 & 0 \end{bmatrix},$$

which is singular. Thus, the system is uncontrollable. The observability matrix **Q** is

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -4 & -3 \end{bmatrix},$$

which is nonsingular. Thus, the system is observable.

For the system in Part (b), the controllability matrix **P** is

$$\mathbf{P} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 2 & -8 \\ 1 & -1 \end{bmatrix},$$

which is nonsingular. Thus, the system is controllable. The observability matrix **Q** is

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix},$$

which is singular. Thus, the system is unobservable.

Note that the two systems in Example 10.20 have the same transfer function representation but different controllability and observability properties. This implies that controllability and observability are functions of the state of the system and cannot be determined from a transfer function.

10.7.2 Control Design for Full-State Feedback

Consider an nth-order dynamic system given by Equation 10.65. If all of the states are measurable, then they can be fed back and used for computing the control input

$$u = -\mathbf{K}\mathbf{x} = -[k_1 \quad k_2 \quad \cdots \quad k_n] \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}, \tag{10.69}$$

where **K** is the feedback gain matrix. The control law defined by Equation 10.69 is called full-state feedback. Figure 10.78 shows the block diagram of a closed-loop system with full-state feedback.

Substituting Equation 10.69 into Equation 10.65 gives the state equation of the closed-loop system, that is,

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}. \tag{10.70}$$

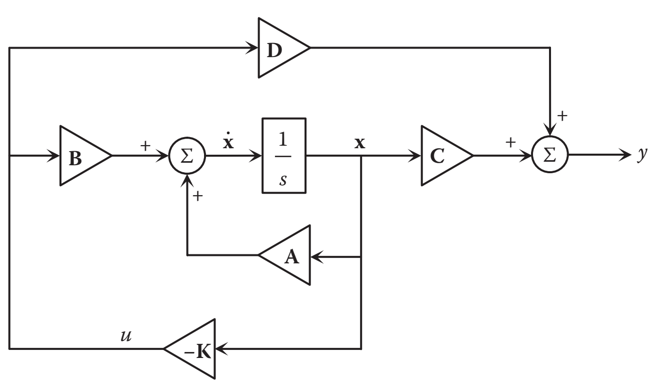


FIGURE 10.78 Block diagram of a closed-loop system with full-state feedback.

Note that the closed-loop poles are the eigenvalues of the matrix **A** – **BK**, and the closed-loop characteristic equation is

$$\left|s\mathbf{I}-(\mathbf{A}-\mathbf{BK})\right|=0, \tag{10.71}$$

where the left-hand side is an nth-order polynomial in s containing the gains k_1, k_2, …, k_n. These gains, and hence the feedback gain matrix **K** can be determined using the pole placement method.

As discussed in Section 10.2, the performance of a controlled system is associated with the closed-loop poles. If a feedback gain matrix **K** is determined based on desired pole locations, then the closed-loop system with the feedback control law $u = -\mathbf{K}\mathbf{x}$ will achieve the desired performance. This is the basic idea of pole placement. Assume that the desired locations of the closed-loop poles are s_1, s_2, …, and s_n. Note that poles of a system are the roots of the characteristic equation of the system. Thus, the desired characteristic equation is

$$(s-s_1)\,(s-s_2)\ldots(s-s_n)=0, \tag{10.72}$$

which is essentially the same as the closed-loop characteristic equation given by Equation 10.71,

$$\left|s\mathbf{I}-(\mathbf{A}-\mathbf{BK})\right|=(s-s_1)(s-s_2)\ldots(s-s_n). \tag{10.73}$$

Equating the coefficients of like powers of s on both sides yields the values of the gains k_1, k_2, …, and k_n.

Example 10.22: Full-State Feedback Control Design

Consider the DC motor–driven cart discussed in Example 10.12, in which a PD controller was designed to achieve the requirements: overshoot $M_\mathrm{p} < 10\%$ and rise time $t_\mathrm{r} < 0.15$ s.

a. Find a full-state feedback controller such that the closed-loop system achieves the same requirements.
b. Use MATLAB to find the control gain matrix **K**.

Solution

a. The transfer function of the cart is given by

$$G(s)=\frac{Y(s)}{U(s)}=\frac{3.778}{s^2+16.883s},$$

where the output y is the position of the cart and the input u is the voltage applied to the DC motor. Using the position and the velocity as the state variables, that is, $x_1 = y$ and $x_2 = \dot{y}$, we find the state-space model as

$$\dot{\mathbf{x}}=\begin{bmatrix}0 & 1\\ 0 & -16.883\end{bmatrix}\mathbf{x}+\begin{bmatrix}0\\ 3.778\end{bmatrix}u,$$

$$y=[1\quad 0]\mathbf{x}.$$

For a second-order system, the gain matrix **K** is 1 × 2 and

$$\mathbf{K}=[k_1\quad k_2].$$

Applying Equation 10.71 yields the theoretical characteristic polynomial

$$\begin{aligned}|s\mathbf{I}-\mathbf{A}+\mathbf{BK}| &= \left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -16.883 \end{bmatrix} + \begin{bmatrix} 0 \\ 3.778 \end{bmatrix}[k_1 \quad k_2]\right| \\ &= \begin{vmatrix} s & -1 \\ 3.778k_1 & s+16.883+3.778k_2 \end{vmatrix} \\ &= s^2+(16.883+3.778k_2)s+3.778k_1.\end{aligned}$$

The time-domain specifications, $M_p < 10\%$ and $t_r < 0.15$ s, indicate that $\zeta > 0.59$ and $\omega_n > 12.33$ rad/s. Choose the same values of ω_n and ζ as in Example 10.12, that is, $\omega_n = 13.5$ rad/s and $\zeta = 0.65$. The desired closed-loop poles are then located at $p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -8.775 \pm 10.26j$. Applying Equation 10.72 gives the desired characteristic polynomial

$$(s-p_1)(s-p_2) = s^2+2\zeta\omega_n s+\omega_n^2 = s^2+17.55s+182.25.$$

Equating the two characteristic polynomials, we have

$$\begin{aligned}16.883+3.778k_2 &= 17.55,\\ 3.778k_1 &= 182.25,\end{aligned}$$

which gives $\mathbf{K} = [48.24 \quad 0.18]$. So the full-state feedback controller is

$$u = -\mathbf{Kx} = -48.24x_1 - 0.18x_2.$$

b. The following is the MATLAB session used to compute a full-state feedback control gain matrix **K** using pole placement design:

```
>> A = [0 1; 0 -16.883];
>> B = [0; 3.778];
>> p = [-8.775+10.26j -8.775-10.26j]; % desired poles
>> K = place(A,B,p);
```

The command `place` returns the gain matrix **K**, for which the full-state feedback $u = -\mathbf{Kx}$ places the closed-loop poles at the desired locations.

It is interesting to see that the values of the gains k_1 and k_2 are the same as the values of the proportional and derivative gains k_P and k_D found in Example 10.12. This is because of the way we selected the states. In this example, $u = -48.24y - 0.18\dot{y}$, and in Example 10.12, $u = 48.24(r - y) + 0.18\,d(r - y)/dt$. If the reference signal r is zero, the two controllers will end up with the same expression. This implies that the closed-loop system with full-state feedback is a regulation system. For tracking control, the control law $u = -\mathbf{Kx}$ needs to be modified (more details can be found in control texts). Note that if a different set of state variables is selected, the corresponding gain matrix **K** will be different. The reader can solve Problem 5 in Problem Set 10.7 to verify this conclusion.

The full-state feedback control method requires that all state variables are measured. However, this is usually not a practical assumption. To make a full-state feedback controller practically implementable, an estimator or observer can be designed to compute an estimate of the state variables based on the measurements of the system. Then, the control law calculations are based on the estimated state rather than the actual state.

PROBLEM SET 10.7

1. For the system shown in Figure 10.79, derive the state-space equations using the state variables indicated. Make sure to give the **A**, **B**, **C**, and D matrices. Also determine the poles of the system.
2. Repeat Problem 1 for the system shown in Figure 10.80.
3. Determine the controllability and observability for each of the following systems:

a. $$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} -5 & -3 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u, \quad y = [0 \quad 1 \quad 6] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

b. $$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} -1 & -1 & -2 \\ 4 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u, \quad y = [2 \quad 2 \quad 1] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

c. $$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 7 & -6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u, \quad y = [1 \quad 1 \quad 0] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

d. $$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = [1 \quad 1 \quad 1] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

4. Consider the two-degree-of-freedom mass–spring system shown in Figure 10.81, in which two masses are to be controlled by two equal and opposite forces f. The equations of motion of the system are derived as

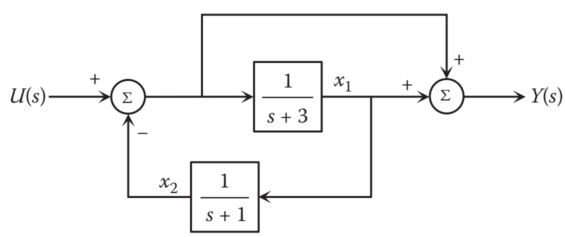


FIGURE 10.79 Problem 1.

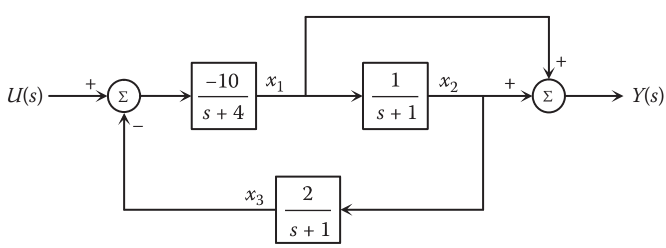


FIGURE 10.80 Problem 2.

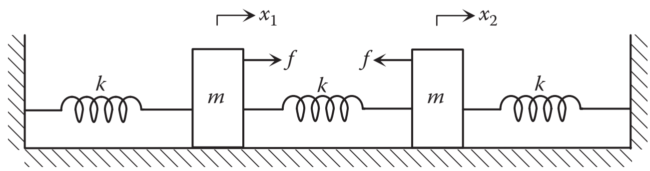


FIGURE 10.81 Problem 4.

$$m\ddot{x}_1 + 2kx_1 - kx_2 = f,$$
$$m\ddot{x}_2 - kx_1 + 2kx_2 = -f.$$

Show that the system is uncontrollable. Using the concept of mode discussed in Section 9.4, associate a physical meaning with the controllable and uncontrollable modes.

5. Reconsider Example 10.22. Using the approach in Section 4.4.1, find the controllable canonical form for the plant transfer function and then design a full-state feedback controller that places the closed-loop poles at the same locations as in the example.
6. A regulation system has a plant with the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{-2}{s^3 + 2s^2 + 5s + 10}.$$

a. Transform the plant transfer function into the state-space form with the state vector $\mathbf{x} = [y \quad \dot{y} \quad \ddot{y}]^T$.
b. Determine the state-feedback gain matrix **K** such that the closed-loop poles are located at $p_{1,2} = -3 \pm 3\text{j}$ and $p_3 = -5$.
c. Verify the result in Part (b) by using the MATLAB command `place`.

7. Consider the system

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = [1 \quad 0] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}.$$

a. Design a state-feedback controller so that the closed-loop poles have a damping ratio $\zeta = 0.8$ and a natural frequency $\omega_n = 5$ rad/s.
b. Verify the result in Part (a) by using the MATLAB command `place`.

8. Consider the system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{(s+2)(s+3)(s+4)}.$$

a. Design a state-feedback controller so that the closed-loop response has an overshoot of less than 5% and a rise time under 0.5 s. Set one of the closed-loop poles at -10.
b. Verify the result in Part (a) by using the MATLAB command `place`.

10.8 INTEGRATION OF SIMULINK AND SIMSCAPE INTO CONTROL DESIGN

In this chapter, we introduced root locus, Bode plot, and state-space techniques, all of which are model-based control design methods. The main steps in a model-based control design method are plant modeling, controller analysis and synthesis, computer simulation, and real-time implementation. To control a dynamic system, a mathematical model is first derived by applying physical laws (such as Newton's second law, Kirchhoff's law, conservation of mass, etc.) or identified using experimental data. Then, closed-loop stability and performance requirements are determined by analyzing the dynamics of the plant, and a controller is designed based on the mathematical model of the plant to meet all requirements. Before implementing the controller on a real dynamic system, computer simulation is usually conducted to verify the closed-loop stability and performance requirements. Examples given in this section illustrate how to integrate Simulink and Simscape into control design to investigate the closed-loop system characteristics.

10.8.1 Control System Simulation Using Simulink

Simulink is a graphical tool that allows us to simulate a feedback control system. A Simulink model representing the block diagram in Figure 10.1 can be constructed, in which the plant could be a simple linear time-invariant or even a nonlinear model. Compared with the Simulink examples of system modeling discussed in Chapters 5 through 7, there is another important and necessary component in control system simulation, which is the controller. Several examples in Sections 10.3 and 10.4 have shown us how to build a Simulink block diagram of a control system, in which the plant is represented by a transfer function. In this section, we will consider simulation of a state feedback control system, in which the plant is represented in state-space form.

Example 10.23: Full-State Feedback Control of a DC Motor–Driven Cart

Consider the full-state feedback control system discussed in Example 10.22, in which the state-space representation of the plant is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -16.883 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 3.778 \end{bmatrix}u,$$

$$y = [1 \quad 0]\mathbf{x},$$

and the mathematical model of the controller is

$$u = -\mathbf{K}\mathbf{x}$$

with $\mathbf{K} = [48.24 \;\; 0.18]$. Build a Simulink block diagram of the feedback control system and find the closed-loop response if the cart is initially 1 m away from the equilibrium position.

Solution

There are two ways to construct a Simulink block diagram of a full-state feedback control system. If we treat the state-space model as its scalar counterpart,

$$\dot{x} = ax + bu,$$

$$y = cx,$$

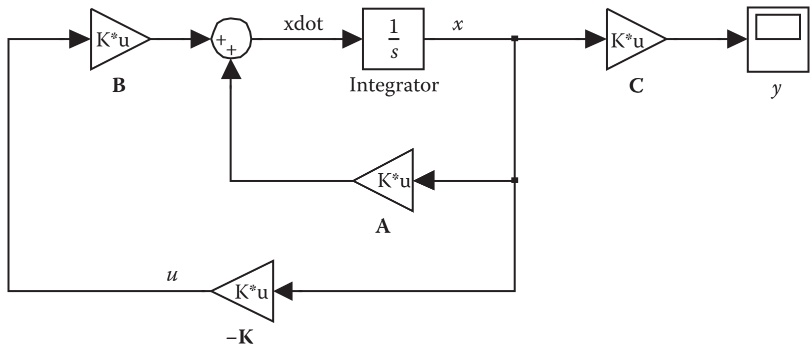


FIGURE 10.82 Simulink block diagram of a full-state feedback control system.

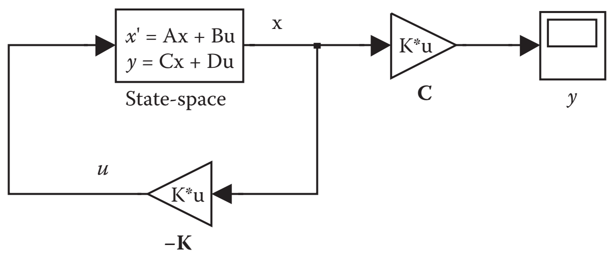


FIGURE 10.83 Alternative Simulink block diagram of a full-state feedback control system.

then we can build a block diagram as shown in Figure 10.82, in which the state-space model is represented by using one `Integrator` block and three `Gain` blocks (**A**, **B**, and **C**). Note that all the gains, including the control gain **K**, are in matrix form. This should be specified explicitly in Simulink. Double-click on each `Gain` block, define the corresponding matrix, and choose `Matrix (K*u)` for the `Multiplication` parameter. Recall that we have chosen $x_1 = y$ and $x_2 = \dot{y}$. Thus, the physical position variable is the same as the first state variable. To specify the nonzero initial position, double-click on the `Integrator` block and type `[1; 0]` for the `Initial conditions` parameter.

Figure 10.83 presents an alternative Simulink block diagram, in which the state-space model is built using the `State-Space` block instead of the `Integrator` block. Note that all state-variables must be available for a full-state feedback control system because the control signal is $u = -\mathbf{K}\mathbf{x}$. To simulate this, double-click on the `State-Space` block and define `C` as an identity matrix and `D` as a zero matrix with compatible dimensions (`eye(2)` for `C` and `zeros(2,1)` for `D` in this example). The parameter of `Initial conditions` has the same value as defined in Figure 10.82. To obtain the output y, a `Gain` block is included to define the real matrix C. It should be pointed out that the value of the `Gain` block corresponding to the full-state feedback controller is $-\mathbf{K}$, not $\mathbf{K}$.

Running both simulations yields the same curve as shown in Figure 10.84, which is the resulting displacement response $y(t)$ due to the nonzero initial condition of 1 m. It is interesting to note that the curve in Figure 10.84 is a mirror image of the unit-step response curve in Figure 10.35 about the x-axis. The reason is left to the reader to find out.

10.8.2 Integration of Simscape into Control System Simulation

Instead of Simulink blocks, such as `Transfer Fcn`, `State-Space`, etc., Simscape models can also be integrated into control system simulation to model the open-loop plant dynamics and test closed-loop system performance.

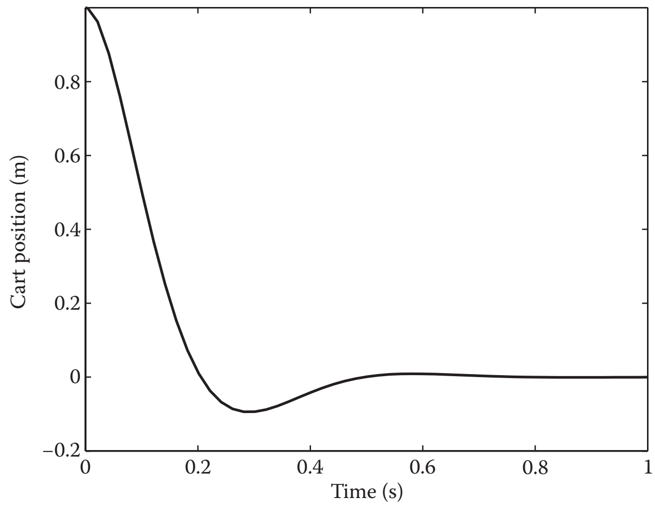


FIGURE 10.84 Closed-loop response of the cart system with full-state feedback control.

Example 10.24: Control of a Single-Degree-of-Freedom Mass–Spring–Damper System

Consider a single-degree-of-freedom mass–spring–damper system as shown in Figure 5.29, where m = 2 kg, b = 2 N·s/m, and k = 50 N/m. A PD controller, $f = -k_p x - k_D \dot{x}$, is designed to adjust the input force f so that the mass block can be maintained in the equilibrium position regardless of disturbance forces applied to the block. The performance requirements of the closed-loop system are overshoot $M_p < 5\%$ and rise time $t_r < 0.25$ s.

a. Design a PD controller to meet the performance requirements.
b. Build a block diagram of the feedback control system, in which the plant is constructed using Simscape blocks and the controller is constructed using Simulink blocks. Find the closed-loop response if the mass block is initially 0.1 m away from the equilibrium position.

Solution

a. The dynamics of the plant is described by

$$m\ddot{x} + b\dot{x} + kx = f$$

where the control force is

$$f = -k_p x - k_D \dot{x}.$$

Combining the two equations gives the dynamics of the closed-loop system,

$$m\ddot{x} + (b + k_D)\dot{x} + (k + k_p)x = 0,$$

which is a second-order system. Thus, the coefficients in the above differential equation can be related to the natural frequency and damping ratio of the closed-loop system via

$$\frac{b+k_D}{m} = 2\zeta\omega_n,$$

$$\frac{k+k_p}{m} = \omega_n^2.$$

The requirement for overshoot indicates

$$\zeta > 0.69.$$

Pick $\zeta = 0.75$ and substitute into the requirement for rise time to obtain

$$\omega_n > 9.26 \text{ rad/s}.$$

Pick $\omega_n = 10$ rad/s. Simultaneous solution of the two relations given above yields $k_p = 150$ and $k_D = 28$.

b. Figure 10.85 is the block diagram of the resulting feedback control system built using Simulink and Simscape. The plant is constructed based on the physical mass–spring–damper system and the details on Simscape modeling can be found in Example 5.4. The controller is constructed using Simulink blocks and its structure is similar to the PD control discussed in Section 10.4 with the reference signal r set as 0. To specify a nonzero initial

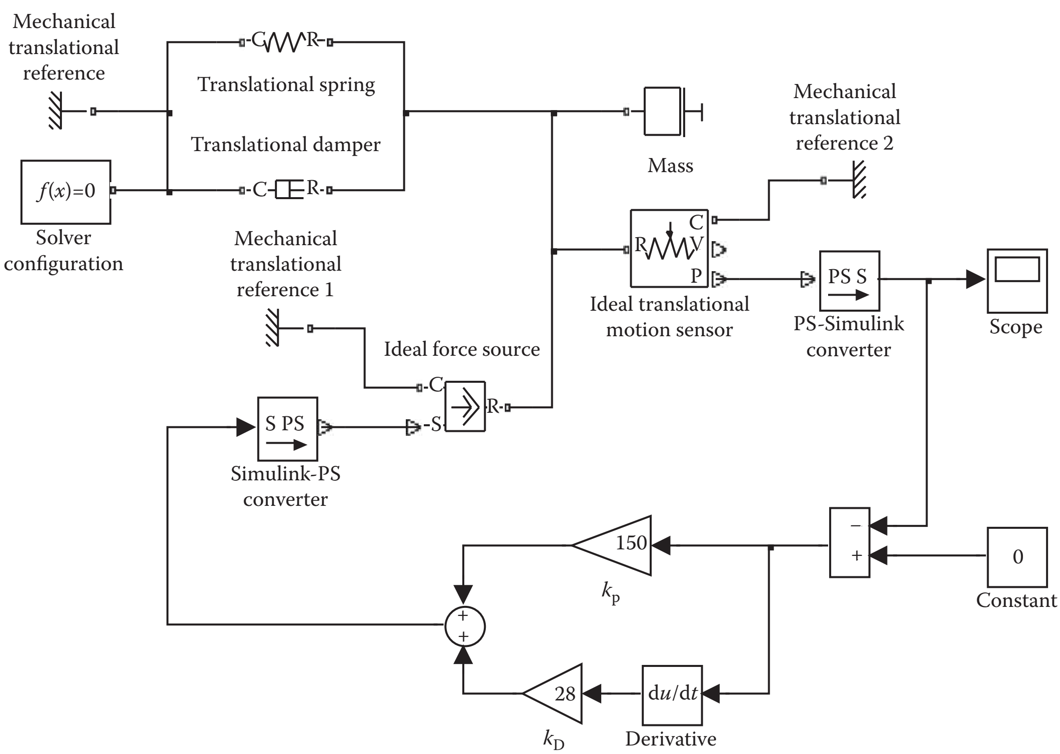


FIGURE 10.85 Simscape block diagram of the feedback control system in Example 10.24.

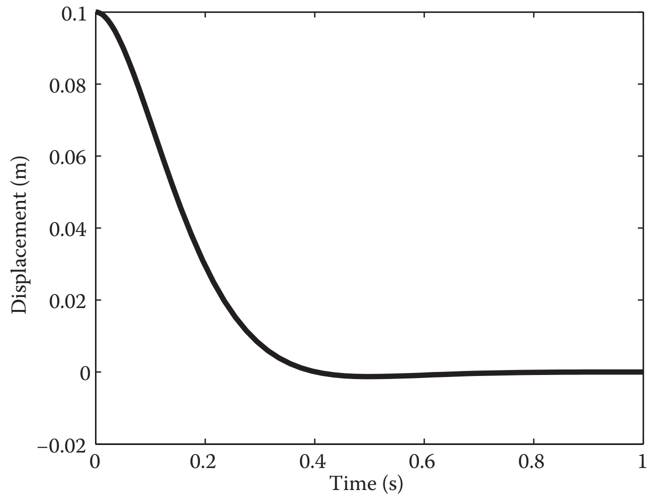


FIGURE 10.86 Closed-loop response of the mass–damper–spring system with PD control.

position, double-click on the `Translational Spring` block, type `0.1` for the Initial deformation, and choose the unit as `m`. This implies that the spring is initially elongated by 0.1 m. Also, double-click on the `Ideal Translational Motion Sensor` block, type `0.1` for the Initial position, and choose the unit as `m`. The corresponding displacement response of the system is shown in Figure 10.86.

PROBLEM SET 10.8

1. Consider the control system shown in Figure 10.40 (Problem Set 10.4, Problem 3). Using the results obtained in Part (a) of the cited problem, build a Simulink block diagram to simulate the feedback control system and find the unit-step response of the closed-loop system.
2. Repeat Problem 1 for the control system shown in Figure 10.41 (Problem Set 10.4, Problem 4).
3. Repeat Problem 1 for the control system shown in Figure 10.42 (Problem Set 10.4, Problem 5).
4. Repeat Problem 1 for the control system shown in Figure 10.43 (Problem Set 10.4, Problem 6).
5. Consider the control system shown in Figure 10.61 (Problem Set 10.5, Problem 7). Using the results obtained in Part (b) of the cited problem, build a Simulink block diagram to simulate the feedback control system and find the unit-step response of the closed-loop system.
6. Repeat Problem 5 for the control system shown in Figure 10.62 (Problem Set 10.5, Problem 8).
7. Consider Problem 6 in Problem Set 10.7. Using the state-space model obtained in Part (a) and the full-state feedback controller obtained in Part (b), build a Simulink block diagram to simulate the resulting feedback control system. Find the closed-loop response if the initial conditions are $y(0) = 0.1$, $\dot{y}(0) = 0$, and $\ddot{y}(0) = 0$.
8. Consider Problem 7 in Problem Set 10.7. Using the full-state feedback controller obtained in Part (b), build a Simulink block diagram to simulate the resulting feedback control system. Find the closed-loop response if the initial conditions are $x_1(0) = 0.1$ and $x_2(0) = 0$.
9. Consider the rotational mass–spring–damper system in Example 5.20. A PD controller, $\tau = -k_{\mathrm{P}}\theta - k_{\mathrm{D}}\dot{\theta}$, is designed to adjust the input torque τ so that the rotational disk can

quickly return to the equilibrium position regardless of disturbances applied to the system. The performance requirements of the closed-loop system are overshoot $M_p < 5\%$ and rise time $t_r < 0.004$ s.

a. Design a PD controller to meet the performance requirements.
b. Build a block diagram of the feedback control system, in which the plant is constructed using Simscape blocks and the controller is constructed using Simulink blocks. Find the closed-loop response if the disk is initially 0.1 rad away from the equilibrium position.

10. Consider the mass–spring–damper system shown in Figure 5.118 (Problem Set 5.6, Problem 3). Assume that f is a control force to maintain the system at equilibrium regardless of disturbances applied to the system.
 a. Design a full-state feedback controller such that the closed-loop poles are located at $-10 \pm 10j$, -15, and -16. Assume the state vector is $\mathbf{x} = [x_1 \quad x_2 \quad \dot{x}_1 \quad \dot{x}_2]^T$.
 b. Build a Simulink block diagram of the feedback control system. Find the closed-loop response if mass 1 is initially 0.1 m from the equilibrium position.

10.9 SUMMARY

This chapter presented an introduction to feedback control systems. The essential components of a feedback control include a system we want to control, a controller we need to design, an actuator used to drive the controlled system, and a sensor used to measure the system output. Generally, the controlled system and the actuator are intimately connected, and they can be combined as one component called the plant. Unlike open-loop control, the output signal of the plant in a feedback control system is measured and fed back for use in computing the control signal. In contrast to open-loop control, feedback can be used to stabilize unstable systems, reduce steady-state errors to disturbances, improve reference tracking performance, and reduce sensitivity to parameter variations.

Stability and performance are two important considerations in control. A linear time-invariant system is said to be stable if and only if all its poles have negative real parts and is unstable otherwise. In terms of the pole locations in the s-plane, the imaginary axis is the stability boundary between the stable left-half s-plane and the unstable right-half s-plane. Solving for the poles of a high-order linear system by hand is not an easy task. Routh's stability criterion is a method of obtaining information about pole locations without solving for the poles. A system is stable if and only if all the elements in the first column of the Routh array are positive.

The locations of poles in the s-plane are also associated with performance measures, which are rise time t_r, overshoot M_p, peak time t_p, and settling time t_s in the time domain, and bandwidth ω_{BW} and resonant peak M_r in the frequency domain. For a second-order system with poles at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$, the correspondences between the system parameters and the time-domain specifications are given by

$$t_r \approx \frac{1.12 - 0.078\zeta + 2.230\zeta^2}{\omega_n},$$

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}},$$

$$t_p = \frac{\pi}{\omega_d},$$

$$t_s = -\frac{\ln(\Delta\sqrt{1-\zeta^2})}{\zeta\omega_n},$$

where Δ is a small value, such as 1%, 2%, and 5%. The resonant peak M_r is similar to the overshoot M_p, both of which are related to the damping ratio ζ, whereas the bandwidth ω_{BW} is similar to the rise time t_r, both of which are related to the natural frequency ω_n.

The PID controller is a generic feedback control structure widely used in industries and is described by the transfer function

$$\frac{U(s)}{E(s)} = k_p + \frac{k_I}{s} + k_D s.$$

In general, a larger proportional gain k_p results in a faster response and a smaller steady-state error. However, an excessively large proportional gain k_p leads to lightly damped oscillations and even instability. A larger integral gain k_I reduces steady-state errors more quickly, but reduces damping leading to a larger overshoot. A larger derivative control decreases the overshoot, but slows down the speed of response. The PID gains can be tuned using the reaction curve method or the ultimate sensitivity method developed by Ziegler and Nichols.

Three different methods were introduced in this chapter for stability analysis and proportional feedback control design: root locus, Bode plot, and state-space methods. The root locus and Bode plot work with graphs obtained from open-loop transfer functions, whereas the state-space method works directly with mathematical models in state-space form.

For a negative feedback system with $KL(s)$ as the open-loop transfer function, a root locus is a graph of the closed-loop poles or the roots of the closed-loop characteristic equation

$$1 + KL(s) = 0,$$

with respect to the control gain K. The rules for sketching a root locus are presented in Section 10.5. Using the root locus technique, it is very easy to determine the stability of a closed-loop system when the proportional gain K varies from 0 to ∞. For a particular value of K, the closed-loop system is stable if and only if all of the poles are in the left-half s-plane.

The Bode plot is a graph of the frequency response function, using a linear scale for magnitude (in decibels) and phase (in degrees) and a logarithmic scale for frequency (in rad/s). For a frequency response function in the form of

$$KG(j\omega) = K\frac{(j\omega - z_1)(j\omega - z_2)\cdots(j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2)\cdots(j\omega - p_n)},$$

the Bode plot can be easily drawn by hand using the rules described in Section 10.6. The stability of the corresponding closed-loop system can be determined by the GM and PM, both of which can be found directly by inspecting the open-loop Bode plot.

A dynamic system described in state-space form is stable if all eigenvalues of the state matrix have negative real parts. If all the states are measurable, a full-state feedback controller given by

$$u = -\mathbf{Kx}$$

can be designed to improve stability and performance. The feedback gain matrix $\mathbf{K}$ can be determined using the pole placement method. Closed-loop poles are selected depending on the desired transient response. Equating the theoretical and desired closed-loop characteristic polynomials

$$\left|s\mathbf{I} - (\mathbf{A} - \mathbf{BK})\right| = (s - s_1)(s - s_2)\ldots(s - s_n)$$

yields the values of the elements k_1, k_2, …, and k_n of the gain matrix $\mathbf{K}$.

REVIEW PROBLEMS

1. Consider the feedback control system shown in Figure 10.87. Determine the range of K for closed-loop stability.
2. Consider the feedback control system shown in Figure 10.88.
   a. Design a PD controller such that the closed-loop poles are at $p_{1,2} = -1 \pm \sqrt{3}\mathrm{j}$.
   b. Estimate the rise time, overshoot, peak time, and 1% settling time for the unit-step response of the closed-loop system.
   c. Use MATLAB to plot the unit-step response of the closed-loop system. Verify the estimates obtained in Part (b).
3. Consider the feedback control system shown in Figure 10.89.
   a. Assuming $C(s) = k_p$, determine the value of the proportional gain that makes the closed-loop system marginally stable. Find the frequency of the sustained oscillation.
   b. Using the gain and the frequency obtained in Part (a), apply the ultimate sensitivity method of Ziegler–Nichols tuning rules to design a PID controller.
   c. Plot the unit-step response of the resulting closed-loop system. Find the values of the rise time t_r, overshoot M_p, peak time t_p, and 1% settling time t_s.
4. Consider the feedback control system shown in Figure 10.90.
   a. Determine the value of the gain K such that the undamped natural frequency ω_n and the damping ratio ζ of the dominant closed-loop poles are roughly 2 rad/s and 0.5, respectively.
   b. Determine the values of all closed-loop poles.
   c. Plot the unit-step response of the resulting closed-loop system. Find the values of the rise time t_r, overshoot M_p, peak time t_p, and 1% settling time t_s.
5. Consider a unity negative feedback system with the open-loop transfer function

$$KG(s) = \frac{K}{s(s+2)(s+4)}.$$

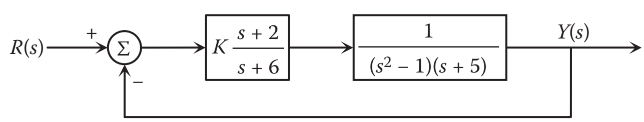


FIGURE 10.87 Problem 1.

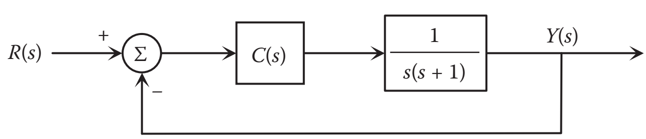


FIGURE 10.88 Problem 2.

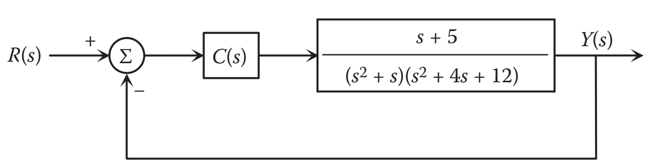


FIGURE 10.89 Problem 3.

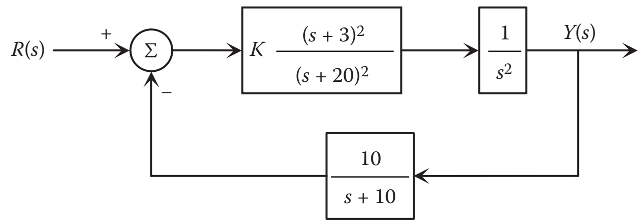


FIGURE 10.90 Problem 4.

a. Use MATLAB to draw the Bode plots for $K = 1$. Determine the range of K for which the closed-loop system will be stable.
b. Determine the range of K for closed-loop stability by sketching the root locus.
c. Using Routh's criterion, determine the range of K for closed-loop stability.

6. Consider the unity negative feedback system with a PD controller shown in Figure 10.91.
   a. Determine the value of the proportional gain k_p such that the damping ratio of the closed-loop system is 0.7.
   b. What is the GM of the system if k_p is set to the value obtained in Part (a)? Answer this question without creating the Bode plots.
   c. Verify your answer in Part (b) by creating the Bode plots using MATLAB.
7. Consider the system

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = [1 \quad 0] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}.$$

   a. Design a state-feedback controller so that the closed-loop unit-step response has an overshoot of less than 5% and a peak time under 1.5 s.
   b. Verify the results of Part (a) in MATLAB.
8. Consider the cart–inverted-pendulum system shown in Example 5.13. Assume that the mass of the cart is 0.8 kg, the mass of the pendulum is 0.2 kg, and the length of the pendulum is 0.6 m.
   a. Determine the poles of the linearized system. Is the linearized system stable or unstable?
   b. Design a full-state feedback controller for the linearized system such that the closed-loop poles are located at $p_{1,2} = -2.90 \pm 2.15j$, $p_3 = -10$, and $p_4 = -20$.
   c. Assume that the initial angle of the inverted pendulum is 5° away from the vertical reference line. Using the state feedback gain matrix **K** obtained in Part (b), examine the responses of the nonlinear and linearized closed-loop systems using Simulink.
9. Consider the two-degree-of-freedom quarter-car model shown in Figure 5.34, in which the force f, applied between the car body and the wheel–tire–axle assembly, is controlled by feedback and represents the active components of the suspension system. Assume that

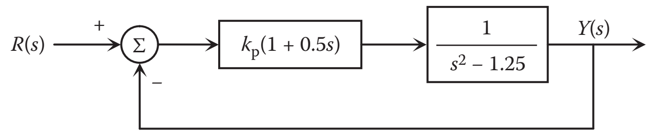


FIGURE 10.91 Problem 6.

$f = 20568x_1 - 30493x_2 - 1278\dot{x}_1 + 3189\dot{x}_2$. Build a Simulink block diagram of the feedback control system. Find the displacement responses $x_1(t)$ and $x_2(t)$ if initially $x_1 = -0.05$ m and $x_2 = -0.05$ m. Ignore the displacement input $z(t)$. What are the system responses $x_1(t)$ and $x_2(t)$ without control?

10. Consider the DC motor–driven wheeled mobile robot shown in Figure 6.83, in which the voltage applied to the DC motor is computed by a controller. Assume that $v_a = 2.56i - 0.37x + 4.61\dot{x} + 0.37x_r$, where x_r is a reference trajectory that the cart should follow. Build a block diagram of the feedback control system, in which the mobile robot is constructed using Simscape blocks and the controller is constructed using Simulink blocks. Find the displacement response $x(t)$ of the mobile robot if a unit-step reference command signal is sent to the system.

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Appendix A

TABLE A.1
The International System (SI) of Units

Basic Units

Quantity	Unit Name	Symbol or Unit
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Voltage	volts	V
Temperature	Kelvin	K
Amount of substance	mole	mol
Solid angle	radian	rad
SI Derived Units		
(Circular) frequency (ω)	rad/s	
Acceleration (a)	m/s^2	
Angular acceleration (α)	rad/s^2	
Angular velocity (ω)	rad/s	
Area (A)	m^2	
Area moment of inertia (I)	m^4	
Density, mass density (ρ)	kg/m^3	
Dynamic viscosity (b,c)	N·s/m	
Electric capacitance (C)	farad (F)	A·s/V
Electric charge (q)	coulomb (C)	A·s
Electric resistance (R)	ohm (Ω)	V/A
Force (f)	newton (N)	kg·m/s^2
Frequency	hertz (Hz)	1/s
Inductance (L)	henry (H)	V·s/A
Magnetic flux	weber (Wb)	V·s
Magnetic flux density	tesla (T)	Wb/m^2
Mass moment of inertia (I)		kg·m^2
Power	watt (W)	J/s
Pressure, mechanical stress	pascal (Pa)	N/m^2
Specific heat (c, c_p, c_v)		J/(kg·K)
Specific volume (v)		m^3/kg
Thermal conductivity (k)		W/(s·m·K)
Velocity, speed (v)		m/s
Volume (V)		m^3
Wave length (λ)		1/m
Work (W), energy (E), heat (Q)	joule (J)	N·m

TABLE A.2
Conversion Factors

Density	1 g/cm^3 = 62.43 lb_m/ft^3
Energy	1 cal = 4.184 J
Force	1 lb_f = 4.45 N
Length	1 in. = 2.54 cm
	1 ft = 0.3048 m
	1 mi. = 1609 m = 5280 ft
Mass	1 lb_m = 0.4536 kg = 16 oz.
	1 slug = 32.174 lb_m
	1 ton = 2000 lb_m
Power	1 W = 3.413 Btu/h
Pressure	1 atm = 1.0132×10^5 Pa
Temperature	°C = (°F – 32)/1.8
	°F = °C(1.8) + 32
	K = °C + 273.16
	°R = °F + 459.69
	1 K = 1.8°R
Thermal conductivity	1 W/(m.°C) = 0.5778 Btu/(h·ft·°F)
Volume	1 liter (L) = 1000 cm^3 = 0.0353 ft^3 = 1.0564 quart
	1 ft^3 = 28.316 L
	1 gal = 3.785 L = 4 quarts
	1 quart = 2 pints = 67.2 $in.^3$ = 0.9466 L
	1 pint = 16 oz.

Appendix B: Useful Formulas

Trigonometric Expansions

Sum to Product	Product to Sum
$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$	$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$	$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$
$\tan(a \pm b) = \dfrac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$	$\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$
	$\cos a \sin b = \frac{1}{2}[\sin(a+b) - \sin(a-b)]$

Double-Angle and Half-Angle Formulas

Double-Angle Formulas	Half-Angle Formulas
$\sin^2 a = \frac{1}{2}(1 - \cos 2a)$	$\cos \frac{1}{2} a = \sqrt{\frac{1}{2}(1 + \cos a)}$
$\cos^2 a = \frac{1}{2}(1 + \cos 2a)$	$\sin \frac{1}{2} a = \sqrt{\frac{1}{2}(1 - \cos a)}$
$\sin 2a = 2 \sin a \cos a$	
$\cos 2a = 2\cos^2 a - 1$	

Hyperbolic Functions

$\sinh a = \frac{1}{2}(e^a - e^{-a})$	$\sinh(a \pm b) = \sinh a \cosh b \pm \cosh a \sinh b$
$\cosh a = \frac{1}{2}(e^a + e^{-a})$	$\cosh(a \pm b) = \cosh a \cosh b \pm \sinh a \sinh b$
$\cosh^2 a - \sinh^2 a = 1$	$\sinh^2 a = \frac{1}{2}(\cosh 2a - 1)$
	$\cosh^2 a = \frac{1}{2}(1 + \cosh 2a)$

Integration

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int e^x \, dx = e^x + c$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$$

$$\int e^{g(x)} g'(x) \, dx = e^{g(x)} + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

$$\int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \tan x \, dx = -\ln|\cos x| + c = \ln|\sec x| + c$$

$$\int \cot x \, dx = \ln|\sin x| + c = -\ln|\csc x| + c$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + c$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + c$$

$$\int \ln x \, dx = x \ln x - x + c$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$\int \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \ln\left|\frac{x+a}{x-a}\right| + c = \frac{1}{a} \tanh^{-1} \frac{x}{a}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = \sinh^{-1} \frac{x}{a} + c$$

$$\int e^{ax} \sin bx \, dx = \frac{1}{a^2 + b^2} e^{ax} (a \sin bx - b \cos bx) + c$$

$$\int e^{ax} \cos bx \, dx = \frac{1}{a^2 + b^2} e^{ax} (a \cos bx + b \sin bx) + c$$

Mechanical Engineering

Modeling and Analysis of Dynamic Systems, Second Edition introduces MATLAB®, Simulink®, and Simscape™ and then uses them throughout the text to perform symbolic, graphical, numerical, and simulation tasks. Written for junior or senior level courses, the textbook meticulously covers techniques for modeling dynamic systems, methods of response analysis, and provides an introduction to vibration and control systems. These features combine to provide students with a thorough knowledge of the mathematical modeling and analysis of dynamic systems.

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Taylor & Francis Group
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6000 Broken Sound Parkway, NW
Suite 300, Boca Raton, FL 33487
711 Third Avenue
New York, NY 10017
2 Park Square, Milton Park
Abingdon, Oxon OX14 4RN, UK

K16666

ISBN: 978-1-4665-7493-9
90000
9 781466 574939

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